Proceedings of IEEE National Telecom Conference, Vol. 2, pp D4.1.1 – D4.1.4, Nov. 81.

ON INFORMATION FLOW IN RELAY NETWORKS

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Preliminary investigations in [1] has shown that a max-flow min-cut interpretation for the capacity expressions of the classes of degraded and semideterministic relay channels can be found. In this paper, we show that such an interpretation can also be found for fairly general classes of discrete memoryless relay networks.

The general discrete memoryless relay network (Figure 1) consists of a source X_0 that sends information to a sink Y_0 with the help of N intermediate relay sender-receiver pairs (X_i, Y_i) The dependence of the received symbols $(y_0, y_1, \ldots, y_N) \in \mathcal{Y}_0 \times \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_N$ on the transmitted symbols $(x_0, x_1, \ldots, x_N) \in \mathcal{X}_0 \times \mathcal{X}_1 \times \ldots \times \mathcal{X}_N$ is specified by the probability transition matrix $p(y_0, y_1, \ldots, y_N | x_0, x_1, \ldots, x_N)$.

An (M,n) code for network consists of a set of intergers $\mathcal{M} = \{1,2,\ldots,M\}$ an encoding function $x_0: \mathcal{M} \; X_0^n$, a set of relay functions $\{f_{ij}\}$ such that

$$x_{ij} = f_{ij}(Y_{i1}, Y_{i2}, \ldots, Y_{ij-1}), 1 \le j \le n,$$

$$1 \le i \le N, \; \underline{x}_i \triangleq (x_{ij}, \ldots, x_{in}) \;,$$

i.e., $x_{ij} \triangleq$ jth component of $\underline{x}_i$, and a decoding function $g: y_0^n \to M$. For generality, all functions are allowed to be stochastic functions. The input x_{ij} is allowed to depend only on the past received signals at the ith node, i.e., $(y_{i1}, \ldots, y_{ij-1})$. The network is memoryless in the sense that $(y_{0i}, y_{1i}, \ldots, y_{Ni})$ depends on the past $(x_0^i, x_1^i, \ldots, x_N^i)$ only through the present transmitted symbols $(x_{0i}, x_{1i}, \ldots, x_{Ni})$. Therefore, the joint probability mass function on $\mathcal{M} \times \mathcal{X}_0^n \times \mathcal{X}_1^n \times \ldots \times \mathcal{X}_N^n \times \mathcal{Y}_0^n \times \mathcal{Y}_1^n \times \ldots \times \mathcal{Y}_N^n$ is given by

$$p(w, \underline{x}_0, \underline{x}_1, \ldots, \underline{x}_N, \underline{y}_0, \underline{y}_1, \ldots, \underline{y}_N) =$$

$$p(w) \prod_{i=1}^{n} p(x_{0i} | w) p(x_{1i} | y_1^{i-1}) \ldots p(x_{Ni} | y_N^{i-1}) \cdot \quad \text{(cont.)}$$

$$\cdot p(y_{0i}, \ldots, y_{Ni} | x_{0i}, \ldots, x_{Ni}) \qquad (1)$$

where p(w) is the probability distribution on the message $w \in M$. If the message $w \in \mathcal{M}$ is sent, let

$$\lambda(W) \triangleq \Pr\{g(\underline{Y}_0) \neq W | W = w\} \qquad (2)$$

denote the conditional probability of error. Define the average probability of error of the code, assuming a uniform distribution over the set of all messages $w \in \mathcal{M}$, as

$$\bar{P}_e^{\,n} = \frac{1}{M} \sum_w \lambda(w). \qquad (3)$$

Let $\lambda_n \triangleq \max_{w \in M} \lambda(w)$ be the maximal probability of error for the (M,n) code. The rate R of an (M,n) code is defined to be

$$R = \frac{1}{n} \log M \text{ bits/transmission.} \qquad (4)$$

The rate R is said to be achievable by the network if, for any $\varepsilon > 0$, and for all n sufficiently large, there exists an (M,n) code with $M \ge 2^{nR}$ such that $\bar{P}_e^{\,n} < \varepsilon$. The capacity C of the network is the supremum of the set of achievable rates.

In [2-4], the relay network with N = 1 (called the relay channel) was investigated. The capacity was established in [3,4] for several special cases including the degraded, reversely degraded, and semi-deterministic classes. General lower and upper bounds to capacity were given in [3]. The capacity of the general relay channel is not known.

In this paper we extend these results to establish the capacity of deterministic relay networks with no interference and degraded relay networks. We first give a general upper bound to the capacity of any relay network. This upper bound is a natural generalization of Theorem 4 in [3].

<u>Theorem 1</u>: For the general discrete memoryless relay network

$(\mathcal{X}_0 \times \mathcal{X}_1 \times \ldots \times \mathcal{X}_N, p(y_0, y_1, \ldots, y_N | x_0, x_1, \ldots, x_N), \mathcal{Y}_0 \times \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_N)$ having intermediate nodes (Figure 1), the capacity C is bounded above by

$$C \le \max_{p(x_0, x_1, \ldots, x_n)} \min \Big[I(\underline{X}(S); \underline{Y}(\bar{S}) | \underline{X}(S^c)), \text{ for all } S$$

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such that $S \subseteq \{0,1,\ldots,N\}$, $S \cup \bar{S} = S \cup S^c = \{0,1,\ldots,N\}$, $S \cap \bar{S} = \{0\}$, $S \cap S^c = \phi$,

$$\bar{S} = S^c \cup \{0\} \Big\} \quad (5)$$

where $\underline{X}(S)$, $\underline{Y}(\bar{S})$ are the vectors defined by:

$\underline{X}(S) = (X_0, X_{i_1}, \ldots, X_{i_k})$, for $S = \{0, i_1, \ldots, i_k\}$

and

$\underline{Y}(\bar{S}) = (Y_0, Y_{j_1}, \ldots, Y_{j_\ell})$, for $\bar{S} = \{0, j_1, \ldots, j_\ell\}$

proof: see [5].

Remarks:

1. For the relay channel (N=1)

$$C \leq \max_{p(x_0,x_1)} \min \left[I(X_0,X_1;Y_0),\ I(X_0;Y_0,Y_1|X_1) \right] \quad (6)$$

which is Theorem 4 in [3]. All known capacity results for the relay channel achieve equality in (6).

2. The upper bound in Theorem 1 has the following max-flow min-cut interpretation. There are 2^N different terms in (5). For any joint probability mass function $p(x_0,x_1,\ldots,x_N)$, the term $I(\underline{X}(S);\underline{Y}(\bar{S})|X(S^c))$ upper bounds the rate of information flow from the senders $\underline{X}(S)$ to the receivers $\underline{Y}(\bar{S})$. This corresponds to the "capacity" of the cut in which the nodes belonging to S are on one side and nodes belonging to $\bar{S}$ are on the other. The maximum over all $p(x_0,x_1,\ldots,x_N)$ of the minimum of the cut capacities gives the upper bound.

The right member of (5) is achieved for the following two classes of relay network:

1. Deterministic relay network with no interference:

This is depicted in Figure 2 for N=2. In general, each relay $1 \leq j \leq N$ receives N symbols $y_j \triangleq (y_{0j},\ldots,y_{kj},\ldots,y_{Nj})$, where $y_{kj} = h_{kj}(x_k)$, a deterministic function of x_k, $k \neq j$. The sink y_0 receives (N+1) symbols.

The capacity of the relay network depicted in Figure 2 is given by.

Theorem 2:

$$C = \sup_{p(x_0)p(x_1)p(x_2)} \min \Big\{ H(Y_{00},Y_{01},Y_{02}), H(Y_{00},Y_{01}) + H(Y_{20},Y_{21}), H(Y_{00},Y_{02}) + H(Y_{10},Y_{12}), H(Y_{00}) + H(Y_{10}) + H(Y_{20}) \Big\}$$

Proof:

The proof uses the standard techniques of random coding, superposition and time sharing (see [1]) in addition to proving that only one of the two links joining the two relays may be needed to achieve the capacity. The details of the proof can be found in [5].

Theorem 2 can be generalized to any $N \geq 2$ as follows.

Theorem 3: The capacity C of the deterministic N-node relay network with no interference is given by

$$C = \sup_{p(x_0)p(x_1)p(x_2)} \min \Big\{ \sum_{\alpha \in S} H(\underline{Y}_{\alpha,\bar{S}}), \text{ for all } S \subseteq \{0,1,\ldots,N\} \text{ such that } S \cap \bar{S} = \{0\}, \text{ and } S \cup \bar{S} = \{0,1,\ldots,N\} \Big\}, \quad (7)$$

where $H(\underline{Y}_{\alpha,\bar{S}}) \triangleq H(Y_{\alpha,0}, Y_{\alpha,j_1}, \ldots, Y_{\alpha,j_\ell})$, for $\bar{S} = \{0, j_1, \ldots, j_\ell\}$, $\alpha \in S$, and $Y_{i,j} \triangleq$ the received at the jth node corresponding to the ith node sender.

2. Degraded Relay Networks:

The discrete memoryless relay network $(\mathcal{X}_0 \times \mathcal{X}_1 \times \ldots \times \mathcal{X}_N, p(y_0,y_1,\ldots,y_N| x_0,x_1,\ldots,x_N), \mathcal{Y}_0 \times \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_N)$ is said to be degraded if $p(y_0,y_1,\ldots,y_N|x_0,x_1,\ldots,x_N)$ can be written in the form $p(y_0,y_1,\ldots,y_N|x_0,x_1,\ldots,x_N) =$

$$p(y_1|x_0,x_1)p(y_2|y_1,x_1,x_2)\ldots p(y_i|y_{i-1},x_{i-1},x_i)\ldots p(y_N|y_{N-1},x_{N-1},x_N)p(y_0|Y_N,x_N). \quad (8)$$

Thus Y_0 is a random degradation of the relay signal Y_N and each Y_i is a random degradation of Y_{i-1} for $2 \leq i \leq N$. Note that for each fixed assignment of $(x_0,x_1,\ldots,x_N)$, the relay outputs $(Y_1,\ldots,Y_N,Y_0)$ are conditionally distributed as a Markox chain $Y_1 \rightarrow Y_2 \rightarrow \ldots \rightarrow Y_N \rightarrow Y_0$.

We first give the capacity for N=2.

Theorem 4:

The capacity C of the degraded relay network having two intermediate nodes (N=2) is given by

$$C = \sup_{p(x_0,x_1,x_2)} \min \Big\{ I(X_0;Y_1|X_1,X_2),\ I(X_0,X_1;Y_2|X_2),\ I(X_0X_1,X_2;Y_0) \Big\} \quad (9)$$

where the supremum is over all joint probability mass functions $p(x_0,x_1,x_2)$ on $\mathcal{X}_0 \times \mathcal{X}_1 \times \mathcal{X}_2$.

Proof:

The proof involves three partitions (see Figure 3).

1) A random partition of $[1,2^{nR}]$ into 2^{nR_1} bins $S_{s_{1,i}}$ $\in [1,2^{nR_1}]$.

2) A random partition of $[1,2^{nR}]$ into $2^{nR'_2}$ bins $S'_{s'_{2,j}}$, $s'_{2,j} \in [1,2^{nR'_2}]$.

3) A random partition of $[1,2^{nR_1}]$ into $2^{nR''_2}$ bins $S''_{s''_{2,i}}, s'_{2,j} \in [1,2^{nR''_2}]$. These partitions allow us to send information to the sink using the technique of random binning [1].

The information in the network during block i is summarized in Table 1.

The details of the proof are ginen in [5].

Theorem 5: The capacity C of the N-node degraded relay netwrok defined by

$$C = \max_{p(x_0,\ldots,x_N)} \min_{0 \le i \le N} I(X_0,X_1,\ldots,X_i;Y_{i+1}|X_{i+1},\ldots,X_N)$$

where

$$Y_{N+1} \triangleq Y_0. \tag{10}$$

The maximum is over all joint probability mass functions $p(x_0,x_1,\ldots,x_N)$ $\mathcal{X}_0 \times \mathcal{X}_1 \times \ldots \times \mathcal{X}_N$.

Note: It is shown in [5] that the result in Theorem 5 can be used to establish the capacity of arbitrary relay networks with feedback.

Acknowledgment

Many valuable discussions with T. Cover has resulted in the investigation of the Max Flow - Min Cut interpretation of network capacity reported in this paper.

The work in this paper is partially supported under NSF Grant 80-26102.

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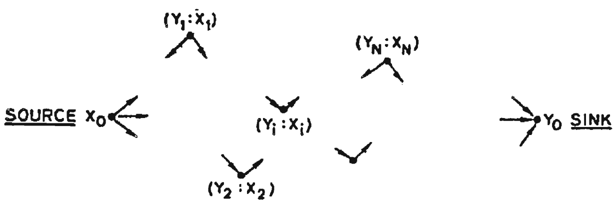

Figure 1

General Discrete Memoryless Relay Network

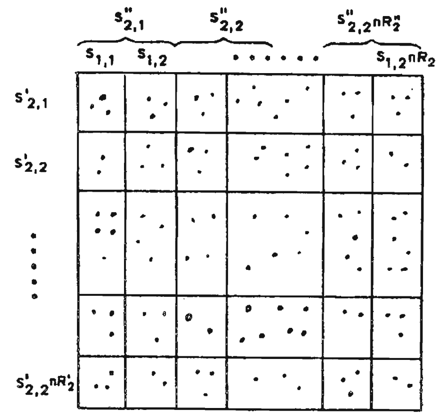

Figure 3

Partitions of Codewords

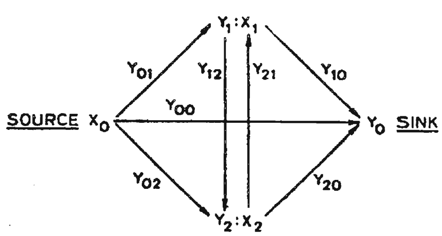

Figure 2

Deterministic Relay Network with No Interference

	Source	First Relay	Second Relay	Sink
Information at the end of block (i-1)	$w_{i-1}, s_{1,i}, s_{2,i+1}$	$w_{i-1}, s_{1,i}, s_{2,i+1}$	$w_{i-2}, s_{1,i-1}, s_{2,i}$	$w_{i-3}, s_{1,i-2}, s_{2,i-1}$
Transmission	$\underline{x}_0(w_i \mid s_{1,i}, s_{2,i})$	$\underline{x}_1(s_{1,i} \mid s_{2,i})$	$\underline{x}_2(s_{2,i})$	---------
Received Signal	------------	$\underline{y}_1(i)$	$\underline{y}_2(i)$	$\underline{y}_0(i)$
Information at the end of block i	$w_i, s_{1,i+1}, s_{2,i+2}$	$w_i, s_{1,i+1}, s_{2,i+2}$	$s_{1,i}, w_{i-1}, s_{2,i+1}$	$s_{2,i}, s_{1,i-1}, w_{i-2}$

Table 1

Information in the Network during Block i