Internal DLA and the Gaussian free field

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Abstract

In previous works, we showed that the internal DLA cluster on $\mathbb{Z}^d$ with t particles is almost surely spherical up to a maximal error of $O(\log t)$ if $d = 2$ and $O(\sqrt{\log t})$ if $d \geq 3$. This paper addresses "average error": in a certain sense, the average deviation of internal DLA from its mean shape is of *constant* order when $d = 2$ and of order $r^{1-d/2}$ (for a radius r cluster) in general. Appropriately normalized, the fluctuations (taken over time and space) scale to a variant of the Gaussian free field.

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Contents

1 Introduction

1.1 Overview

We study scaling limits of *internal diffusion limited aggregation* ("internal DLA"), a growth model introduced in [MD86, DF91]. In internal DLA, one inductively constructs an **occupied set** $A_t \subset \mathbb{Z}^d$ for each time $t \geq 0$ as follows: begin with $A_0 = \emptyset$ and $A_1 = \{0\}$, and let A_{t+1} be the union of A_t and the first place a random walk from the origin hits $\mathbb{Z}^d \setminus A_t$. A continuum analogue of internal DLA is the famous Hele-Shaw model for fluid insertion.[1]

The purpose of this paper is to study the growing family of sets A_t. Following the pioneering work of [LBG92], it is by now well known that, for large t, the set A_t approximates an origin-centered Euclidean lattice ball $\mathbf{B}_r := B_r(0) \cap \mathbb{Z}^d$ (where $r = r(t)$ is such that $B_r(0)$ has volume t). The authors recently showed that this is true in a fairly strong sense [JLS09, JLS12a, JLS12b]: the maximal distance from a point where $1_{A_t} - 1_{\mathbf{B}_r}$ is non-zero to $\partial B_r(0)$ is a.s. $O(\log t)$ if $d = 2$ and $O(\sqrt{\log t})$ if $d \geq 3$. In fact, if C is large enough, the probability that this maximal distance exceeds $C \log t$ (or $C\sqrt{\log t}$ when $d \geq 3$) decays faster than any

[1] It follows from [LP10] that the internal DLA cluster formed from a finite set of point sources in $\mathbb{Z}^d$ has a scaling limit which solves an obstacle problem in $\mathbb{R}^d$. Hele-Shaw flow solves the same obstacle problem [GV06]. In contrast, the Witten-Sander model of *external* DLA [WS81], in which random walkers start "at infinity" and stop when reaching a site neighboring the cluster, is analogous to the (ill-posed) reverse time direction of Hele-Shaw flow.

fixed (negative) power of t. Some of these results are obtained by different methods in [AG10a, AG10b].

This paper will ask what happens if, instead of considering the maximal distance from $\partial B_r(0)$ at time t, we consider the "average error" at time t (allowing inner and outer errors to cancel each other out). It turns out that in a distributional "average fluctuation" sense, the set A_t deviates from $B_r(0)$ by only a constant number of lattice spaces when $d = 2$ and by an even smaller amount when $d \geq 3$. Appropriately normalized, the fluctuations of A_t, taken over time and space, define a distribution on $\mathbb{R}^d$ that converges in law to a variant of the Gaussian free field (GFF): a random distribution on $\mathbb{R}^d$ that we will call the **augmented Gaussian free field**. (It can be constructed by defining the GFF in spherical coordinates and replacing variances associated to spherical harmonics of degree k by variances associated to spherical harmonics of degree $k+1$; see §1.5.) The "augmentation" appears to be related to a damping effect produced by the mean curvature of the sphere (as discussed below).[2]

To our knowledge, no central limit theorem of this kind has been previously conjectured in either the physics or the mathematics literature. The appearance of the GFF and its "augmented" variants is a particular surprise. (It implies that internal DLA fluctuations — although very small — have long-range correlations and that, up to the curvature-related augmentation, the fluctuations in the direction transverse to the boundary of the cluster are of a similar nature to those in the tangential directions.) Nonetheless, the heuristic idea is easy to explain. Before we state the central limit theorems precisely (§1.3 and §1.4), let us explain the intuition behind them.

Write a point $x \in \mathbb{R}^d$ in polar coordinates as ru for $r \geq 0$ and $u \in \mathbb{R}^d$ on the unit sphere ($|u| = 1$). Suppose that at each time t the boundary of A_t is approximately parameterized by $r_t(u)u$ for a function r_t defined on the unit sphere. Write

$$r_t(u) = (t/\omega_d)^{1/d} + \rho_t(u)$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. The $\rho_t(u)$ term measures the deviation from circularity of the cluster A_t in the direction u. How do we expect ρ_t to evolve in time? To a first approximation, the angle at which a random walk exits A_t is a uniform point on the unit sphere. If we run many such random walks, we obtain a sort of Poisson point process on the sphere, which has a scaling limit given by space-time white noise on the sphere. However there is a smoothing effect coming from the fact that places where ρ_t is negative are more likely to be hit by the random walks than places where ρ_t is positive, and hence $|\rho_t|$ is more likely to shrink in time. There is also secondary damping effect coming from the mean curvature of the sphere, which implies that even if (after a certain time) particles began to hit

[2] Consider continuous time internal DLA on the half cylinder $(\mathbb{Z}/m\mathbb{Z})^{d-1} \times \mathbb{Z}_+$, with particles started uniformly on $(\mathbb{Z}/m\mathbb{Z})^{d-1} \times \{0\}$. Though we do not prove this here, we expect the cluster boundaries to be approximately flat cross-sections of the cylinder, and we expect the fluctuations to scale to the *ordinary* GFF on the half cylinder as $m \to \infty$.

all angles with equal probability, the magnitude of the ρ_t fluctuations would shrink with increasing t as the existing fluctuations were averaged over larger spheres.

The white noise should correspond to adding independent Brownian noise terms to the spherical Fourier modes of ρ_t. The rate of smoothing/damping in time should be approximately given by $\Lambda\rho_t$ for some linear operator Λ mapping the space of functions on the unit sphere to itself. Since the random walks approximate Brownian motion (which is rotationally invariant), we would expect Λ to commute with orthogonal rotations, and hence have spherical harmonics as eigenfunctions. With the right normalization and parameterization, it is therefore natural to expect the spherical Fourier modes of ρ_t to evolve as independent Brownian motions subject to linear "restoration forces" (a.k.a. Ornstein-Uhlenbeck processes) where the magnitude of the restoration force depends on the degree of the corresponding spherical harmonic. It turns out that the restriction of the (ordinary or augmented) GFF on $\mathbb{R}^d$ to a centered volume t sphere evolves in time t in a similar way.

Of course, as stated above, the "spherical Fourier modes of ρ_t" have not really been defined (since the boundary of A_t is complicated and generally *cannot* be parameterized by $r_t(u)u$). In the coming sections, we will define related quantities that (in some sense) encode these spherical Fourier modes and are easy to work with. These quantities are the martingales obtained by summing discrete harmonic polynomials over the cluster A_t.

The heuristic just described provides intuitive interpretations of the results given below. Theorem 1.3, for instance, identifies the weak limit as $t \to \infty$ of the internal DLA fluctuations from circularity at a fixed time t: the limit is the two-dimensional augmented Gaussian free field restricted to the unit circle $\partial B_1(0)$, which can be interpreted in a distributional sense as the random Fourier series

$$\frac{1}{\sqrt{2\pi}}\left[\alpha_0/\sqrt{2} + \sum_{k=1}^{\infty} \alpha_k \frac{\cos k\theta}{\sqrt{k+1}} + \beta_k \frac{\sin k\theta}{\sqrt{k+1}}\right] \tag{1}$$

where α_k for $k \geq 0$ and β_k for $k \geq 1$ are independent standard Gaussians. The ordinary two-dimensional GFF restricted to the unit circle is similar, except that $\sqrt{k+1}$ is replaced by $\sqrt{k}$.

The series (1) — unlike its counterpart for the one-dimensional Gaussian free field, which is a variant of Brownian bridge — is a.s. divergent, which is why we use the dual formulation explained in §1.4. The dual formulation of (1) amounts to a central limit theorem, saying that for each $k \geq 1$ the real and imaginary parts of

$$M_k = \frac{1}{r} \sum_{z \in A_{\pi r^2}} \left(\frac{z}{r}\right)^k$$

converge in law as $r \to \infty$ to normal random variables with variance $\frac{\pi}{2(k+1)}$ (and that M_j and M_k are asymptotically uncorrelated for $j \neq k$). See [FL12, §6.2] for numerical data on the moments M_k in large simulations.

1.2 FKG inequality statement and continuous time

Before we set about formulating our central limit theorems precisely, we mention a previously overlooked fact. Suppose that we run internal DLA in continuous time by adding particles at Poisson random times instead of at integer times: this process we will denote by $A_{T(t)}$ (or often just A_T) where $T(t)$ is the counting function for a Poisson point process in the interval $[0,t]$ (so $T(t)$ is Poisson distributed with mean t). We then view the entire history of the IDLA growth process as a (random) function on $[0,\infty)\times\mathbb{Z}^d$, which takes the value 1 or 0 on the pair (t,x) accordingly as $x \in A_{T(t)}$ or $x \notin A_{T(t)}$. Write Ω for the set of functions $f : [0,\infty)\times\mathbb{Z}^d \to \{0,1\}$ such that $f(t,x) \leq f(t',x)$ whenever $t \leq t'$, endowed with the coordinate-wise partial ordering. Let $\mathbb{P}$ be the distribution of $\{A_{T(t)}\}_{t\geq 0}$, viewed as a probability measure on Ω.

Theorem 1.1. (FKG inequality) *For any two increasing functions $F, G \in L^2(\Omega,\mathbb{P})$, the random variables $F(\{A_{T(t)}\}_{t\geq 0})$ and $G(\{A_{T(t)}\}_{t\geq 0})$ are nonnegatively correlated.*

One example of an increasing function is the total number $\#A_{T(t)}\cap X$ of occupied sites in a fixed subset $X \subset \mathbb{Z}^d$ at a fixed time t. One example of a decreasing function is the smallest t for which all of the points in X are occupied. Intuitively, Theorem 1.1 means that on the event that one point is absorbed at a late time, it is conditionally more likely for all other points to be absorbed late. The FKG inequality is an important feature of the discrete and continuous Gaussian free fields [She07], so it is interesting (and reassuring) that it appears in internal DLA at the discrete level.

Note that sampling a continuous time internal DLA cluster at time t is equivalent to first sampling a Poisson random variable T with expectation t and then sampling an ordinary internal DLA cluster of size T. (By the central limit theorem, $|t-T|$ has order $\sqrt{t}$ with high probability.) Although using continuous time amounts to only a modest time reparameterization (chosen independently of everything else) it is aesthetically natural. Our use of "white noise" in the heuristic of the previous section implicitly assumed continuous time. (Otherwise the total integral of ρ_t would be deterministic, so the noise would have to be conditioned to have mean zero at each time.)

1.3 Main results in dimension two

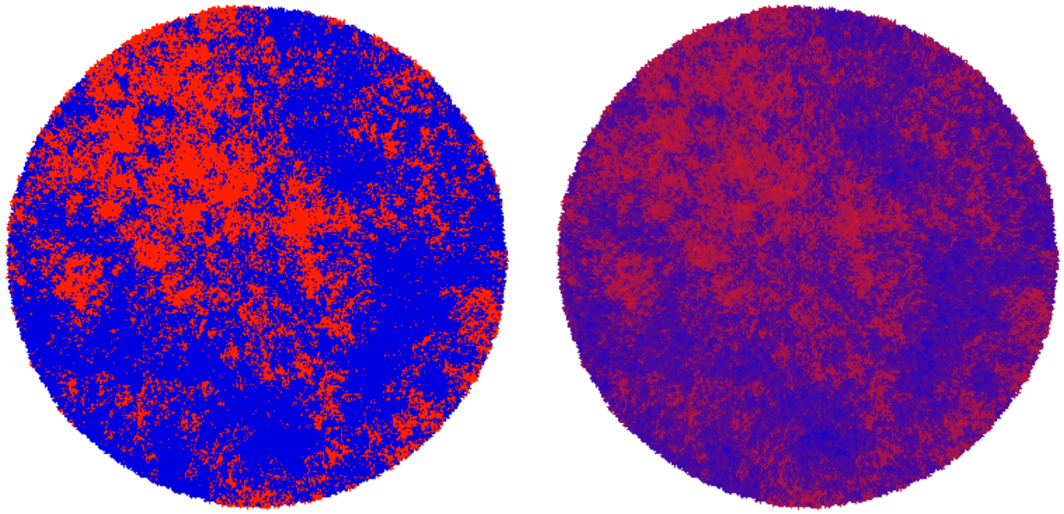
For $x \in \mathbb{Z}^2$ write

$$F(x) := \inf\{t : x \in A_{T(t)}\}$$

and

$$L(x) := \sqrt{F(x)/\pi} - |x|.$$

In words, $L(x)$ is the difference between the radius of the area t disk — at the time t that x was absorbed into A_T — and $|x|$. It is a measure of how much later or



(a) (b)

Figure 1: (a) Continuous-time IDLA cluster $A_{T(t)}$ for $t = 10^5$. Early points (where L is negative) are colored red, and late points (where L is positive) are colored blue. (b) The same cluster, with the function $L(x)$ represented by red-blue shading.

earlier x was absorbed into A_T than it would have been if the sets $A_{T(t)}$ were *exactly* centered discs of area t. By the main result of [JLS12a], almost surely

$$\limsup_{x\in\mathbb{Z}^2} \frac{L(x)}{\log|x|} < \infty.$$

The coloring in Figure 1(a) indicates the sign of the function $L(x)$, while Figure 1(b) illustrates the magnitude of $L(x)$ by shading. Note that the use of continuous time means that the average of $L(x)$ over x may differ substantially from 0. Indeed we see that — in contrast with the corresponding discrete-time figure of [JLS12a] — there are noticeably fewer early points than late points in Figure 1(a), which corresponds to the fact that in this particular simulation $T(t)$ was smaller than t for most values of t. Since for each fixed $x \in \mathbb{Z}^2$ the quantity $L(x)$ is a decreasing function of $A_t(x)$, the FKG inequality holds for L as well. The positive correlation between values of L at nearby points is readily apparent from the figure.

To state a limit theorem for the lateness function, consider its rescaling for $R > 0$

$$G_R((x_1,x_2)) := L((\lfloor Rx_1\rfloor, \lfloor Rx_2\rfloor)).$$

Identify $\mathbb{R}^2$ with $\mathbb{C}$ and let H_0 be the linear span of the set of functions on $\mathbb{C}$ of the form $\mathrm{Re}(az^k)f(|z|)$ for $a \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, and f smooth and compactly supported on $\mathbb{R}_{>0}$. The space H_0 is obviously dense in $L^2(\mathbb{C})$, and it turns out to be a convenient space of test functions. The augmented GFF (and its restriction to $\partial B_1(0)$) will be defined precisely in §1.4 and §1.5.

Theorem 1.2. (Weak convergence of the lateness function) *As $R \to \infty$, the function G_R converges to the augmented Gaussian free field h in the following sense: for each set of test functions $\phi_1, \ldots, \phi_k$ in H_0, the joint law of the inner products (ϕ_j, G_R) converges to the joint law of (ϕ_j, h).*

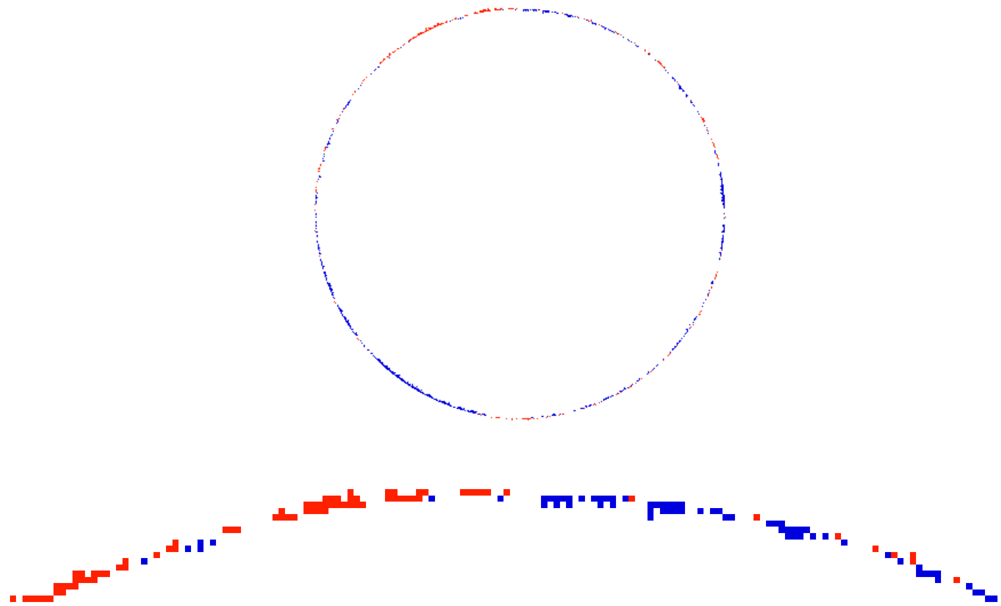


Figure 2: Top: Symmetric difference between IDLA cluster $A_{T(t)}$ at continuous time $t = 10^5$ and the disk of radius $\sqrt{t/\pi}$. Bottom: closeup of a portion of the boundary. Sites outside the disk are colored red if they belong to $A_{T(t)}$; sites inside the disk are colored blue if they do not belong to $A_{T(t)}$.

Our next result addresses the fluctuations from circularity at a fixed time, as illustrated in Figure 2.

Theorem 1.3. (Fluctuations from circularity) *Consider the distribution with point masses on $\mathbb{R}^2$ given by*

$$E_t := r^{-1} \sum_{x \in \mathbb{Z}^2} \left(1_{x \in A_{T(t)}} - 1_{x \in \mathbf{B}_r} \right) \delta_{x/r}, \tag{2}$$

where $r = \sqrt{t/\pi}$. As $t \to \infty$, the E_t converge to the restriction of the augmented GFF to $\partial B_1(0)$, in the sense that for each set of test functions $\phi_1, \ldots, \phi_k$ in H_0, the joint law of (ϕ_j, E_t) converges to the joint law of $\Phi_h(\phi_j, \pi)$ (a Gaussian process defined in §1.4).

1.4 Main results in general dimensions

In this section, we will extend Theorem 1.3 to general dimensions and to a range of times (instead of a single time). That is, we will try to understand scaling limits of the discrepancies of the sort depicted in Figure 2 (interpreted in some sense as random distributions) in general dimensions and taken over a range of times. However, some caution is in order. By classical results in number theory (see the survey [IKKN04] for their history), the size of $\mathbf{B}_r = B_r(0) \cap \mathbb{Z}^d$ is approximately the volume of $B_r(0)$ — but with errors of order r^{d-2} (i.e., both $O(r^{d-2})$ and $\Omega(r^{d-2})$) in all dimensions $d \geq 5$. The errors in dimension $d = 4$ are of order r^{d-2} times logarithmic factors that grow to infinity. It remains a famous open problem in number theory to estimate the errors when $d \in \{2,3\}$. (When $d = 2$ this is called Gauss's circle problem.)

These number theoretic results imply that $\#\mathbf{B}_{r(t)}$ is, as a function of t, much more irregular than the size $T(t)$ of the cluster obtained in continuous time internal DLA, at least when $d \geq 5$. The results also imply that even if points were added to A_t precisely in order of increasing radius, the difference between the radius of A_t and the radius of $B_{r(t)}(0)$ would fail to be $o(r^{-1})$ when $d \geq 5$ and fail to be $O(r^{-1})$ when $d = 4$.

On the other hand, we will see that the kinds of fluctuations that emerge from internal DLA randomness are of the order that one would obtain by spreading an extra $r^{d/2} \sim \sqrt{t}$ particles over a constant fraction of the spherical boundary, which is also what one obtains by changing the radius (along some or all of the boundary) by $r^{1-d/2}$. This implies that the higher dimensional analog of Theorem 1.3 cannot be true exactly the way it is stated if $d \geq 4$. Indeed, suppose that we define E_t analogously to (2) as

$$E_t = r^{-d/2} \sum_{x \in \mathbb{Z}^d} \big(1_{x \in A_{T(t)}} - 1_{x \in \mathbf{B}_r}\big)\delta_{x/r},$$

and let ϕ be a test function that is equal to 1 in a neighborhood of $\partial B_1(0)$. Then the results mentioned above imply that

$$(E_t, \phi) = r^{-d/2}\big(T(t) - \#\mathbf{B}_r\big)$$

cannot converge in law to a finite random variable if $d \geq 4$.

It is therefore a challenge to formulate a central limit theorem for the (small) fluctuations of internal DLA that is not swamped by these (potentially large) number theoretic irregularities. We will see below that this can be achieved by replacing $\mathbf{B}_r$ with different ball approximations (the so-called "divisible sandpiles") that are in some sense even "rounder" than the lattice balls themselves. We will also have to define and interpret the (augmented) GFF in a particular way.

Given smooth real-valued functions f and g on $\mathbb{R}^d$, write

$$(f, g)_\nabla = \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g(x) dx.$$

Here and below dx denotes Lebesgue measure on $\mathbb{R}^d$. Given a bounded domain $D \subset \mathbb{R}^d$, let $H(D)$ be the Hilbert space closure in $(\cdot,\cdot)_\nabla$ of the set of smooth compactly supported functions on D. We define $H = H(\mathbb{R}^d)$ analogously except that the functions are taken modulo additive constants. The Gaussian free field (GFF) is defined formally by

$$h := \sum_{i=1}^{\infty} \alpha_i f_i, \tag{3}$$

where the f_i are any fixed $(\cdot,\cdot)_\nabla$ orthonormal basis for H and the α_i are i.i.d. mean zero, unit variance normal random variables. (One also defines the GFF on D similarly, using $H(D)$ in place of H.) The augmented GFF will be defined similarly below, but with a slightly different inner product.

Since the sum (3) a.s. does not converge within H, one has to think a bit about how h is defined. Note that for any *fixed* $f = \sum \beta_i f_i \in H$, the quantity $(h,f)_\nabla := \sum(\alpha_i f_i, f)_\nabla = \sum \alpha_i \beta_i$ is almost surely finite and has the law of a centered Gaussian with variance $\|f\|_\nabla = \sum |\beta_i|^2$. However, there a.s. exist some functions $f \in H$ for which the sum does not converge, and $(h,\cdot)_\nabla$ cannot be considered as a continuous functional on all of H. Rather than try to define $(h,f)_\nabla$ for all $f \in H$, it is often more convenient and natural to focus on some subset of f values (with dense span) on which $f \mapsto (h,f)_\nabla$ is a.s. a continuous function (in some topology). Here are some sample approaches to defining a GFF on D:

1. h **as a random distribution:** For each smooth, compactly supported ϕ, write $(h,\phi) := (h, -\Delta^{-1}\phi)_\nabla$, which (by integration by parts) is formally the same as $\int h(x)\phi(x)dx$. This is almost surely well defined for all such ϕ and makes h a random distribution [She07]. (If $D = \mathbb{R}^d$ and $d = 2$, one requires $\int \phi(x)dx = 0$, so that (h,ϕ) is defined independently of the additive constant. When $d > 2$ one may fix the additive constant by requiring that the mean of h on $B_r(0)$ tends to zero as $r \to \infty$ [She07].)

2. h **as a random continuous $(d+1)$-real-parameter function:** For each $\varepsilon > 0$ and $x \in \mathbb{R}^d$, let $h_\varepsilon(x)$ denote the mean value of h on $\partial B_\varepsilon(x)$. For each fixed x, this $h_\varepsilon(x)$ is a Brownian motion in time parameterized by $-\log\varepsilon$ in dimension 2, or $-\varepsilon^{2-d}$ in higher dimensions [She07]. For each fixed ε, the function h_ε can be thought of as a regularization of h (a point of view used extensively in [DS10]).

3. h **as a family of "distributions" on origin-centered spheres:** For each polynomial function ψ on $\mathbb{R}^d$ and each time t, define $\Phi_h(\psi,t)$ to be the integral of $h\psi$ over $\partial B_r(0)$ where $B_r(0)$ is the origin-centered ball of volume t. We actually lose no generality in requiring ψ to be a harmonic polynomial on $\mathbb{R}^d$, since the restriction of any polynomial to $\partial B_r(0)$ agrees with the restriction of a (unique) harmonic polynomial.

The difference between these three approaches boils down to what test functions or measures we want to be able to integrate h against. In the first case we consider smooth test functions, in the second uniform measures on spheres, and in the third uniform measures on origin-centered spheres weighted by harmonic polynomials.

The last approach is the least intuitive, but it turns out to be particularly natural for our purposes. We define the augmented GFF h as a distribution corresponding to this class of test measures by defining the random variables $\Phi_h(\psi, t)$ for all $t > 0$ and harmonic polynomials ψ, as follows. Let Φ_h be the centered Gaussian function for which

$$\operatorname{Cov}\bigl(\Phi_h(\psi_1, t_1), \Phi_h(\psi_2, t_2)\bigr) = \int_{B_r(0)} \psi_1(x)\psi_2(x)dx, \tag{4}$$

for all harmonic polynomials ψ_1 and ψ_2, where $B_r(0)$ is the origin-centered ball of volume $\min\{t_1, t_2\}$. In particular, taking $\psi_1 = \psi_2 = \psi$, we find that

$$\operatorname{Var}\bigl(\Phi_h(\psi, t)\bigr) = \int_{B_r(0)} \psi(x)^2 dx. \tag{5}$$

Though not immediately obvious from the above, we will see in §1.5 that this definition is very close to that of the ordinary GFF. Now, for each integer m and harmonic polynomial ψ, there is a discrete harmonic polynomial $\psi_{(m)}$ on $\frac{1}{m}\mathbb{Z}^d$ (defined precisely in §2.2) that approximates ψ in the sense that $\psi - \psi_{(m)}$ is a polynomial of degree at most $k-2$, where k is the degree of ψ. In particular, if we fix ψ and limit our attention to x in a fixed bounded subset of $\mathbb{R}^d$, then we have $|\psi_{(m)}(x) - \psi(x)| = O(1/m^2)$.

Discrete harmonic functions obey a mean value property: for each $r > 0$ there is a function w supported on the discrete ball $B = B_r(0) \cap \frac{1}{m}\mathbb{Z}^d$, such that w closely approximates the indicator function 1_B, and $\sum_{x\in B} w(x)(f(x) - f(0)) = 0$ for all functions f that are discrete harmonic on B; see the remark following Theorem 1.4. To measure the deviation of the IDLA cluster from circularity (more precisely, its deviation from w) we define

$$\Phi_A^m(\psi, t) := m^{-d/2}\left(\left[\sum_{x\in A_{T(m^d t)}} \psi_{(m)}(x/m)\right] - m^d t \psi_{(m)}(0)\right). \tag{6}$$

When $\psi_{(m)}(0) = 0$, this random variable measures to what extent the mean value property for the discrete harmonic polynomial $\psi_{(m)}$ fails for the set $A_{T(m^d t)}$. When $\psi_{(m)}$ is a constant function, it measures fluctuations in the size of the cluster.

Theorem 1.4. *Let h be the augmented GFF, and Φ_h as discussed above. Then as $m \to \infty$, the random functions Φ_A^m converge in law to Φ_h (w.r.t. the smallest topology that makes $\Phi \mapsto \Phi(\psi, t)$ continuous for each ψ and t). In other words, for any finite collection of pairs $(\psi_1, t_1), \ldots, (\psi_k, t_k)$, the joint law of the $\Phi_A^m(\psi_i, t_i)$ converges to the joint law of the $\Phi_h(\psi_i, t_i)$.*

Remark. The reason for the variance formula (5) in the definition of augmented GFF boils down to a very simple calculation: Supposing $\psi(0) = 0$, consider the discrete time process

$$M(n) = \sum_{x \in A_n} \psi_{(1)}(x).$$

Since $\psi_{(1)}$ is discrete harmonic, M is a martingale, and

$$\mathbb{E}M(n)^2 = \mathbb{E}\sum_{j=1}^{n}((M(j) - M(j-1))^2 = \mathbb{E}\sum_{j=1}^{n}\psi_{(1)}(X_j)^2$$

where $\{X_j\} = A_j \setminus A_{j-1}$. Because A_n is close to the origin-centered ball $B_{r(n)}$ of volume n, the right side divided by $\int_{B_{r(n)}} \psi(x)^2 dx$ tends to 1 as $n \to \infty$ Except for minor complications about continuous time, the proof in Section 2.3 proceeds exactly on these lines.

Theorem 1.4 does not really address the discrepancies between A_T and $\mathbf{B}_r$ (which, as we noted earlier, can be very large, in particular in the case that ψ is a constant function). Rather, it can be interpreted as a measure of the discrepancy between A_T and the so-called *divisible sandpile*, which is a function $w_t : \mathbb{Z}^d \to [0, 1]$ defined for all $t \geq 0$. The quantity $w_t(x)$ represents the amount of mass that ends up at x if one begins with t units of mass at the origin and then "spreads" the mass around according to certain rules that ensure that the final amount of mass at each site is at most one. We will not give the construction here, but just list the properties of w_t that are important to us. For proofs of these properties, see [JLS12b, Lemma 6], which in turn is a restatement of [LP09, Theorem 1.3].

For fixed x, the quantity $w_t(x)$ is a continuously increasing function of t, and moreover there exists a constant c depending only on the dimension d, such that $w_t(x) = 1$ if $|x| < r(t) - c$ and $w_t(x) = 0$ if $|x| > r(t) + c$. An important property of w_t is that for any function f on $\mathbb{Z}^d$ that is discrete harmonic on $\mathbf{B}_{r(t)+c}$ we have $\sum_{x \in \mathbb{Z}^d} w_t(x)(f(x) - f(0)) = 0$. It is natural to replace (2) with

$$\tilde{E}_t := r^{-d/2} \sum_{x \in \mathbb{Z}^d} \big(1_{x \in A_{T(t)}} - w_t(x)\big)\delta_{x/r}, \tag{7}$$

and interpret Theorem 1.4 as a statement about the distributional limit of $\tilde{E}_t$.

Even with this replacement, Theorem 1.4 differs from Theorem 1.3, since it addresses only harmonic polynomial test functions ψ and also requires that we replace them with approximations $\psi_{(m)}$ on the discrete level. It is natural to ask, in general dimensions, what happens when we try to modify the statement of Theorem 1.4 (interpreted as a sort of distributional limit statement for (7)) to make it read like the distributional convergence statement of Theorem 1.3. We will discuss this in more detail in §3.4, but we can summarize the situation roughly as follows:

The restriction to harmonic ψ (as opposed to a more general test function ϕ) seems to be necessary in large dimensions because otherwise the derivative of the

Modification	When it matters
Replacing w_t in (7) with $1_{\mathbf{B}_r}$	No effect when $d = 2$. Invalidates result when $d > 3$.
Keeping w_t in (7) but **Replacing** $\psi_{(m)}$ with ψ	No effect if $d \in \{2, 3, 4, 5\}$. Unclear if $d > 5$.
Keeping w_t in (7) but **Replacing** $\psi_{(m)}$ with general smooth test function ϕ.	No effect if $d \in \{2, 3\}$. Probably invalidates result if $d > 3$.

test function along $\partial B_1(0)$ appears to have a non-trivial effect on (7) (see §3.4). This is because (7) has a lot of positive mass just outside of the unit sphere and a lot of negative mass just inside the unit sphere. It may be possible to formulate a version of Theorem 1.4 (involving some modification of the "mean shape" described by w_t) that uses test functions that are constant in the radial direction in a neighborhood of the $\partial B_1(0)$ (instead of using only harmonic test functions), but we will not address this point here. Deciding whether Theorem 1.2 as stated extends to higher dimensions requires some number theoretic understanding of the extent to which the discrepancies between w_t and $1_{\mathbf{B}_r}$ (as well as the errors that come from replacing a $\psi_{(m)}$ with a smooth test function ϕ) average out when one integrates over a range of times. We will not address these points here either.

1.5 Comparing the GFF and the augmented GFF

Using the last of the three approaches to GFF discussed in Section 1.4, we will compare the functionals $\Phi_g(\psi, t)$ and $\Phi_h(\psi, t)$, where g is the ordinary GFF and h is the augmented GFF.

We may write a general vector in $\mathbb{R}^d$ as ru where $r \in [0, \infty)$ and $u \in S^{d-1} := \partial B_1(0)$. We write the Laplacian in spherical coordinates as

$$\Delta = r^{1-d} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} + r^{-2} \Delta_{S^{d-1}}. \tag{8}$$

A polynomial $\psi \in \mathbb{R}[x_1, \ldots, x_d]$ is called *harmonic* if $\Delta\psi$ is the zero polynomial. Suppose that ψ is harmonic and homogeneous of degree k. Letting $f = \psi|_{S^{d-1}}$, we have $\psi(ru) = f(u)r^k$ for all $u \in S^{d-1}$ and $r \geq 0$. Setting (8) to zero at $r = 1$ yields

$$\Delta_{S^{d-1}} f = -k(k + d - 2)f,$$

i.e., f is an eigenfunction of $\Delta_{S^{d-1}}$ with eigenvalue $-k(k + d - 2)$. Note that the expression $-k(k + d - 2)$ is unchanged when the nonnegative integer k is replaced with the negative integer $k' := -(d - 2) - k$. Thus $f(u)r^{k'}$ is also harmonic on $\mathbb{R}^d \setminus \{0\}$.

Lemma 1.5. *Let $\psi \in \mathbb{R}[x_1, \ldots, x_d]$ be a homogeneous harmonic polynomial of degree $k \geq 0$, normalized so that*

$$\int_{S^{d-1}} \psi(u)^2 du = 1. \tag{9}$$

Let R be such that the ball $B_R(0)$ in $\mathbb{R}^d$ has volume t. Then

$$\operatorname{Var} \Phi_g(\psi, t) = \frac{R^{2k+d}}{2k+d-2} \tag{10}$$

and

$$\operatorname{Var} \Phi_h(\psi, t) = \frac{R^{2k+d}}{2k+d}. \tag{11}$$

Proof. By scaling, the integral of ψ^2 over $\partial B_r(0)$ is given by $r^{d-1}r^{2k}$. By the definition (5) of the augmented GFF, the variance of $\Phi_h(\psi, t)$ equals the L^2 norm of ψ on $B_R(0)$:

$$\operatorname{Var} \Phi_h(\psi, t) = \int_{B_R(0)} \psi(z)^2 dz = \int_0^R r^{d-1} r^{2k} dr = \frac{R^{d+2k}}{d+2k}.$$

Next we compute the variance of $\Phi_g(\psi, t)$. Consider the function ψ_R that equals ψ on the ball $B_R(0)$ and is extended harmonically outside $B_R(0)$ by the formula

$$\psi_R(ru) = R^{k-k'} f(u) r^{k'}$$

for $r > R$. Then $-\Delta \psi_R = c\psi\sigma_R$ for a constant $c = \frac{k-k'}{R}$, where σ_R is the surface measure on the sphere $\partial B_R(0)$. Hence

$$\Phi_g(\psi, t) := (g, \psi\sigma_R) = (g, -\frac{1}{c}\Delta\psi_R) = \frac{1}{c}(g, \psi_R)_\nabla$$

so that

$$\operatorname{Var} \Phi_g(\psi, t) = \frac{1}{c^2}(\psi_R, \psi_R)_\nabla. \tag{12}$$

The calculation that remains is to find the Dirichlet energy $(\psi_R, \psi_R)_\nabla$. A standard identity states that the Dirichlet energy of f, as a function on S^{d-1}, is given by the L^2 inner product $(-\Delta f, f) = k(k+d-2)$. The square of $\|\nabla\psi\|$ is given by the square of its component along S^{d-1} plus the square of its radial component. We thus find that the Dirichlet energy of ψ on $B_R(0)$ is given by

$$\begin{aligned}
\int_{B_R(0)} \|\nabla\psi(z)\|^2 dz &= k(k+d-2)\int_0^R r^{d-1} r^{2(k-1)} dr + \int_0^R r^{d-1} r^{2(k-1)} k^2 dr \\
&= \frac{k(k+d-2)}{2k+d-2} R^{2k+d-2} + \frac{k^2}{2k+d-2} R^{2k+d-2} \\
&= kR^{2k+d-2}.
\end{aligned}$$

Likewise, the Dirichlet energy of ψ_R outside of $B_R(0)$ can be computed as

$$R^{2(k-k')}k(k+d-2)\int_R^\infty r^{d-1}r^{2(k'-1)}dr + R^{2(k-k')}\int_R^\infty r^{d-1}r^{2(k'-1)}(k')^2dr,$$

which (recalling $k' = -(d-2)-k$) simplifies to

$$-\frac{k^2+k(d-2)+(k')^2}{2k'+(d-2)}R^{2k+d-2} = (k+d-2)R^{2k+d-2}.$$

Combining the inside and outside contributions, we obtain $(\psi_R, \psi_R)_\nabla = (2k+d-2)R^{2k+d-2}$. Recalling that $c = \frac{k-k'}{R} = \frac{2k+d-2}{R}$, the result now follows from (12). $\square$

In some ways, the augmented GFF is very similar to the ordinary GFF: when we restrict attention to an origin-centered annulus, it is possible to construct independent Gaussian random distributions h_1, h_2, and h_3 such that h_1 has the law of a constant multiple of the GFF, $h_1 + h_2$ has the law of the augmented GFF, and $h_1 + h_2 + h_3$ has the law of the ordinary GFF.

In light of Theorem 1.3, the following implies that (up to absolute continuity) the scaling limit of fixed-time A_t fluctuations can be described by the GFF itself.

Proposition 1.6. *When $d = 2$, the law ν of the restriction of the GFF to the unit circle (modulo additive constant) is absolutely continuous w.r.t. the law μ of the restriction of the augmented GFF restricted to the unit circle.*

Proof. The relative entropy of a Gaussian of density $e^{-x^2/2}$ with respect to a Gaussian of density $\sigma^{-1}e^{-x^2/(2\sigma^2)}$ is given by

$$F(\sigma) = \int e^{-x^2/2}\left((\sigma^{-2}-1)x^2/2 + \log\sigma\right)dx = (\sigma^{-2}-1)/2 + \log\sigma.$$

Note that $F'(\sigma) = -\sigma^{-3} + \sigma^{-1}$, and in particular $F'(1) = 0$. Thus the relative entropy of a centered Gaussian of variance 1 with respect to a centered Gaussian of variance $1+a$ is $O(a^2)$. This implies that the relative entropy of μ with respect to ν — restricted to the jth component α_j — is $O(j^{-2})$. The same holds for the relative entropy of ν with respect to μ. Because the α_j are independent in both μ and ν, the relative entropy of one of μ and ν with respect to the other is the sum of the relative entropies of the individual components, and this sum is finite. $\square$

2 General dimension

2.1 FKG inequality: Proof of Theorem 1.1

We recall that increasing functions of a Poisson point process are non-negatively correlated [GK97]. (This is easily derived from the more well known statement [FKG71] that increasing functions of independent Bernoulli random variables are non-negatively correlated.) Let μ be the simple random walk probability measure on the space Ω' of walks W beginning at the origin. Then the randomness for internal DLA is given by a rate-one Poisson point process on $\mu \times \nu$ where ν is Lebesgue measure on $[0, \infty)$. A realization of this process is a random collection of points in $\Omega' \times [0, \infty)$. It is easy to see (for example, using the abelian property of internal DLA discovered by Diaconis and Fulton [DF91]) that adding an additional point (w, s) increases the value of $A_{T(t)}$ for all times t. The $A_{T(t)}$ are hence increasing functions of the Poisson point process, and are non-negatively correlated. Since F and G are increasing functions of the $A_{T(t)}$, they are also increasing functions of the point process — and are thus non-negatively correlated.

2.2 Discrete harmonic polynomials

Let $\psi(x_1, \ldots, x_d)$ be a polynomial that is harmonic on $\mathbb{R}^d$, that is

$$\sum_{i=1}^{d} \frac{\partial^2 \psi}{\partial x_i^2} = 0.$$

Let $m \geq 1$. In this section we give a recipe for constructing a polynomial $\psi_{(m)}$ that is discrete harmonic on the lattice $\frac{1}{m}\mathbb{Z}^d$ and such that $\psi_{(m)} - \psi$ has degree at most $k-2$, where k is the degree of ψ.

We begin by constructing $\psi_{(1)}$. The requirement of discrete harmonicity is that

$$\sum_{i=1}^{d} D_i^2 \psi_{(1)} = 0$$

where

$$D_i^2 \psi_{(1)} = \psi_{(1)}(x + \mathbf{e}_i) - 2\psi_{(1)}(x) + \psi_{(1)}(x - \mathbf{e}_i)$$

is the symmetric second difference in direction $\mathbf{e}_i$. The construction described below is nearly the same as the one given by Lovász in [Lov04], except that we have tweaked it in order to obtain a smaller error term: if ψ has degree k, then $\psi - \psi_{(1)}$ has degree at most $k-2$ instead of $k-1$. Discrete harmonic polynomials have been studied classically, primarily in two variables: see for example Duffin [Duf56], who gives a construction based on discrete contour integration.

Consider the linear map

$$\Xi : \mathbb{R}[x_1, \ldots, x_d] \to \mathbb{R}[x_1, \ldots, x_d]$$

defined on monomials by

$$\Xi(x_1^{k_1} \cdots x_d^{k_d}) = P_{k_1}(x_1) \cdots P_{k_d}(x_d)$$

where P_k is the one-variable polynomial defined by

$$P_k(y) = \prod_{j=-(k-1)/2}^{(k-1)/2} (y + j).$$

Lemma 2.1. *If $\psi \in \mathbb{R}[x_1, \dots, x_d]$ is a polynomial of degree k that is harmonic on $\mathbb{R}^d$, then the polynomial $\psi_{(1)} = \Xi(\psi)$ is discrete harmonic on $\mathbb{Z}^d$, and $\psi - \psi_{(1)}$ is a polynomial of degree at most $k-2$.*

Proof. An easy calculation shows that

$$D^2 P_k = k(k-1)P_{k-2}$$

from which we see that

$$D_i^2 \Xi[\psi] = \Xi[\frac{\partial^2}{\partial x_i^2}\psi].$$

If ψ is harmonic, then the right side vanishes when summed over $i = 1, \dots, d$, which shows that $\Xi[\psi]$ is discrete harmonic.

Note that $P_k(y)$ is even for k even and odd for k odd. In particular, $P_k(y) - y^k$ has degree at most $k-2$, which implies that $\psi - \psi_{(1)}$ has degree at most $k-2$. □

To obtain a discrete harmonic polynomial $\psi_{(m)}$ on the lattice $\frac{1}{m}\mathbb{Z}^d$, let k be the degree of ψ and write $\psi = \phi + \xi$, where ϕ is homogeneous of degree k, and ξ has degree at most $m-1$. Since ψ is harmonic on $\mathbb{R}^d$, both ϕ and ξ are harmonic on $\mathbb{R}^d$. Now set

$$\psi_{(m)}(x) := m^{-k}\phi_{(1)}(mx) + m^{1-k}\xi_{(1)}(mx).$$

Lemma 2.1 ensures that $\psi_{(m)}$ is discrete harmonic on $\frac{1}{m}\mathbb{Z}^d$, and that $\phi - \phi_{(1)}$ and $\xi - \xi_{(1)}$ have degree at most $k-2$. Hence

$$\psi(x) - \psi_{(m)}(x) = \phi(x) - m^{-k}\phi(mx) + \xi(x) - m^{1-k}\xi(mx) + \varepsilon(x)$$

with ε of degree at most $k-2$. Since ϕ is homogeneous and ξ has degree at most $k-1$, the right side has degree at most $k-2$ as desired. Moreover, by adding a constant to $\psi_{(m)}$ we may also assume that

$$\psi(0) = \psi_{(m)}(0).$$

2.3 General-dimensional CLT: Proof of Theorem 1.4

We have defined a discrete time IDLA cluster $A_t = A_{\lfloor t \rfloor}$ in which new particles arrive at integer times, and a continuous time cluster $A_{T(t)}$ where they arrive at Poisson random times. Both of these are "jump" processes: the former changes suddenly at integer times, and the latter at Poisson times. For the proof of Theorem 1.4, we introduce a smoother continuous time process $\widetilde{A}_t$ (used already in [JLS12a]) that interpolates $\{A_n\}_{n\in\mathbb{N}}$.

To define $\widetilde{A}$, let $\mathcal{G}$ denote the grid comprised of the edges connecting nearest neighbor vertices of $\mathbb{Z}^d$. (As a set, $\mathcal{G}$ consists of the points in $\mathbb{R}^d$ with at most one non-integer coordinate.) Now suppose that at each integer time n, a new particle is added at the origin and performs a Brownian motion $\{B_t^{(n)}\}_{t\geq n}$ on $\mathcal{G}$ (instead of simple random walk on $\mathbb{Z}^d$), starting at $B_n^{(n)} = 0$ and stopping at time T_n when it first hits the set $\mathbb{Z}^d \setminus A_n$. By applying a deterministic time change to the Brownian motion, we can ensure that $T_n < n+1$. Then for $t \in [n, n+1)$ we set

$$\widetilde{A}_t := A_n \cup \{B^{(n)}_{t \wedge T_n}\}.$$

Thus $\widetilde{A}_t$ consists of $A_{\lfloor t \rfloor}$ plus a single additional point, the location of the currently active particle; note that $\widetilde{A}_t$ is a multiset at those times t when $B_t^{(n)} \in A_n$.

Now let f be a discrete harmonic function on $\mathbb{Z}^d$ with $f(0) = 0$. Extend f linearly along each segment of the grid $\mathcal{G}$, and define

$$Y(t) = \sum_{x \in \widetilde{A}_t} f(x), \qquad Z(t) = \sum_{x \in \widetilde{A}_t} f(x)^2. \tag{13}$$

For $n \leq s < t \leq n+1$ we have $Y(t) - Y(s) = F(t) - F(s)$, where $F(t) = f(B^{(n)}_{t \wedge T_n})$. Since f is discrete harmonic and linear on segments of $\mathcal{G}$, we have that F is a martingale, and hence Y is a martingale. Let

$$S(t) := \limsup_{\substack{0 = t_0 < t_1 < \cdots < t_n = t \\ |t_{i+1} - t_i| \to 0}} \sum_{i=0}^{n-1} (Y(t_{i+1}) - Y(t_i))^2.$$

be the quadratic variation of Y on the interval $[0,t]$. Write $\mathcal{F}_t = \sigma(\widetilde{A}_s | s \leq t)$. Then for $n \leq s < t \leq n+1$ we have

$$\begin{aligned} \mathbb{E}[S(t) - S(s) | \mathcal{F}_s] &= \mathbb{E}[(F(t) - F(s))^2 | \mathcal{F}_s] \\ &= \mathbb{E}[F(t)^2 - F(s)^2 | \mathcal{F}_s] \\ &= \mathbb{E}[Z(t) - Z(s) | \mathcal{F}_s]. \end{aligned}$$

Thus the process

$$N(t) := S(t) - Z(t) \tag{14}$$

is a martingale, a fact that will be useful in the proof below.

Finally, to accommodate Poisson arrivals in the above discussion, write $t_n = \inf\{t : T(t) \geq n\}$ for the time of the n-th particle's arrival at 0. Let $\widetilde{T}$ be the random function that coincides with T at all times t_n and is linear on each interval $[t_n, t_{n+1}]$ for $n \in \mathbb{N}$. Then $\widetilde{A}_{\widetilde{T}(n)} = A_{T(n)}$ for $n \in \mathbb{N}$. If we define $\widetilde{Y}$ and $\widetilde{Z}$ by substituting $\widetilde{T}(t)$ for t in (13), then $\widetilde{Y}(t)$ is a martingale adapted to the filtration $\widetilde{F}_t := \sigma(t^+, \{\widetilde{A}_{\widetilde{T}(s)} | s \leq t\})$, where $t^+ = \inf\{t_n | t_n \geq t\}$. The quadratic variation of $\widetilde{Y}$ is given by $\widetilde{S}(t) := S(\widetilde{T}(t))$, and hence

$$\widetilde{N}(t) := \widetilde{S}(t) - \widetilde{Z}(t) = N(\widetilde{T}(t)).$$

Proof of Theorem 1.4. Fix $m > 0$ and a harmonic polynomial $\psi \in \mathbb{R}[x_1, \dots, x_d]$. We consider first the case $\psi(0) = 0$. The process

$$M_m(t) := m^{-d/2} \sum_{x \in \widetilde{A}_{\widetilde{T}(m^d t)}} \psi_{(m)}(x/m)$$

is a martingale in t. This M_m is identical to Φ_A^m of (6) except that it uses the modified process $\widetilde{A}_{\widetilde{T}}$ in place of A_T. The difference $M_m(t) - \Phi_A^m(t)$ equals $m^{-d/2}\psi_{(m)}(X_t)$ for a single point $X_t \in \frac{1}{m} A_{T(m^d t)}$. With probability tending to 1 as $m \to \infty$ we have $X_t \in B$ where B is the origin-centered ball of volume $2t$. Since $\psi_{(m)} \leq 2\psi$ is bounded on B independently of m, it follows that $M_m - \Phi_A^m \to 0$ in law as $m \to \infty$. Thus, it suffices to prove the theorem with M_m in place of Φ_A^m.

By the martingale representation theorem (see [RY05, Theorem V.1.6]), we can write $M_m(t) = \beta(S_m(t))$, where β is a standard Brownian motion and $S_m(t)$ is the quadratic variation of M_m on the interval $[0, t]$. To show that $M_m(t)$ converges in law as $m \to \infty$ to a Gaussian with variance $V := \int_{B_{r(t)}(0)} \psi(x)^2 dx$, it suffices to show that for fixed t the random variable $S_m(t)$ converges in law to V.

By standard Riemann integration and the A_t fluctuation bounds in [JLS12a, JLS12b] (the weaker bounds of [LBG92] would also suffice here) along with the fact that $T(tm^d)/m^d \to t$ in law, we know that

$$Z_m(t) := m^{-d} \sum_{x \in A_{T(t) m^d}} \psi_{(m)}(x/m)^2 \to V$$

in law as $m \to \infty$. Thus it suffices to show that

$$N_m(t) := S_m(t) - Z_m(t) \tag{15}$$

converges in law to zero. We have $N_m(t) = N(T(t)m^d)$, where $N = S - Z$ is the martingale (14) associated to the process $Y(s) := m^{-d} \sum_{x \in \widetilde{A}_s} \psi_{(m)}(x/m)$. Let $s = tm^d$. The expected square of $N(s)$ is the sum of the expectations of the squares of its s increments, each of which is $O(m^{-2d})$, so $\mathbb{E}N(s)^2 = O(m^{-d})$. Thus the process $\{N(s)\}_{s \geq 0}$ tends to zero in law as $m \to \infty$, and so does its time change $\{N_m(t)\}_{t \geq 0}$.

When $\psi(0) \neq 0$, the second term in (6) introduces an asymptotically independent source or randomness which scales to a Gaussian of variance $\psi(0)^2 t$ (simply by the central limit theorem for the Poisson point process), and hence (5) remains correct in this case.

Similarly, suppose we are given $0 = t_0 < t_1 < t_2 < \ldots < t_\ell$ and functions $\psi_1, \psi_2, \ldots \psi_\ell$. The same argument as above, using the martingale in t,

$$m^{-d/2} \sum_{j=1}^{\ell} \sum_{x \in \widetilde{A}_{\widetilde{T}(m^d (t \wedge t_j))}} \psi_{j,(m)}(x/m)$$

implies that $\sum_{j=1}^{\ell} \Phi_A^m(\psi_j, t_j)$ converges in law to a Gaussian with variance

$$\sum_{j=1}^{\ell} \int_{B_{r(t_j)} \setminus B_{r(t_{j-1})}} \left(\sum_{i=j}^{\ell} \psi_i(x) \right)^2 dx.$$

The theorem now follows from a standard fact about Gaussian random variables on a finite dimensional vector spaces (proved using characteristic functions): namely, a sequence of random variables on a vector space converges in law to a multivariate Gaussian if and only if all of the one-dimensional projections converge. The law of h is determined by the fact that it is a centered Gaussian with covariance given by (4). $\square$

3 Dimension two

3.1 Two dimensional central limit theorem

Recall that A_t for $t \in \mathbb{Z}_+$ denotes the discrete-time IDLA cluster with exactly t sites, and $A_T = A_{T(t)}$ for $t \in \mathbb{R}_+$ denotes the continuous-time cluster whose cardinality is Poisson-distributed with mean t.

For $z \in \mathbb{Z}^2$, let

$$F_0(z) := \inf\{t : z \in A_t\}$$

be the first time that z joins the cluster. Consider the *lateness function*

$$L_0(z) := \sqrt{F_0(z)/\pi} - |z|.$$

The random variable $L_0(z)$ is negative if z joins the cluster early and positive if z joins the cluster late. The goal of this section is to prove a central limit theorem for functionals of L_0, Theorem 3.1 below.

Fix $N < \infty$, and consider a test function of the form

$$\phi(re^{i\theta}) = \sum_{|k| \leq N} a_k(r) e^{ik\theta} \tag{16}$$

where the a_k are smooth functions supported in an interval $0 < r_0 \leq r \leq r_1 < \infty$. We will assume, furthermore, that ϕ is real-valued. That is, the complex numbers a_k satisfy

$$a_{-k}(r) = \overline{a_k(r)}.$$

Theorem 3.1. *Let*

$$X_R := \frac{1}{R^2} \sum_{z \in (\mathbb{Z}+i\mathbb{Z})/R} L_0(Rz) \frac{\phi(z)}{|z|^2}. \tag{17}$$

Then $X_R \to N(0, V_0)$ in law as $R \to \infty$, where

$$V_0 = \sum_{0<|k|\le N} 2\pi \int_0^\infty \left| \int_\rho^\infty a_k(r) \left(\frac{\rho}{r}\right)^{|k|+1} \frac{dr}{r} \right|^2 \frac{d\rho}{\rho}. \tag{18}$$

Before proving Theorem 3.1, we explain how it can be interpreted as saying that $L_0(Rz)$ tends weakly to the Gaussian random distribution h_{nr} associated to the Hilbert space H^1_{nr} with norm

$$\|\eta\|^2_{\mathrm{nr}} = \sum_{0<|k|<\infty} 2\pi \int_0^\infty [|r\partial_r \eta_k|^2 + (|k|+1)^2 |\eta_k|^2] \frac{dr}{r}$$

where

$$\eta_k(r) = \frac{1}{2\pi} \int_0^{2\pi} \eta(re^{i\theta}) e^{-ik\theta} d\theta.$$

(The subscript nr means nonradial: H^1_{nr} is the orthogonal complement of radial functions in the Sobolev space H^1.) We will see below that the factor of $1/|z|^2$ in (17) is natural from the point of view of a change of variables $y = \log r$ where $z = re^{i\theta}$.

We first consider a simpler space. For fixed $q \ge 0$, let H_q be the Hilbert space of compactly supported functions $f : \mathbb{R} \to \mathbb{R}$ with inner product

$$(f, g)_q := \int_{-\infty}^\infty (f'(y)g'(y) + q^2 f(y)g(y))dy.$$

Lemma 3.2. *For $\psi \in L^2(\mathbb{R})$, denote*

$$\|\psi\|_{q,*} = \sup \int_{-\infty}^\infty \psi(y) f(y) dy$$

where the supremum is over all $f \in H_q$ subject to the constraint

$$(f, f)_q \le 1.$$

Then

$$\|\psi\|^2_{q,*} = \int_{-\infty}^\infty \left| \int_s^\infty \psi(y) e^{q(s-y)} dy \right|^2 ds. \tag{19}$$

Proof. In the case $q = 0$, replace f in $\displaystyle\int \psi f \, dy$ with

$$f(y) = \int_{-\infty}^y f'(s) ds$$

change order of integration and apply the Cauchy-Schwarz inequality. For the case $q > 0$, multiply by the appropriate factors e^{qy} and e^{2qy} to deduce this from the case $q = 0$. ∎

Lemma 3.3. *Let ϕ be a test function of the form (16), and define*

$$\|\phi\|_* = \sup_\eta \int_{\mathbb{R}^2} \eta(z) \frac{\phi(z)}{|z|^2} dz$$

where the supremum is taken over all $\eta \in H^1_{\mathrm{nr}}$ with $\|\eta\|_{\mathrm{nr}} \le 1$. Then

$$\|\phi\|_*^2 = V_0$$

where V_0 is given by (18).

Proof. Recall that $a_{-k} = \bar{a_k}$ and $\eta_{-k} = \bar{\eta_k}$. Writing the integral in polar coordinates $z = re^{i\theta}$ and substituting $y = \log r$, we obtain

$$\begin{aligned}\int_0^{2\pi} \int_0^\infty \eta(z) \frac{\phi(z)}{r^2} r dr d\theta &= 2\pi \sum_{0<|k|\le N} \int_0^\infty a_k(r) \bar{\eta}_k(r) \frac{dr}{r} \\ &= 2\pi \sum_{0<|k|\le N} \int_{-\infty}^\infty \psi_k(y) \bar{f}_k(y)\, dy\end{aligned}$$

where $\psi_k(y) := a_k(e^y)$ and $f_k(y) := \eta_k(e^y)$. The constraint $\|\eta\|_{\mathrm{nr}} \le 1$ is equivalent to

$$\sum_{0<|k|<\infty} \int_{-\infty}^\infty (|f_k'|^2 + (|k|+1)^2 |f_k|^2) dy \le \frac{1}{2\pi}$$

hence

$$\|\phi\|_* = \frac{2\pi}{\sqrt{2\pi}} \left(\sum_{0<|k|\le N} \|\psi_k\|^2_{|k|+1,*} \right)^{1/2}$$

Changing variables back to $r = e^y$ in Lemma 3.2, the square of the right side equals V_0. □

Remark. We can now give the promised interpretation of Theorem 3.1. For each continuous linear functional Ψ on H^1_{nr}, the random variable $\Psi(h_{\mathrm{nr}})$ is a centered Gaussian of variance $\|\Psi\|^2$, where

$$\|\Psi\| = \sup\{\Psi(\eta) : \|\eta\|_{\mathrm{nr}} \le 1\}.$$

By the definition of $\|\phi\|_*$, the functional $\Psi_\phi(\eta) := \int_{\mathbb{R}^2} \eta(z) \frac{\phi(z)}{|z|^2} dz$ has norm $\|\Psi_\phi\| = \|\phi\|_*$, so $\Psi_\phi(h_{\mathrm{nr}})$ has variance $\|\phi\|_*^2 = V_0$.

To begin the proof of Theorem 3.1, let $p_0(z) = 1$, and for $k \ge 1$ let $p_k(z) = q_k(z) - q_k(0)$, where

$$q_k(z) = \Xi[z^k]$$

is the discrete harmonic polynomial associated to $z^k = (x+iy)^k$ as described in §2.2. The sequence p_k begins

$$1,\ z,\ z^2,\ z^3 - \frac{1}{4}\bar{z},\ z^4 - z\bar{z},\ \ldots.$$

For instance, to compute p_3, we expand

$$z^3 = x^3 - 3xy^2 + i\left[3x^2y - y^3\right]$$

and apply Ξ to each monomial, obtaining

$$p_3(z) = (x-1)x(x+1) - 3x(y-\frac{1}{2})(y+\frac{1}{2}) + i\left[3(x-\frac{1}{2})(x+\frac{1}{2})y - (y-1)y(y+1)\right]$$

which simplifies to $z^3 - \frac{1}{4}\bar{z}$. One readily checks that this defines a discrete harmonic function on $\mathbb{Z} + i\mathbb{Z}$. (In fact, z^3 is itself discrete harmonic, but z^k is not for $k \geq 4$.) To define p_k for negative k, we set $p_{-k}(z) = \overline{p_k(z)}$.

Define

$$\psi(z,t,R) = \sum_{k=-N}^{N} a_k(\sqrt{t/\pi R^2})p_k(z)(\sqrt{t/\pi})^{-|k|}$$

and

$$\psi_0(z,t,R) = \psi(z,t,R) - a_0(\sqrt{t/\pi R^2})$$

Lemma 3.4. *If $c_1R^2 \leq t \leq c_2R^2$ and $||z| - \sqrt{t/\pi}| \leq C\log R$, then*

$$|\psi(z,t,R) - \phi(z/R)| \leq C(\log R)/R$$

This lemma follows easily from the fact that the coefficients a_k are smooth and the bound $|p_k(z) - z^k| \leq C|z|^{k-1}$ for $k \geq 1$.

Lemma 3.5. *(Van der Corput)*

(a) $|\#\{z \in \mathbb{Z} + i\mathbb{Z} : \pi|z|^2 \leq t\} - t| \leq Ct^{1/3}$.

(b) For $k \geq 1$,

$$t^{-k/2}\left|\sum_{z\in\mathbb{Z}+i\mathbb{Z}} z^k\, 1_{\pi|z|^2\leq t}\right| \leq Ct^{1/3}.$$

(c) For $k \geq 1$,

$$t^{-k/2}\left|\sum_{z\in\mathbb{Z}+i\mathbb{Z}} p_k(z)\, 1_{\pi|z|^2\leq t}\right| \leq Ct^{1/3}.$$

Part (a) of this lemma was proved by van der Corput in the 1920s (See [GS10], Theorem 87 p. 484). Part (b) follows from the same method, and we defer the proof to §3.3. Part (c) follows from part (b) and the stronger estimate of Lemma 2.1, $|p_k(z) - z^k| \le C|z|^{k-2}$ for $k \ge 2$ (and $p_1(z) - z = 0$).

Now we have assembled the necessary ingredients to prove Theorem 3.1. Write the lateness function in the form

$$\begin{aligned} L_0(z) &= \frac{1}{2\sqrt{\pi}} \int_0^\infty (1 - 1_{A_t}) t^{1/2} \frac{dt}{t} - \frac{1}{2\sqrt{\pi}} \int_0^\infty (1 - 1_{\pi|z|^2 \le t}) t^{1/2} \frac{dt}{t} \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty (1_{\pi|z|^2 \le t} - 1_{A_t}) t^{1/2} \frac{dt}{t}. \end{aligned}$$

The random variable X_R appearing in Theorem 3.1 then takes the form

$$\begin{aligned} X_R &= \sum_{z \in \mathbb{Z}+i\mathbb{Z}} L_0(z) \frac{\phi(z/R)}{|z|^2} \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \sum_{z \in \mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2 \le t} - 1_{A_t}) \frac{\phi(z/R)}{|z|^2} t^{1/2} \frac{dt}{t} \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \sum_{z \in \mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2 \le t} - 1_{A_t}) \frac{\psi(z,t,R)}{t/\pi} t^{1/2} \frac{dt}{t} + E_R. \end{aligned}$$

To estimate the error term E_R, note first that the coefficients a_k are supported in a fixed annulus, the integrand above is supported in the range $c_1 R^2 \le t \le c_2 R^2$. Furthermore, by [JLS12a], there is an absolute constant C such that for all sufficiently large R and all t in this range, the difference $1_{\pi|z|^2 \le t} - 1_{A_t}$ is supported on the set of $z \in \mathbb{Z}^2$ such that $||z| - \sqrt{t/\pi}| \le C \log R$. Thus

$$\sum_{z \in \mathbb{Z}+i\mathbb{Z}} |1_{\pi|z|^2 \le t} - 1_{A_t}| \le K R \log R.$$

Moreover, Lemma 3.4 applies and

$$|E_R| \le C \int_{c_1 R^2}^{c_2 R^2} (R \log R) \frac{\log R}{R} t^{-1/2} \frac{dt}{t} = O((\log R)^2 / R).$$

Next, Lemma 3.5(a) says (since $\#A_t = t$)

$$\left| \sum_{z \in \mathbb{Z}+i\mathbb{Z}} 1_{\pi|z|^2 \le t} - 1_{A_t} \right| \le C t^{1/3}.$$

Thus replacing ψ by ψ_0 gives an additional error of size at most

$$C \int_{c_1 R^2}^{c_2 R^2} t^{1/3} t^{-1/2} \frac{dt}{t} = O(R^{-1/3}).$$

In all,

$$X_R = \frac{\sqrt{\pi}}{2} \int_0^\infty \sum_{z \in \mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2 \le t} - 1_{A_t}) \psi_0(z,t,R) t^{-1/2} \frac{dt}{t} + O(R^{-1/3}) \tag{20}$$

For $s = 0, 1, \ldots,$ consider the process

$$M(s) = \frac{\sqrt{\pi}}{2} \int_0^\infty \sum_{z \in \mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2 \le t} - 1_{A_{s \wedge t}}) \psi_0(z,t,R) t^{-1/2} \frac{dt}{t}$$

Note that $M(s) \to X_R$ as $s \to \infty$. Note also that Lemma 3.5(c) implies

$$M(0) = O(R^{-1/3}).$$

Because p_k are discrete harmonic and $p_k(0) = 0$ for all $k \ne 0$, $M(s) - M(0)$ is a martingale. It remains to show that $M(s) - M(0) \longrightarrow N(0, V_0)$ in law. As outlined below, this will follow from the martingale central limit theorem (see, e.g., [Bro71, HH80] or [Dur95, p. 414]).

For sufficiently large R, the difference $M(s+1) - M(s)$ is nonzero only for s in the range $c_1 R^2 \le s \le c_2 R^2$; and $|F_0(z) - \pi|z|^2| \le CR\log R$. We now show that this implies

$$|M(s+1) - M(s)|^2 = O(1/R^2) \tag{21}$$

and

$$\sum_{s=0}^{\infty} |M(s+1) - M(s)|^2 = V_0 + O((\log R)/R) \tag{22}$$

so that the martingale central limit theorem applies.

To prove (21), observe that

$$M(s+1) - M(s) = -\frac{\sqrt{\pi}}{2} \int_{F_0(z)}^\infty \psi_0(z,t,R) t^{-1/2} \frac{dt}{t}$$

where z is the $(s+1)$th point of A_t. Then $|z| \le \sqrt{t/\pi} + K\log R$ implies $|p_k(z)|(t/\pi)^{-|k|/2} \le C$, and hence

$$|\psi_0(z,t,R)| \le C$$

Recalling that $\psi_0 = 0$ unless $c_1 R^2 \le t \le c_2 R^2$, we have

$$|M(s+1) - M(s)| \le C \int_{c_1 R^2}^{c_2 R^2} t^{-1/2} \frac{dt}{t} = O(1/R)$$

which confirms (21).

Because A_t fills the lattice $\mathbb{Z} + i\mathbb{Z}$ as $t \to \infty$, we have

$$\sum_{s=0}^{\infty} |M(s+1) - M(s)|^2$$
$$= \sum_{z \in \mathbb{Z}+i\mathbb{Z}} \left| \frac{\sqrt{\pi}}{2} \int_{F_0(z)}^{\infty} \sum_{0<|k|\le N} a_k(\sqrt{t/\pi R^2}) p_k(z) (t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 .$$

We prove (22) in three steps: replace $p_k(z)$ by z^k (or $\bar{z}^{|k|}$ if $k < 0$); replace the lower limit $F_0(z)$ by $\pi|z|^2$; replace the sum of z over lattice sites with the integral with respect to Lebesgue measure in the complex z-plane.

We begin the proof of (22) by noting that the error term introduced by replacing p_k with z^k is

$$|p_k(z) - z^k| (t/\pi)^{-|k|/2} \le C_k t^{-1} = O(1/R^2)$$

In the integral this is majorized by

$$\int_{c_1 R^2}^{c_2 R^2} t^{-1/2} \frac{dt}{t} \int_{c_1 R^2}^{c_2 R^2} \frac{1}{R^2} t^{-1/2} \frac{dt}{t} = O(1/R^4)$$

Since there are $O(R^2)$ such terms, this change contributes order $R^2/R^4 = 1/R^2$ to the sum.

Next, we change the lower limit from $F_0(z)$ to $\pi|z|^2$. Since $|F_0(z) - \pi|z|^2| \le CR \log R$, the integral inside $|\cdots|^2$ is changed by

$$\int_{F_0(z)}^{\pi|z|^2} 1_{c_1 R^2 \le c_2 R^2} t^{-1/2} \frac{dt}{t} = O((\log R)/R^2)$$

Thus the change in the whole expression is majorized by the order of the cross term

$$(1/R)(\log R)/R^2 = (\log R)/R^3$$

Again there are R^2 terms in the sum over z, so the sum of the errors is $O((\log R)/R)$.

Lastly, we replace the value at each site z_0 by the integral

$$\int_{Q_{z_0}} \left| \frac{\sqrt{\pi}}{2} \int_{\pi r^2}^{\infty} \sum_{0<|k|\le N} a_k(\sqrt{t/\pi R^2}) r^k e^{ik\theta} (t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 r dr d\theta$$

where Q_{z_0} is the unit square centered at z_0 and $z = re^{i\theta}$. Because the square has area 1, the term in the lattice sum is the same as this integral with $z = re^{i\theta}$ replaced by z_0 at each occurrence. Since $|z - z_0| \le \sqrt{2}$,

$$|z^k - z_0^k| \le 4k(|z| + |z_0|)^{k-1} = O(R^{k-1})$$

After we divide by $(\sqrt{t/\pi})^k$, the order of error is $1/R$. Adding all the errors contributes at most order $1/R$ to the sum. We must also take into account the change in the lower limit of the integral, $\pi|z_0|^2$ is replaced by $\pi|z|^2 = \pi r^2$. Since $|z - z_0| \le \sqrt{2}$,

$$||z|^2 - |z_0|^2| \le \sqrt{2}(|z| + |z_0|) \le CR$$

Recall that in the previous step we previously changed the lower limit by $O(R \log R)$. Thus by the same argument, this smaller change gives rise to an error of order $1/R$ in the sum over z_0.

The proof of (22) is now reduced to evaluating

$$\int_0^{2\pi} \int_0^\infty \left| \frac{\sqrt{\pi}}{2} \int_{\pi r^2}^\infty \sum_{0<|k|\le N} a_k(\sqrt{t/\pi R^2}) r^{|k|} e^{ik\theta} (t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 r dr d\theta$$

Integrating in θ and changing variables from r to $\rho = r/R$,

$$= \frac{\pi^2}{2} \sum_{0<|k|\le N} \int_0^\infty \left| \int_{\pi \rho^2 R^2}^\infty a_k(\sqrt{t/\pi R^2}) (R\rho)^{|k|+1} (t/\pi)^{-|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 \frac{d\rho}{\rho}$$

Then change variables from t to to $r = \sqrt{t/\pi R^2}$ to obtain

$$= 2\pi \sum_{0<|k|\le N} \int_0^\infty \left| \int_\rho^\infty a_k(r) (\rho/r)^{|k|+1} \frac{dr}{r} \right|^2 \frac{d\rho}{\rho} = V_0.$$

This completes the proof of Theorem 3.1.

3.2 Proof of Theorem 1.2

Next we adapt Theorem 3.1 to the continuous time cluster A_T. The corresponding lateness function $L(z)$ was defined in §1.3. Letting ϕ be a test function of the form (16), the a_0 coefficient now figures in the limit formula as follows.

Theorem 3.6. *As $R \to \infty$,*

$$\frac{1}{R^2} \sum_{z \in (\mathbb{Z}+i\mathbb{Z})/R} L(Rz) \frac{\phi(z)}{|z|^2} \longrightarrow N(0, V)$$

in law, where

$$V = \sum_{|k|\le N} 2\pi \int_0^\infty \left| \int_\rho^\infty a_k(r) \left(\frac{\rho}{r}\right)^{|k|+1} \frac{dr}{r} \right|^2 \frac{d\rho}{\rho}. \tag{23}$$

Analogously to the remark following Theorem 3.1, we can interpret Theorem 3.6 as saying that that $L(Rz)$ tends weakly to the Gaussian random distribution h associated to the Hilbert space H^1 with norm

$$\|\eta\|^2 = \sum_{k=-\infty}^{\infty} 2\pi \int_0^\infty [|r\partial_r \eta_k|^2 + (|k|+1)^2 |\eta_k|^2] \frac{dr}{r}$$

where the term $k = 0$ corresponding to the radial function η_0 is now included in the sum. This random distribution is precisely the 2-dimensional augmented GFF. To see why, consider the harmonic polynomial $\psi(z) = \frac{1}{\sqrt{2\pi}} z^k$ and the corresponding random variable $\Phi_h(\psi, t)$ obtained by integrating $h\psi$ over the surface of the origin-centered circle $\partial B_R(0)$ enclosing area t. If $\phi(z)/|z|^2 = \delta(|z|-R)\psi(z)$ (note that this ϕ is not in the class of test functions for which we prove convergence; we are using it only for the purpose of checking that h is the augmented GFF) then (23) becomes

$$V = 2\pi \int_0^\infty \left| \int_\rho^\infty \delta(r-R) \frac{1}{\sqrt{2\pi}} r^{k+2} \left(\frac{\rho}{r}\right)^{k+1} \frac{dr}{r} \right|^2 \frac{d\rho}{\rho}.$$

The inner integral vanishes unless $\rho \le R$, leaving

$$V = \int_0^R \rho^{2(k+1)} \frac{d\rho}{\rho} = \frac{R^{2k+2}}{2k+2}$$

in agreement with the variance calculation (11) in the case $d = 2$.

As in the proof of Theorem 1.4, the convergence in law of all one-dimensional projections to the appropriate normal random variables implies the corresponding result for the joint distribution of any finite collection of such projections. Hence, Theorem 3.6 is a restatement of Theorem 1.2.

By way of comparison, the usual Gaussian free field is the one associated to the Dirichlet norm

$$\int_{\mathbb{R}^2} |\nabla \eta|^2 dx dy = \sum_{k=-\infty}^{\infty} 2\pi \int_0^\infty [|r\partial_r \eta_k|^2 + k^2 |\eta_k|^2] \frac{dr}{r}.$$

Comparing these two norms, we see that the second term in $\|\eta\|^2$ has an additional $+1$, hence our choice of the term "augmented Gaussian free field." As derived in §1.5, this $+1$ results in a smaller variance $\frac{1}{2k+d} R^{2k+d}$ in each spherical mode of degree k of the augmented GFF, as compared to $\frac{1}{2k+d-2} R^{2k+d}$ for the usual GFF. The surface area of the sphere is implicit in the normalization (9), and is accounted for here in the factors 2π above.

The proof of Theorem 3.6 follows the same idea as the proof of Theorem 3.1. We replace A_t by the continuous time cluster A_T (for $T = T(t)$), and we need to find the limit as $R \to \infty$ of

$$\frac{\sqrt{\pi}}{2} \int_0^\infty (t - T(t)) a_0(\sqrt{t/\pi R^2}) t^{-1/2} \frac{dt}{t} + \frac{\sqrt{\pi}}{2} \int_0^\infty \sum_{z \in \mathbb{Z} + i\mathbb{Z}} (1_{\pi|z|^2 \le t} - 1_{A_T}) \psi_0(z, t, R) t^{-1/2} \frac{dt}{t}$$

The error terms in the estimation showing this quantity is within $O(R^{-1/3})$ of

$$\frac{1}{R^2} \sum_{z \in (\mathbb{Z}+i\mathbb{Z})/R} L(Rz) \frac{\phi(z)}{|z|^2}$$

are nearly the same as in the previous proof. We describe briefly the differences. The difference between Poisson time and ordinary counting is

$$|\#A_T - \#A_t| = |T(t) - t| \le Ct^{1/2} \log t = O(R \log R) \quad \text{almost surely}$$

if $t \approx R^2$. It follows that for $|z| \approx R$,

$$|F(z) - \pi |z|^2| = O(R \log R) \quad \text{almost surely}$$

as in the previous proof for $F_0(z)$. Further errors are also controlled since we then have the estimate analogous to the one above for A_t, namely

$$\sum_{z \in \mathbb{Z}+i\mathbb{Z}} |1_{\pi|z|^2 \le t} - 1_{A_T}| \le CR \log R$$

We consider the continuous time martingale

$$M(s) = \frac{\sqrt{\pi}}{2} \int_0^\infty (s \wedge t - T(s \wedge t)) a_0(\sqrt{t/\pi R^2}) t^{-1/2} \frac{dt}{t} + \frac{\sqrt{\pi}}{2} \int_0^\infty \sum_{z \in \mathbb{Z}+i\mathbb{Z}} (1_{\pi|z|^2 \le t} - 1_{A_{T(s \wedge t)}}) \psi_0(z, t, R) t^{-1/2} \frac{dt}{t}$$

Instead of using the martingale central limit theorem, we use the martingale representation theorem. This says that the martingale $M(s)$ when reparameterized by its quadratic variation has the same law as Brownian motion. We must show that almost surely the quadratic variation of M on $0 \le s < \infty$ is $V + O(R^{-1/3})$.

$$\begin{aligned} &\lim_{\epsilon \to 0} \mathbb{E}\left((M(s+\epsilon) - M(s))^2 | A_{T(s)}\right)/\epsilon \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\sqrt{\pi}}{2} \int_s^\infty \sum_{|k| \le N} a_k(\sqrt{t/\pi R^2}) e^{ik\theta}) (s/t)^{|k|/2} t^{-1/2} \frac{dt}{t} \right|^2 d\theta \\ &\quad + O(R^{-1/3}) \end{aligned}$$

Integrating with respect to s gives the quadratic variation $V + O(R^{-1/3})$ after a suitable change of variable as in the proof of Theorem 3.1.

3.3 Van der Corput bounds

This section is devoted to the proof of part (b) of Lemma 3.5.

We prove an generalization of part (b) to all dimensions. To formulate it, let P_k be a harmonic polynomial on $\mathbb{R}^d$ that is homogeneous of degree k. Normalize so that

$$\max_{x\in B} |P_k(x)| = 1$$

where B is the unit ball in $\mathbb{R}^d$. In this discussion k will be fixed and the constants are allowed to depend on k and d. We are going to show that for $k \geq 1$,

$$\left| \frac{1}{R^d} \sum_{|x|<R,\, x\in\mathbb{Z}^d} P_k(x)/R^k \right| \leq CR^{-1-\alpha}$$

where

$$\alpha = 1 - 2/(d+1).$$

In dimension $d = 2$ we take $P_k(x) = (x_1 + ix_2)^k$; in this case $\alpha = 1/3$, and $R^d R^{-1-\alpha} = R^{2/3} \approx t^{1/3}$, so we recover the claim of part (b).

The van der Corput theorem is the case $k = 0$. It says

$$(1/R^d) \left| \# \left\{ x \in \mathbb{Z}^d : |x| < R \right\} - \text{vol}\,(|x| < R) \right| \leq CR^{-1-\alpha}$$

Let $\epsilon = 1/R^\alpha$.

Consider ρ a smooth, radial function on $\mathbb{R}^d$ with integral 1 supported in the unit ball. Then define $\chi = 1_B$ characteristic function of the unit ball. Denote

$$\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon), \quad \chi_R(x) = R^{-d}\chi(x/R)$$

Then

$$\left| \sum_{x\in\mathbb{Z}^d} (\chi_R * \rho_\epsilon(x) - \chi_R(x)) P_k(x)/R^k \right| \leq CR^{-1-\alpha}$$

This is because $\chi_R * \rho_\epsilon(x) - \chi_R(x)$ is nonzero only in the annulus of width 2ϵ around $|x| = R$ in which (by the van der Corput bound) there are $O(R^{d-1}\epsilon)$ lattice points.

The Poisson summation formula implies

$$\sum_{x\in\mathbb{Z}^d} \chi_R * \rho_\epsilon(x) P_k(x)/R^k = \sum_{\xi\in 2\pi\mathbb{Z}^d} [\hat{\chi}_R(\xi)\hat{\rho}_\epsilon(\xi)] * \hat{P}_k(\xi)/R^k$$

in the sense of distributions. The Fourier transform of a polynomial is a derivative of the delta function, $\hat{P}_k(\xi) = P_k(i\partial_\xi)\delta(\xi)$. Because $k \geq 1$ and $P_k(x)$ is harmonic, its average with repect to any radial function is zero. This is expressed in the dual variable as the fact that when $\xi = 0$,

$$P_k(i\partial_\xi)[\hat{\chi}_R(\xi)\hat{\rho}_\epsilon(\xi)] = 0$$

So we our sum equals

$$\sum_{\xi\neq 0,\,\xi\in 2\pi\mathbb{Z}^d} [\hat{\chi}_R(\xi)\hat{\rho}_\epsilon(\xi)] * \hat{P}_k(\xi)/R^k$$

Next look at

$$\hat{\chi}_R(\xi) = \hat{\chi}(R\xi)$$

$$P_k(i\partial_\xi)\hat{\chi}(R\xi) = R^k \int_{|x|<1} P_k(x)e^{-iRx\cdot\xi}dx$$

All the terms in which fewer derivatives fall on $\hat{\chi}_R$ and more fall on ρ_ϵ give much smaller expressions: the factor R corresponding to each such differentiation is replaced by an ϵ.

The asymptotics of this oscillatory integral above are well known. For any fixed polynomial P they are of the same order of magnitude as for $P \equiv 1$, namely

$$|P_k(i\partial_\xi)\hat{\chi}(R\xi)|/R^k \leq C_k|R\xi|^{-(d+1)/2}$$

This is proved by the method of stationary phase and can also be derived from well known asymptotics of Bessel functions.

It follows that our sum is majorized by (replacing the letter d by n so that it does not get mixed up with the differential dr)

$$\begin{aligned}\int_1^\infty (Rr)^{-(n+1)}\frac{r^{n-1}dr}{(1+\epsilon r)^N} &\approx \int_1^{1/\epsilon}(Rr)^{-(n+1)}r^n\frac{dr}{r}\\ &\approx R^{-(n+1)/2}\epsilon^{-(n-1)/2}\\ &= R^{-1-\alpha}.\end{aligned}$$

3.4 Fixed time fluctuations: Proof of Theorem 1.3

Theorem 1.3 follows almost immediately from the $d = 2$ case of Theorem 1.4 and the estimates above. Consider $(\phi, \tilde{E}_t)$ where $\tilde{E}_t$ is as in (7). What happens if we replace ϕ with a function $\tilde{\phi}$ that is discrete harmonic on the rescaled mesh $m^{-1}\mathbb{Z}^d$ within a $\log m/m$ neighborhood of $B_1(0)$? Clearly, if ϕ is smooth, we will have $\phi - \tilde{\phi} = O(m^{-1}\log m)$. Since there are at most $O(m^{d-1}\log m)$ non-zero terms in (7), the discrepancy in

$$(\phi, \tilde{E}_t) - (\tilde{\phi}, \tilde{E}_t) = O\big(m^{-d/2}m^{d-1}(m^{-1}\log m)\log m\big) = O\big(m^{d/2-2}(\log m)^2\big), \quad (24)$$

which tends to zero as long as $d \in \{2, 3\}$.

The fact that replacing E_t with $\tilde{E}_t$ has a negligible effect follows from the above estimates when $d = 2$. This may also hold when $d = 3$, but we will not prove it here. Instead we remark that Theorem 1.3 holds in three dimensions provided that we replace (2) with (7), and that the theorem as stated probably fails in higher dimensions even if we make a such a replacement. The reason is that (7) is positive at points slightly outside of $\mathbf{B}_r$ (or outside of the support of w_t) and negative at points slightly inside. If we replace a discrete harmonic polynomial ψ with a function that agrees with ψ on $B_1(0)$ but has a different derivative along portions of $\partial B_1(0)$, this may produce a non-trivial effect (by the discussion above) when $d \geq 4$.

Finally, we note that replacing ψ_m by ψ introduces an error of order m^{-2}, and the same argument as above gives

$$(\psi, \tilde{E}_t) - (\tilde{\psi}_m, \tilde{E}_t) = O\big(m^{-d/2} m^{d-1} m^{-2} \log m\big) = O\big(m^{d/2-3} (\log m)\big), \tag{25}$$

which tends to zero when $d \in \{2, 3, 4, 5\}$.

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