PARALLEL CHIP-FIRING ON THE COMPLETE GRAPH: DEVIL'S STAIRCASE AND POINCARÉ ROTATION NUMBER

LIONEL LEVINE

Abstract. We study how parallel chip-firing on the complete graph K_n changes behavior as we vary the total number of chips. Surprisingly, the activity of the system, defined as the average number of firings per time step, does not increase smoothly in the number of chips; instead it remains constant over long intervals, punctuated by sudden jumps. In the large n limit we find a "devil's staircase" dependence of activity on the number of chips. The proof proceeds by reducing the chip-firing dynamics to iteration of a self-map of the circle S^1, in such a way that the activity of the chip-firing state equals the Poincaré rotation number of the circle map. The stairs of the devil's staircase correspond to periodic chip-firing states of small period.

1. Introduction

In this paper we explore a connection between the Poincaré rotation number of a circle map $S^1 \to S^1$ and the behavior of a discrete dynamical system known variously as *parallel chip-firing* [4, 5] or the (deterministic) *fixed energy sandpile* [2, 15]. We use this connection to shed light on two intriguing features of parallel chip-firing, *mode locking* and *short period attractors*. Ever since Bagnoli, Cecconi, Flammini, and Vespignani [1] found evidence of mode locking and short period attractors in numerical experiments in 2003, these two phenomena have called out for a mathematical explanation.

In parallel chip-firing on the complete graph K_n, each vertex $v \in [n] = \{1, \ldots, n\}$ starts with a pile of $\sigma(v) \geq 0$ chips. A vertex with n or more chips is *unstable*, and can *fire* by sending one chip to each vertex of K_n (including one chip to itself). The *parallel update* rule fires all unstable vertices simultaneously, yielding a new chip configuration $U\sigma$ given by

$$U\sigma(v) = \begin{cases} \sigma(v) + r(\sigma), & \sigma(v) \leq n-1 \\ \sigma(v) - n + r(\sigma), & \sigma(v) \geq n. \end{cases} \tag{1}$$

Date: January 6, 2010.

2000 *Mathematics Subject Classification.* 26A30, 37E10, 37E45, 82C20.

Key words and phrases. Circle map, devil's staircase, fixed-energy sandpile, mode locking, non-ergodicity, parallel chip-firing, rotation number, short period attractors.

The author is supported by a National Science Foundation Postdoctoral Research Fellowship.

Here

$$r(\sigma) = \#\{v | \sigma(v) \geq n\}$$

is the number of unstable vertices. Write $U^0\sigma = \sigma$, and $U^t\sigma = U(U^{t-1}\sigma)$ for $t \geq 1$.

Note that the total number of chips in the system is conserved. In particular, only finitely many different states are reachable from the initial configuration σ, so the sequence $(U^t\sigma)_{t\geq 0}$ is eventually periodic: there exist integers $m \geq 1$ and $t_0 \geq 0$ such that

$$U^{t+m}\sigma = U^t\sigma \qquad \forall t \geq t_0. \tag{2}$$

The *activity* of σ is the limit

$$a(\sigma) = \lim_{t\to\infty} \frac{\alpha_t}{nt}. \tag{3}$$

where

$$\alpha_t = \sum_{s=0}^{t-1} r(U^s\sigma)$$

is the total number of firings performed in the first t updates. By (2), the limit in (3) exists and equals $\frac{1}{mn}(\alpha_{t+m} - \alpha_t)$ for any $t \geq t_0$. Since $0 \leq \alpha_t \leq nt$, we have $0 \leq a(\sigma) \leq 1$.

Following [1], we ask: how does the activity change when chips are added to the system? If σ_n is a chip configuration on K_n, write $\sigma_n + k$ for the configuration obtained from σ_n by adding k chips at each vertex. The function

$$\tilde{s}_n(k) = a(\sigma_n + k)$$

is called the *activity phase diagram* of σ_n. Surprisingly, for many choices of σ_n, the function $\tilde{s}_n$ looks like a staircase, with long intervals of constancy punctuated by sudden jumps (Figure 1). This phenomenon is known as *mode locking*: if the system is in a preferred mode, corresponding to a wide stair in the staircase, then even a relatively large perturbation in the form of adding extra chips will not change the activity. For a general discussion of mode locking, see [12].

To quantify the idea of mode locking in our setting, suppose we are given an infinite family of chip configurations $\sigma_2, \sigma_3, \ldots$ with σ_n defined on K_n. Suppose that $0 \leq \sigma_n(v) \leq n-1$ for all $v \in [n]$, and that for all $0 \leq x \leq 1$

$$\frac{1}{n}\#\{v \in [n] \,|\, \sigma_n(v) < nx\} \to F(x) \qquad \text{as } n \to \infty \tag{4}$$

for a continuous function $F : [0,1] \to [0,1]$. Then according to Theorem 8, the activity phase diagrams $\tilde{s}_n$, suitably rescaled, converge pointwise to a continuous, nondecreasing function $s : [0,1] \to [0,1]$. Moreover, under a mild additional hypothesis, Proposition 10 says that this limiting function s is a *devil's staircase*: it is locally constant on an open dense subset of $[0,1]$. For each rational number $p/q \in [0,1]$ there is a stair of height p/q, that is, an interval of positive length on which s is constant and equal to p/q.

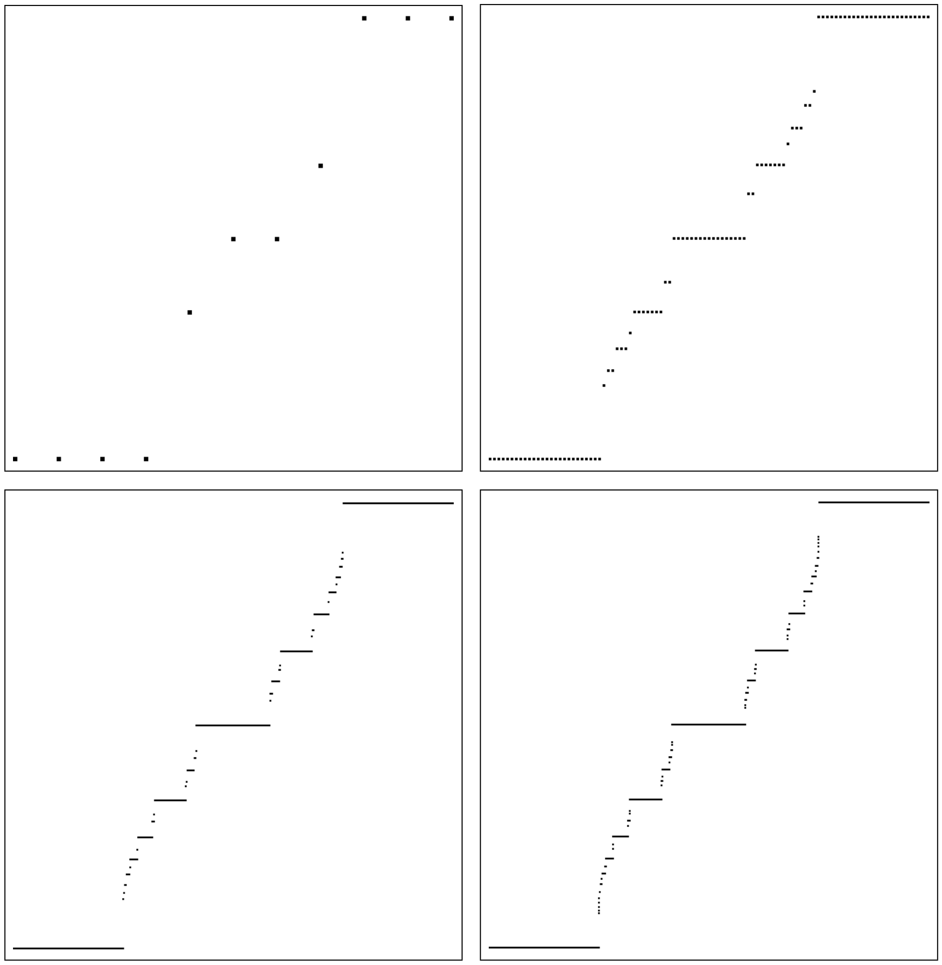

FIGURE 1. The activity phase diagrams $\tilde{s}_n(k) = a(\sigma_n + k)$, for $n = 10$ (top left), 100 (top right), 1000 (bottom left), and 10000, where σ_n is given by (5). On the horizontal axis, k runs from 0 to n. On the vertical axis, $\tilde{s}_n$ runs from 0 to 1.

Related to mode locking, a second feature observed in simulations of parallel chip-firing is non-ergodicity: in trials performed with random initial configurations on the $n \times n$ torus, the activity observed in individual trials differs markedly from the average activity observed over many trials [15]. The experiments of [1] suggested a reason: the chip-firing states in locked modes, corresponding to stairs of the devil's staircase, tend to be periodic with very small period. We study these *short period attractors* in Theorem 18. Under the same hypotheses that yield a devil's staircase in Propositon 10, for each $q \in \mathbb{N}$, at least a constant fraction $c_q n$ of the states $\{\sigma_n + k\}_{k=0}^n$ have eventual period q.

To illustrate these results, consider the chip configuration σ_n on K_n defined by

$$\sigma_n(v) = \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{v-1}{2} \right\rfloor, \qquad v = 1, \ldots, n. \tag{5}$$

Here $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. This family of chip configurations satisfies (4) with

$$F(x) = \begin{cases} 0, & x \leq \frac{1}{4} \\ 2x - \frac{1}{2}, & \frac{1}{4} \leq x \leq \frac{3}{4} \\ 1, & x \geq \frac{3}{4}. \end{cases} \tag{6}$$

The activity phase diagrams of σ_n for $n = 10, 100, 1000, 10000$ are graphed in Figure 1. For example, when $n = 10$ we have

$$(a(\sigma_{10} + k))_{k=0}^{10} = (0,\ 0,\ 0,\ 0,\ 1/3,\ 1/2,\ 1/2,\ 2/3,\ 1,\ 1,\ 1)$$

and when $n = 100$, we have

$(a(\sigma_{100}+k))_{k=0}^{100} =$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1/6, 1/5, 1/5, 1/4, 1/4, 1/4, 2/7, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 1/3, 2/5, 2/5, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 1/2, 3/5, 3/5, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 2/3, 5/7, 3/4, 3/4, 3/4, 4/5, 4/5, 5/6, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1).

As n grows, the denominators of these rational numbers grow remarkably slowly: the largest denominator is 11 for $n = 1000$, and 13 for $n = 10000$. Moreover, for any fixed n the very smallest denominators occur with greatest frequency. For example, when $n = 10000$, there are 1667 values of k for which $a(\sigma_n + k) = \frac{1}{2}$, and 714 values of k for which $a(\sigma_n + k) = \frac{1}{3}$; but for each $p = 1, \ldots, 12$ there is just one value of k for which $a(\sigma_n + k) = \frac{p}{13}$. In Lemma 17, we relate these denominators to the periodicity: if $a(\sigma) = p/q$ in lowest terms, then σ has eventual period q.

The remainder of the paper is organized as follows. In section 2 we show how to construct, given a chip configuration σ on K_n, a circle map $f : S^1 \to S^1$ whose rotation number equals the activity of σ. This construction resembles the one-dimensional particle/barrier model of [10]. In section 3 we use the circle map to prove our main results on mode locking, Theorem 8 and Proposition 10. Short period attractors are studied in section 4, where we show that all states on K_n have eventual period at most n (Proposition 16). Finally, in Theorem 23, we find a small "window" in which all states have eventual period two.

In parallel chip-firing on a general graph, a vertex is unstable if it has at least as many chips as its degree, and fires by sending one chip to each neighbor; at each time step, all unstable vertices fire simultaneously. Many questions remain concerning parallel chip-firing on graphs other than K_n. If the underlying graph is a tree [4] or a cycle [7], then instead of a devil's staircase of infinitely many preferred modes, there are just three: activity

0, $\frac{1}{2}$ and 1. On the other hand, the numerical experiments of [1] for parallel chip-firing on the $n \times n$ torus suggest a devil's staircase in the large n limit. Our arguments rely quite strongly on the structure of the complete graph, whereas the mode locking phenomenon seems to be widespread. It would be very interesting to relate parallel chip-firing on other graphs to iteration of a circle map (or perhaps a map on a higher-dimensional manifold) in order to explain the ubiquity of mode locking.

2. Construction of the Circle Map

We first introduce a framework of generalized chip configurations, which will encompass chip configurations on K_n for all n. To each generalized chip configuration we associate a probability measure on the interval $[0, 2)$, and to each such measure we associate a circle map $S^1 \to S^1$. We will define an update rule U describing the dynamics on each of these objects, so that the following diagram commutes.

$$\begin{tikzcd}
\text{chip configurations on } K_n \arrow[r, "U"] \arrow[d] & \text{chip configurations on } K_n \arrow[d] \\
\text{generalized chip configurations} \arrow[r, "U"] \arrow[d] & \text{generalized chip configurations} \arrow[d] \\
\text{measures on } [0,2) \arrow[r, "U"] \arrow[d] & \text{measures on } [0,2) \arrow[d] \\
\text{lifts of circle maps } S^1 \to S^1 \arrow[r, "U"] & \text{lifts of circle maps } S^1 \to S^1
\end{tikzcd} \tag{7}$$

Write $\mathcal{L}$ for Lebesgue measure on $[0, 1]$. A *generalized chip configuration* is a measurable function

$$\eta : [0, 1] \to [0, \infty).$$

Let

$$r(\eta) = \mathcal{L}\{x | \eta(x) \geq 1\}$$

and define the update rule

$$U\eta(x) = \begin{cases} \eta(x) + r(\eta), & \eta(x) < 1 \\ \eta(x) - 1 + r(\eta), & \eta(x) \geq 1. \end{cases} \tag{8}$$

If σ is a chip configuration on K_n, we define its associated generalized chip configuration $\psi(\sigma)$ by

$$\psi(\sigma)(x) = \frac{\sigma(\lceil nx \rceil) + \lceil nx \rceil - nx}{n} \tag{9}$$

where $\lceil y \rceil$ denotes the least integer $\geq y$. Our first lemma checks that the top square of (7) commutes.

Lemma 1. $U \circ \psi = \psi \circ U$.

Proof. Let σ be a chip configuration on K_n, and write $\eta = \psi(\sigma)$. For $v \in [n]$, if $\sigma(v) \geq n$, then $\eta(x) \geq 1$ for all $x \in (\frac{v-1}{n}, \frac{v}{n}]$. Conversely, if $\sigma(v) \leq n-1$, then $\eta(x) < 1$ for all $x \in (\frac{v-1}{n}, \frac{v}{n}]$. Hence $r(\eta) = r(\sigma)/n$.

Fix $x \in [0,1]$ and let $v = \lceil nx \rceil$. Then

$$\begin{aligned} U(\psi(\sigma))(x) &= \eta(x) - 1_{\{\eta(x) \geq 1\}} + r(\eta) \\ &= \frac{\sigma(v) + v - nx}{n} - 1_{\{\sigma(v) \geq n\}} + \frac{r(\sigma)}{n} \\ &= \frac{U\sigma(v) + v - nx}{n} \\ &= \psi(U\sigma)(x). \quad \square \end{aligned}$$

We remark that Lemma 1 would hold also if we defined ψ using the piecewise constant interpolation $\frac{\sigma(\lceil nx \rceil)}{n}$. We have chosen the piecewise linear interpolation (9) because it is better suited to our construction of the circle map, below: the circle map associated to the piecewise linear interpolation will always be continuous.

Next we observe a simple consequence of the update rule (8), which will allow us to focus on a subset of generalized chip configurations which we call "confined."

Lemma 2. *Let η be a generalized chip configuration, and let $x \in [0,1]$. If $\eta(x) < 2$, then*

$$r(\eta) \leq U\eta(x) < 1 + r(\eta).$$

Proof. The first inequality is immediate from the definition of $U\eta$. For the second inequality, if $\eta(x) < 1$, then

$$U\eta(x) = \eta(x) + r(\eta) < 1 + r(\eta)$$

while if $1 \leq \eta(x) < 2$, then

$$U\eta(x) = \eta(x) - 1 + r(\eta) < 1 + r(\eta). \quad \square$$

Definition. A generalized chip configuration η is *preconfined* if it satisfies

(i) $\eta(x) < 2$ for all $x \in [0,1]$.

If, in addition, there exists $r \in [0,1]$ such that

(ii) $r \leq \eta(x) < 1 + r$ for all $x \in [0,1]$

then η is *confined.*

By Lemma 2, if η is preconfined, then $U\eta$ is confined.

Note that from (1)

$$U\eta(x) \equiv \eta(x) + r(\eta) \pmod 1.$$

Iterating yields the congruence

$$U^t\eta(x) \equiv \eta(x) + \beta_t \pmod 1 \tag{10}$$

where

$$\beta_t = \sum_{s=0}^{t-1} r(U^s\eta).$$

Next we find a recurrence for the sequence β_t. We start with the following lemma.

Lemma 3. *If $U^t\eta$ is preconfined, then $U^{t+1}\eta(x) \geq 1$ if and only if*

$$\eta(x) \in \mathbb{Z} - (\beta_t, \beta_{t+1}].$$

Proof. By Lemma 2, since $U^t\eta$ is preconfined, we have

$$r_t \leq U^{t+1}\eta(x) < 1 + r_t$$

for all $x \in [0,1]$, where $r_t = r(U^t\eta)$. Thus $U^{t+1}\eta(x) \geq 1$ if and only if

$$U^{t+1}\eta(x) \in \mathbb{Z} + [0, r_t).$$

By (10), this condition is equivalent to

$$\eta(x) + \beta_{t+1} \in \mathbb{Z} + [0, r_t).$$

Using the fact that $\beta_{t+1} = \beta_t + r_t$, this in turn is equivalent to

$$\eta(x) \in \mathbb{Z} + [-\beta_{t+1}, r_t - \beta_{t+1}) = \mathbb{Z} - (\beta_t, \beta_{t+1}]. \quad \square$$

The essential information in a generalized chip configuration η is contained in the associated probability measure μ on $[0, \infty)$ given by

$$\mu(A) = \mathcal{L}(\eta^{-1}(A))$$

for Borel sets $A \subset [0, \infty)$. If η arises from a chip configuration σ on K_n, then $\mu([a,b])$ is the proportion of vertices that have between an and bn chips. For $y \in \mathbb{R}$, write $\xi^y\mu$ for the translated measure $\xi^y\mu(A) = \mu(A - y)$. For $Y \subset \mathbb{R}$, write $\mu|_Y$ for the restricted measure $\mu|_Y(A) = \mu(A \cap Y)$. Then the update rule (8) takes the form

$$U\mu = \xi^{\mu[1,\infty)}(\mu|_{[0,1)}) + \xi^{\mu[1,\infty)-1}(\mu|_{[1,\infty)}). \tag{11}$$

Now consider the measure ν on $\mathbb{R}$ given by

$$\nu(A) = \sum_{m \in \mathbb{Z}} \mu(m - A) = \sum_{m \in \mathbb{Z}} \mathcal{L}\{x \,|\, m - \eta(x) \in A\}. \tag{12}$$

By Lemma 3, if $U^t\eta$ is preconfined, then

$$r(U^{t+1}\eta) = \mu(\mathbb{Z} - (\beta_t, \beta_{t+1}]) = \nu(\beta_t, \beta_{t+1}],$$

hence

$$\beta_{t+2} = \beta_{t+1} + \nu(\beta_t, \beta_{t+1}]. \tag{13}$$

This gives a recurrence relating three consecutive terms of the sequence β_t. Our next lemma simplifies this recurrence to one relating just two consecutive terms.

Lemma 4. *If η is preconfined, then for all $t \geq 0$*

$$\beta_{t+1} = f(\beta_t),$$

where $f : [0, \infty) \to [0, \infty)$ is given by

$$f(y) = \beta_1 + \nu(0, y] \tag{14}$$

and ν is given by (12).

Proof. By Lemma 2, $U^t\eta$ is preconfined for all $t \geq 0$, so from (13),

$$\begin{aligned} \beta_{t+1} - \beta_1 &= \sum_{s=0}^{t-1} (\beta_{s+2} - \beta_{s+1}) \\ &= \sum_{s=0}^{t-1} \nu(\beta_s, \beta_{s+1}] \\ &= \nu(0, \beta_t]. \end{aligned}$$

Hence $\beta_{t+1} = f(\beta_t)$. $\square$

The function f appearing in Lemma 4 satisfies

$$\begin{aligned} f(y+1) &= f(y) + \nu(y, y+1] \\ &= f(y) + 1 \end{aligned}$$

for all $y \geq 0$. Thus it has a unique extension to a function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(y+1) = f(y) + 1$ for all $y \in \mathbb{R}$.

Note that f is nondecreasing. If it is also continuous, then it has a well-defined *Poincaré rotation number* [9, 13]

$$\rho(f) = \lim_{t\to\infty} \frac{f^t(y)}{t}$$

which does not depend on y. Here f^t denotes the t-fold iterate $f^t(y) = f(f^{t-1}(y))$, with $f^0 = Id$.

Viewing the circle S^1 as $\mathbb{R}/\mathbb{Z}$, the map f descends to a circle map $\bar{f} : S^1 \to S^1$. The rotation number $\rho(f)$ measures the asymptotic rate at which the sequence

$$y,\ \bar{f}(y),\ \bar{f}(\bar{f}(y)),\ \ldots$$

winds around the circle.

We define the *activity* of a generalized chip configuration η as the limit

$$a(\eta) = \lim_{t\to\infty} \frac{\beta_t}{t}.$$

From Lemma 4 we see that if η is preconfined and f is continuous, then this limit exists and equals the rotation number of f.

Lemma 5. *If η is preconfined, then $\beta_t = f^t(0)$ for all $t \geq 0$.*

Lemma 6. *If η is preconfined and f is continuous, then $a(\eta) = \rho(f)$.*

The conditions that η is preconfined and f is continuous are most succinctly expressed in terms of the associated probability measure: μ is supported on $[0, 2)$ and is *non-atomic*, that is, $\mu(\{y\}) = 0$ for all $y \in [0, 2)$.

To complete the commutative diagram (7), it remains to describe the dynamics U on lifts of circle maps. Call a map $f : \mathbb{R} \to \mathbb{R}$ a *monotone degree one lift* if f is continuous, nondecreasing and satisfies

$$f(y + 1) = f(y) + 1 \tag{15}$$

for all $y \in \mathbb{R}$. Let $\mathcal{F}$ be the set of monotone degree-one lifts $f : \mathbb{R} \to \mathbb{R}$, equipped with the L^∞ topology. We define the update rule $U : \mathcal{F} \to \mathcal{F}$ by

$$Uf = R_{-f(0)} \circ f \circ R_{f(0)}$$

where for $y \in \mathbb{R}$, we write R_y for the translation $x \mapsto x + y$.

Let $\mathcal{M}$ be the space of non-atomic probability measures μ on the interval $[0, 2)$, equipped with the topology of weak-$*$ convergence: $\mu_n \to \mu$ if and only if $\mu_n[0, y) \to \mu[0, y)$ for all $y \in [0, 2)$. For $\mu \in \mathcal{M}$, define $\phi(\mu) : \mathbb{R} \to \mathbb{R}$ by

$$\phi(\mu)(y) = \mu[1, 2) + \sum_{m \in \mathbb{Z}} \mu(m - y, m]$$

where if $b < a$ we define $\mu(a, b] := -\mu[b, a)$.

Theorem 7. *If $\mu \in \mathcal{M}$, then $\phi(\mu) \in \mathcal{F}$. The map $\phi : \mathcal{M} \to \mathcal{F}$ is continuous and preserves the dynamics: $U \circ \phi = \phi \circ U$.*

Proof. $\phi(\mu)$ is clearly nondecreasing in y, and since μ is non-atomic, $\phi(\mu)$ is continuous. Moreover, for any $y \in \mathbb{R}$,

$$\phi(\mu)(y + 1) - \phi(\mu)(y) = \sum_{m \in \mathbb{Z}} \mu(m - y - 1, m - y] = \mu(\mathbb{R}) = 1$$

so $\phi(\mu) \in \mathcal{F}$.

If $\mu_n \in \mathcal{M}$ and $\mu_n \to \mu \in \mathcal{M}$, then the cumulative distribution functions $F_n(y) = \mu_n[0, y)$ converge pointwise to $F(y) := \mu[0, y)$. Since F_n and F are continuous and nondecreasing, this convergence is uniform in y. Since μ_n and μ are supported on the interval $[0, 2)$, we have for $y \in [0, 1]$

$$\begin{aligned} \phi(\mu_n)(y) &= \mu_n[1, 2) + \mu_n(1 - y, 1] + \mu_n(2 - y, 2] \\ &= 2F_n(2) - F_n(1 - y) - F_n(2 - y). \end{aligned}$$

The right side converges uniformly in y to $2F(2) - F(1 - y) - F(2 - y) = \phi(\mu)(y)$. Since $\phi(\mu_n)$ and $\phi(\mu)$ are degree one lifts, we have

$$\sup_{y \in \mathbb{R}} |\phi(\mu_n)(y) - \phi(\mu)(y)| = \sup_{y \in [0,1]} |\phi(\mu_n)(y) - \phi(\mu)(y)| \to 0$$

as $n \to \infty$, which shows that ϕ is continuous.

Let $\beta = \mu[1,2) = \phi(\mu)(0)$. Then

$$\begin{aligned}(U \circ \phi)(\mu)(y) &= (R_{-\beta} \circ \phi(\mu) \circ R_\beta)(y) \\ &= \sum_{m \in \mathbb{Z}} \mu(m - y - \beta, m].\end{aligned}$$

Write $\mu = \mu_0 + \mu_1$, where $\mu_0 = \mu|_{[0,1)}$ and $\mu_1 = \mu|_{[1,2)}$. By (11), we have

$$U\mu = \xi^\beta \mu_0 + \xi^{\beta-1}\mu_1$$

hence

$$\begin{aligned}U\mu[1,2) &= \mu_0[1-\beta, 2-\beta) + \mu_1[2-\beta, 3-\beta) \\ &= \mu[1-\beta, 1) + \mu[2-\beta, 2) \\ &= \sum_{m \in \mathbb{Z}} \mu(m-\beta, m]\end{aligned}$$

where in the last line we have used that μ is non-atomic and supported on $[0,2)$. Moreover

$$\begin{aligned}\sum_{m \in \mathbb{Z}} U\mu(m-y, m] &= \sum_{m \in \mathbb{Z}} \mu_0(m-y-\beta, m-\beta] + \\ &\qquad + \sum_{m \in \mathbb{Z}} \mu_1(m-y+1-\beta, m+1-\beta] \\ &= \sum_{m \in \mathbb{Z}} \mu(m-y-\beta, m-\beta].\end{aligned}$$

We conclude that

$$\begin{aligned}(\phi \circ U)(\mu)(y) &= U\mu[1,2) + \sum_{m \in \mathbb{Z}} U\mu(m-y, m] \\ &= \sum_{m \in \mathbb{Z}} \mu(m-y-\beta, m]\end{aligned}$$

so $\phi \circ U = U \circ \phi$. □

One naturally wonders how to generalize the construction described in this section to chip-firing on graphs other than K_n. A key step may involve identifying invariants of the dynamics. On K_n, these invariants take a very simple form: by (10), for any two vertices $v, w \in [n]$, the difference

$$U^t\sigma(v) - U^t\sigma(w) \mod n$$

does not depend on t. Analogous invariants for parallel chip-firing on the $n \times n$ torus are classified in [6], following the approach of [8].

3. Devil's Staircase

Let f, f_n, g be monotone degree one lifts (15), and denote by $\bar{f}, \bar{f}_n, \bar{g}$ the corresponding circle maps $S^1 \to S^1$. Write $f \leq g$ if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, and $f < g$ if $f(x) < g(x)$ for all $x \in \mathbb{R}$. We will make use of the following well-known properties of the rotation number. For their proofs, see, for example [9, 13].

- **Monotonicity**. If $f \leq g$, then $\rho(f) \leq \rho(g)$.
- **Continuity**. If $\sup |f_n - f| \to 0$, then $\rho(f_n) \to \rho(f)$.
- **Conjugation Invariance**. If g is a homeomorphism, then $\rho(g \circ f \circ g^{-1}) = \rho(f)$.
- **Instability of an irrational rotation number**. If $\rho(f) \notin \mathbb{Q}$, and $f_1 < f < f_2$, then $\rho(f_1) < \rho(f) < \rho(f_2)$.
- **Stability of a rational rotation number**. If $\rho(f) = p/q \in \mathbb{Q}$, and $\bar{f}^q \neq Id : S^1 \to S^1$, then for sufficiently small $\epsilon > 0$, either
$$\rho(g) = p/q \text{ whenever } f \leq g \leq f + \epsilon,$$
or
$$\rho(g) = p/q \text{ whenever } f - \epsilon \leq g \leq f.$$

Let $\sigma_2, \sigma_3, \dots$ be a sequence of chip configurations, with σ_n defined on K_n. Suppose σ_n is stable, i.e.,
$$0 \leq \sigma_n(v) \leq n - 1 \tag{16}$$
for all $v \in [n]$. Moreover, suppose that there is a continuous function $F : [0, 1] \to [0, 1]$, such that for all $0 \leq x \leq 1$
$$\frac{1}{n}\#\{v \in [n] \mid \sigma_n(v) < nx\} \to F(x) \tag{17}$$
as $n \to \infty$. Since the left side is nondecreasing in x, this convergence is uniform in x. We define $\Phi : \mathbb{R} \to \mathbb{R}$ by
$$\Phi(x) = \lceil x \rceil - F(\lceil x \rceil - x). \tag{18}$$
Note that (16) and (17) force $F(0) = 0$ and $F(1) = 1$; by compactness, F is uniformly continuous on $[0, 1]$, and hence Φ is uniformly continuous on $\mathbb{R}$.

The *rescaled activity phase diagram* of σ_n is the function $s_n : [0, 1] \to [0, 1]$ defined by
$$s_n(y) = a(\sigma_n + ny).$$
As $n \to \infty$, the s_n have a pointwise limit identified in our next result.

Theorem 8. *If (16) and (17) hold, then for each $y \in [0, 1]$ we have*
$$s_n(y) \to s(y) := \rho(R_y \circ \Phi)$$
as $n \to \infty$, where Φ is given by (18), and $R_y(x) = x + y$.

In Proposition 10, below, we show that under an additional mild hypothesis, the limiting function $s(y)$ is a devil's staircase. Examples of these staircases for different choices of F are shown in Figure 2.

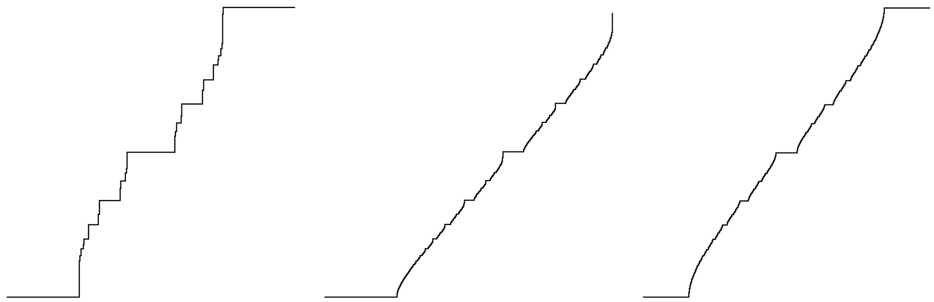

FIGURE 2. The devil's staircase $s(y)$, when (a) $F(x)$ is given by (6); (b) $F(x) = \sqrt{x}$ for $x \in [0,1]$; and (c) $F(x) = x + \frac{1}{2\pi}\sin 2\pi x$. On the horizontal axis y runs from 0 to 1, and on the vertical axis $s(y)$ runs from 0 to 1.

To prepare for the proof of Theorem 8, let $\eta_{n,y}$ be the generalized chip configuration associated to $\sigma_n + ny$, let $\mu_{n,y}$ be the associated measure on $[0,2)$, and let $f_{n,y} = \phi(\mu_{n,y})$ be the corresponding circle map lift.

Lemma 9. *Let $y \in [0,1]$. If (16) and (17) hold, then as $n \to \infty$*

$$\sup |f_{n,y} - \Phi \circ R_y| \to 0$$

uniformly in y.

Proof. The measures $\mu_{n,0}$ are nonatomic, and by (16) they are supported on $[0,1)$, so for $x \in [0,1]$ we have

$$\begin{aligned} f_{n,y}(x-y) &= \mu_{n,y}[1,2) + \sum_{m \in \mathbb{Z}} \mu_{n,y}(m-x+y, m] \\ &= \mu_{n,0}[1-y, 2-y) + \sum_{m \in \mathbb{Z}} \mu_{n,0}(m-x, m-y] \\ &= \mu_{n,0}[1-y,1) + \mu_{n,0}(1-x, 1-y] \\ &= \mu_{n,0}[1-x,1) \\ &= f_{n,0}(x). \end{aligned}$$

Since $f_{n,y}$ and $f_{n,0} \circ R_y$ are degree one lifts that agree on $[-y, 1-y]$, they agree everywhere.

By (17) we have $\mu_{n,0} \to \mu$, where $\mu([a,b]) = F(b) - F(a)$. Since μ is supported on $[0,1)$, we have for $x \in [0,1]$

$$\phi(\mu)(x) = \mu(1-x, 1] = 1 - F(1-x) = \Phi(x).$$

Since Φ and $\phi(\mu)$ are degree one lifts that agree on $[0,1]$, they agree everywhere. Hence

$$\begin{aligned}\sup|f_{n,y} - \Phi\circ R_y| &= \sup|(f_{n,0}-\Phi)\circ R_y| \\ &= \sup|f_{n,0}-\Phi| \\ &= \sup|\phi(\mu_{n,0})-\phi(\mu)|.\end{aligned}$$

The right side does not depend on y, and tends to zero as $n\to\infty$ by the continuity of ϕ (Theorem 7). ☐

Proof of Theorem 8. By Lemma 6, Lemma 9 and the continuity of the rotation number,

$$s_n(y) = a(\eta_{n,y}) = \rho(f_{n,y}) \to \rho(\Phi\circ R_y)$$

By the conjugation invariance of the rotation number, $\rho(\Phi\circ R_y) = \rho(R_y\circ\Phi)$, which completes the proof. ☐

Write $\Phi_y = R_y\circ\Phi$, and let $\bar{\Phi}_y : S^1\to S^1$ be the corresponding circle map. We will call a function $s:[0,1]\to[0,1]$ a *devil's staircase* if it is continuous, nondecreasing, nonconstant, and locally constant on an open dense set. Next we show that if

$$\left(\bar{\Phi}_y\right)^q \neq Id \quad \text{for all } y\in[0,1) \text{ and all } q\in\mathbb{N}, \tag{19}$$

then the limiting function $s(y)$ in Theorem 8 is a devil's staircase.

Proposition 10. *The function $s(y)=\rho(\Phi_y)$ continuous and nondecreasing in y. If $z\in[0,1]$ is irrational, then $s^{-1}(z)$ is a point. Moreover, if (19) holds, then for every rational number $p/q\in[0,1]$ the fiber $s^{-1}(p/q)$ is an interval of positive length.*

Substantially similar results appear in [9, 13]; we include a proof here for the sake of completeness.

Proof. The monotonicity of the rotation number implies that s is nondecreasing. Since $\sup|\Phi_y-\Phi_{y'}| = |y-y'|$, the continuity of the rotation number implies s is continuous.

Since $\Phi(m)=m$ for all $m\in\mathbb{Z}$, we have $s(0)=0$ and $s(1)=1$. By the intermediate value theorem, $s:[0,1]\to[0,1]$ is onto. If $z=s(y)$ is irrational, and $y_1<y<y_2$, then $\Phi_{y_1}<\Phi_y<\Phi_{y_2}$, hence

$$s(y_1) < s(y) < s(y_2)$$

by the instability of an irrational rotation number. It follows that $s^{-1}(z) = \{y\}$.

If $s(y)=p/q\in\mathbb{Q}$, then by the stability of a rational rotation number, since $(\bar{\Phi}_y)^q\neq Id$, there exists an interval I of positive length (take either $I=[y-\epsilon,y]$ or $I=[y,y+\epsilon]$ for small enough ϵ) such that $\rho(\Phi_{y'})=p/q$ for all $y'\in I$. Hence $I\subset s^{-1}(p/q)$. ☐

Note that if $y \le y'$, then $f_{n,y} \le f_{n,y'}$, so $s_n(y) = \rho(f_{n,y})$ is nondecreasing in y. By the continuity of s, it follows that the convergence in Theorem 8 is uniform in y.

Our next result shows that in the interiors of the stairs, we in fact have $s_n(y) = s(y)$ for sufficiently large n.

Proposition 11. *Suppose that (16), (17) and (19) hold. If $s^{-1}(p/q) = [a, b]$, then for any $\epsilon > 0$*

$$[a + \epsilon, b - \epsilon] \subset s_n^{-1}(p/q)$$

for all sufficiently large n.

Proof. Fix $\epsilon > 0$. By Lemma 9, for sufficiently large n we have

$$\sup |f_{n,y} - \Phi \circ R_y| < \epsilon$$

for all $y \in [0, 1]$. Then $f_{n,y} \ge R_{-\epsilon} \circ \Phi \circ R_y$, so for any $y \ge a + \epsilon$ we have by the monotonicity and conjugation invariance of the rotation number

$$\begin{aligned} s_n(y) = \rho(f_{n,y}) &\ge \rho(R_{-\epsilon} \circ \Phi \circ R_y) \\ &= \rho(R_{y-\epsilon} \circ \Phi) = s(y - \epsilon) \ge s(a) = p/q. \end{aligned}$$

Likewise $f_{n,y} \le R_\epsilon \circ \Phi \circ R_y$, hence for any $y \le b - \epsilon$

$$\begin{aligned} s_n(y) = \rho(f_{n,y}) &\le \rho(R_\epsilon \circ \Phi \circ R_y) \\ &= \rho(R_{y+\epsilon} \circ \Phi) = s(y + \epsilon) \le s(b) = p/q. \quad \square \end{aligned}$$

4. Short Period Attractors

In this section we explore the prevalance of parallel chip-firing states on K_n with small period.

Definition. A chip configuration σ on K_n is *preconfined* if it satisfies

(i) $\sigma(v) \le 2n - 1$ for all vertices v of K_n.

If, in addition, σ satisfies

(ii) $\max_v \sigma(v) - \min_v \sigma(v) \le n - 1$

then σ is *confined*.

Equivalently, σ is preconfined (confined) if and only if the generalized chip configuration $\eta(\sigma)$ is preconfined (confined) as defined in section 2.

Recall (2) that for any chip configuration σ on K_n, the sequence $(U^t\sigma)_{t \ge 0}$ is eventually periodic. Denote its transient length by t_0; that is

$$t_0 = \min\{t \ge 0 \,|\, U^t\sigma = U^{t'}\sigma \text{ for some } t' > t\}.$$

We say that "v fires at time t" if $U^t\sigma(v) \ge n$.

Lemma 12. *If $a(\sigma) < 1$, then $U^t\sigma$ is confined for all $t \ge t_0$.*

Proof. If $a(\sigma) < 1$, then at each time step, some vertex does not fire. If a given vertex v fires at time t, then

$$U^{t+1}\sigma(v) < U^t\sigma(v).$$

Since $U^t\sigma(v) \in \mathbb{Z}_{\geq 0}$ for all t, there is some time time T_v at which v does not fire. By Lemma 2, we have $U^t\sigma(v) \leq 2n-1$ for all $t \geq T_v$, and $U^t\sigma$ is confined for all $t > \max_v T_v$.

For any $t \geq t_0$ we have $U^t\sigma = U^{t'}\sigma$ for infinitely many values of t', hence $U^t\sigma$ is confined. □

For a chip configuration σ on K_n and a vertex $v \in [n]$, let

$$u_t(\sigma, v) = \#\{0 \leq s < t \,|\, U^s\sigma(v) \geq n\}$$

be the number of times v fires during the first t updates. During these updates, the vertex v emits a total of $n\, u_t(\sigma, v)$ chips and receives a total of $\alpha_t = \sum_w u_t(\sigma, w)$ chips, so that

$$U^t\sigma(v) - \sigma(v) = \alpha_t - n\, u_t(\sigma, v). \tag{20}$$

An easy consequence that we will use repeatedly is the following.

Lemma 13. *A chip configuration σ on K_n satisfies $U^t\sigma = \sigma$ if and only if*

$$u_t(\sigma, v) = u_t(\sigma, w) \tag{21}$$

for all vertices v and w.

Proof. If (21) holds, then $u_t(\sigma, v) = \alpha_t/n$ for all v, so $U^t\sigma(v) = \sigma(v)$ by (20). Conversely, if $U^t\sigma = \sigma$, then the left side of (20) vanishes, so $u_t(\sigma, v) = \alpha_t/n$ for all vertices v. □

According to our next lemma, if σ is confined, then $u_t(\sigma, v)$ and $u_t(\sigma, w)$ differ by at most one.

Lemma 14. *If σ is confined, and $\sigma(v) \leq \sigma(w)$, then for all $t \geq 0$*

$$u_t(\sigma, v) \leq u_t(\sigma, w) \leq u_t(\sigma, v) + 1.$$

Proof. Induct on t. If $u_t(\sigma, w) = u_t(\sigma, v)$, then by (20)

$$U^t\sigma(v) - U^t\sigma(w) = \sigma(v) - \sigma(w) \leq 0.$$

Thus if v fires at time t, then w must also fire, hence

$$u_{t+1}(\sigma, v) \leq u_{t+1}(\sigma, w) \leq u_{t+1}(\sigma, v) + 1. \tag{22}$$

On the other hand, if $u_t(\sigma, w) = u_t(\sigma, v) + 1$, then since σ is confined, we have by (20)

$$U^t\sigma(v) - U^t\sigma(w) = n + \sigma(v) - \sigma(w) \geq 0.$$

Thus if w fires at time t, then v must also fire, so once again (22) holds. □

Lemma 15. *If σ is confined, then $U^t\sigma = \sigma$ if and only if $n|\alpha_t$.*

Proof. If $U^t\sigma = \sigma$, then $n|\alpha_t$ by Lemma 13. For the converse, write $\ell = \min_v u_t(\sigma, v)$. By Lemma 14 we have

$$u_t(\sigma, v) \in \{\ell, \ell + 1\}$$

for all vertices v. If $n|\alpha_t$, then since

$$\alpha_t = \sum_{v=1}^{n} u_t(\sigma, v),$$

we must have $u_t(\sigma, v) = \ell$ for all v, so $U^t\sigma = \sigma$ Lemma 13. □

Let σ be a confined state on K_n. By the pigeonhole principle, there exist times $0 \leq s < t \leq n$ with

$$\alpha_s \equiv \alpha_t \pmod n.$$

By Lemma 15 it follows that $U^s\sigma = U^t\sigma$, so σ has eventual period at most n.

Our next result improves this bound a bit. Write $m(\sigma)$ for the eventual period of σ, and

$$\nu(\sigma) = \#\{\sigma(v)|v \in [n]\}$$

for the number of distinct heights in σ.

Proposition 16. *For any chip configuration σ on K_n,*

$$m(\sigma) \leq \nu(\sigma).$$

Proof. If $a(\sigma) = 1$, then $m(\sigma) = 1$. Since $\nu(U\sigma) \leq \nu(\sigma)$ and $m(U\sigma) = m(\sigma)$, by Lemma 12 it suffices to prove the proposition for confined states σ. Moreover, by relabeling the vertices we may assume that $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n)$. Write the total activity α_t as

$$\alpha_t = Q_t n + R_t$$

for nonnegative integers Q_t and R_t with $R_t \leq n - 1$. By Lemma 14, since $\sum_v u_t(\sigma, v) = \alpha_t$, we must have

$$u_t(\sigma, v) = \begin{cases} Q_t + 1, & v \leq R_t \\ Q_t, & v > R_t. \end{cases}$$

Moreover if $R_t > 0$, then $\sigma(R_t) > \sigma(R_t + 1)$. Since the set

$$V = \{0\} \cup \{v \in [n-1] | \sigma(v) > \sigma(v+1)\}$$

has cardinality $\nu(\sigma)$, by the pigeonhole principle there exist times $0 \leq s < t \leq \nu(\sigma)$ with $R_s = R_t$. Then

$$u_t(\sigma, v) - u_s(\sigma, v) = Q_t - Q_s$$

independent of v, hence $U^s\sigma = U^t\sigma$ by Lemma 13. Thus $m(\sigma) \leq t - s \leq \nu(\sigma)$. □

Bitar [3] conjectured that any parallel chip-firing configuration on a connected graph of n vertices has eventual period at most n. A counterexample was found by Kiwi et al. [11]. Anne Fey (personal communication) has found a counterexample on a regular graph. Proposition 16 shows that Bitar's conjecture holds on the complete graph; it is also known to hold on trees [4] and on cycles [7]. It would be interesting to find other classes of graphs for which Bitar's conjecture holds.

Next we relate the period of a chip configuration to its activity. We will use the fact that the rotation number of a circle map determines the periods of its periodic points: if $f : \mathbb{R} \to \mathbb{R}$ is a monotone degree one lift (15) with $\rho(f) = p/q$ in lowest terms, then all periodic points of $\bar{f} : S^1 \to S^1$ have period q; see Proposition 4.3.8 and Exercise 4.3.5 of [9].

Note that if a chip configuration σ on K_n is periodic of period m, and $a(\sigma) = p/q$, then

$$\frac{\alpha_m}{mn} = \frac{p}{q} \tag{23}$$

and $\alpha_{km} = k\alpha_m$ for all $k \in \mathbb{N}$.

Lemma 17. *If $a(\sigma) = p/q$ and $(p, q) = 1$, then $m(\sigma) = q$.*

Proof. Since $a(U\sigma) = a(\sigma)$ and $m(U\sigma) = m(\sigma)$, we may assume that σ is periodic. Now if $a(\sigma) = 1$, then $m(\sigma) = 1$, so we may assume $a(\sigma) < 1$. In particular, σ is confined by Lemma 12. Write $m = m(\sigma)$. By Lemma 13 and (23)

$$u_m(\sigma, v) = \frac{\alpha_m}{n} = \frac{pm}{q}$$

for all vertices v. Hence $q|pm$ and hence $q|m$.

Let f be the lift of the circle map $\bar{f}$ associated to σ. By Lemma 6 we have $\rho(f) = a(\sigma)$, so all periodic points of $\bar{f}$ have period q. By Lemma 5 we have

$$f^{qm}(0) = \frac{\alpha_{qm}}{n} = \frac{q\alpha_m}{n} = pm \in \mathbb{Z},$$

so 0 is a periodic point of $\bar{f}$. Thus $f^q(0) \in \mathbb{Z}$. Now by Lemma 5

$$\alpha_q = nf^q(0)$$

hence $U^q\sigma = \sigma$ by Lemma 15. Thus $m|q$. $\square$

Given $1 \le p < q \le n$ with $(p, q) = 1$ and $p/q \le 1/2$, one can check that the chip configuration σ on K_n given by

$$\sigma(v) = \begin{cases} v + p - 1, & v \le q - 1 - p \\ v + n + p - q - 1, & q - p \le v \le q - 1 \\ n + p - 1, & v \ge q. \end{cases}$$

has activity $a(\sigma) = p/q$. For a similar construction on more general graphs in the case $p = 1$, see [14, Prop. 6.8]. By Lemma 17, $m(\sigma) = q$. So for every integer $q = 1, \dots, n$ there exists a chip configuration on K_n of period q.

Despite the existence of states of period as large as n, states of smaller period are in some sense more prevalent. One way to capture this is the following.

Theorem 18. *If $\sigma_2, \sigma_3, \ldots$ is a sequence of chip configurations satisfying (16), (17) and (19), then for each $q \in \mathbb{N}$ there is a constant $c = c_q > 0$ such that for all sufficiently large n, at least cn of the states $\{\sigma_n + k\}_{k=0}^{n}$ have eventual period q.*

Proof. Fix $\epsilon > 0$, and choose $p < q$ with $(p,q) = 1$. By Proposition 10 we have $s^{-1}(p/q) = [a_p, b_p]$ for some $a_p < b_p$. By Proposition 11, for sufficiently large n we have

$$a(\sigma_n + \lfloor ny \rfloor) = s_n(y) = p/q$$

for all $y \in [a_p + \epsilon/2q, b_p - \epsilon/2q]$. Thus all states $\sigma_n + k$ with

$$\left(a_p + \frac{\epsilon}{2q}\right) n < k < \left(b_p - \frac{\epsilon}{2q}\right) n$$

have activity p/q and hence, by Lemma 17, eventual period q. We can therefore take

$$c_q = \sum_{p:(p,q)=1} (b_p - a_p) - \epsilon$$

for any $\epsilon > 0$. $\square$

The rest of this section is devoted to proving the existence of a *period* 2 *window*: any chip configuration on K_n with total number of chips strictly between $n^2 - n$ and n^2 has eventual period 2.

The following lemma is a special case of [14, Prop. 6.2]; we include a proof for completeness. Write $r(\sigma)$ for the number of unstable vertices in σ.

Lemma 19. *If σ and τ are chip configurations on K_n with $\sigma(v) + \tau(v) = 2n - 1$ for all v, then $a(\sigma) + a(\tau) = 1$.*

Proof. If $\sigma(v) + \tau(v) = 2n - 1$ for all v, then for each vertex v, exactly one of $\sigma(v), \tau(v)$ is $\geq n$. Hence $r(\sigma) + r(\tau) = n$, and

$$\begin{aligned} U\sigma(v) + U\tau(v) &= \sigma(v) + r(\sigma) + \tau(v) + r(\tau) - n \\ &= 2n - 1 \end{aligned}$$

for all vertices v. Inducting on t, we obtain

$$r(U^t\sigma) + r(U^t\tau) = n$$

for all $t \geq 0$, and hence $a(\sigma) + a(\tau) = 1$ by (3). $\square$

Given a chip configuration σ on K_n, for $j = 1, \ldots, n$ we define *conjugate* configurations

$$c^j\sigma(v) = \begin{cases} \sigma(v) + j - n, & v \leq j \\ \sigma(v) + j, & v > j. \end{cases}$$

Note that $c^n\sigma = \sigma$.

Lemma 20. *Let σ be a chip configuration on K_n, and fix $j \in [n]$. For all $t \geq 0$, we have for $v \leq j$*

$$u_t(\sigma, v) - 1 \leq u_t(c^j\sigma, v) \leq u_t(\sigma, v),$$

while for $v > j$

$$u_t(\sigma, v) \leq u_t(c^j\sigma, v) \leq u_t(\sigma, v) + 1.$$

Proof. Induct on t. By (20)

$$U^t c^j\sigma(v) = c^j\sigma(v) + \alpha_{t,j} - n\, u_t(c^j\sigma, v) \tag{24}$$

where

$$\alpha_{t,j} = \sum_{w=1}^{n} u_t(c^j\sigma, w).$$

By the inductive hypothesis,

$$-j \leq \alpha_{t,j} - \alpha_{t,0} \leq n - j. \tag{25}$$

Fix a vertex v, and write

$$b = u_t(c^j\sigma, v) - u_t(\sigma, v).$$

By the inductive hypothesis we have $b \in \{-1, 0, 1\}$. From (24) we have

$$U^t c^j\sigma(v) - U^t\sigma(v) = \alpha_{t,j} - \alpha_{t,0} + c^j\sigma(v) - \sigma(v) - nb \tag{26}$$

Case 1: $v \leq j$ and $b = -1$. By (25), the right side of (26) is ≥ 0, so if v is unstable in $U^t\sigma$, then it is also unstable in $U^t c^j\sigma$. Thus

$$u_{t+1}(\sigma, v) - 1 \leq u_{t+1}(c^j\sigma, v) \leq u_{t+1}(\sigma, v), \tag{27}$$

completing the inductive step.

Case 2: $v \leq j$ and $b = 0$. By (25), the right side of (26) is ≤ 0, so if v is unstable in $U^t c^j\sigma$, then it is also unstable in $U^t\sigma$. Thus once again (27) holds.

Case 3: $v > j$ and $b = 0$. By (25), the right side of (26) is ≥ 0, so

$$u_{t+1}(\sigma, v) \leq u_{t+1}(c^j\sigma, v) \leq u_{t+1}(\sigma, v) + 1, \tag{28}$$

completing the inductive step.

Case 4: $v > j$ and $b = 1$. By (25), the right side of (26) is ≤ 0, so once again (28) holds. □

Corollary 21. *For any chip configuration σ on K_n and any $j \in [n]$,*

$$a(c^j\sigma) = a(\sigma).$$

It turns out that the circle maps corresponding to σ and $c^j\sigma$ are conjugate to one another by a rotation. This gives an alternative proof of the corollary, in the case when both σ and $c^j\sigma$ are preconfined.

Lemma 22. *Let σ be a chip configuration on K_n. If $u_2(\sigma, v) \geq 1$ for all v, then $u_{2t}(\sigma, v) \geq t$ for all v and all $t \geq 1$.*

Proof. Induct on t. By the inductive hypothesis, $\alpha_{2t} \geq nt$. Fix a vertex v, and suppose that $u_{2t}(\sigma, v) = t$. Since $u_2(\sigma, v) \geq 1$, either $\sigma(v) \geq n$ or $U\sigma(v) \geq n$. In the former case, by (20)

$$U^{2t}\sigma(v) - \sigma(v) = \alpha_{2t} - nt \geq 0,$$

so v fires at time $2t$. Hence $u_{2t+1}(\sigma, v) \geq t + 1$. Summing over v yields

$$\alpha_{2t+1} \geq \alpha_1(t+1) + (n - \alpha_1)t = nt + \alpha_1.$$

In the latter case, if v does not fire at time $2t$, we have by (20)

$$U^{2t+1}\sigma(v) - U\sigma(v) = \alpha_{2t+1} - \alpha_1 - nt \geq 0,$$

so v fires at time $2t + 1$. Hence $u_{2t+2}(\sigma, v) \geq t + 1$ for all v. □

Given a chip configuration σ on K_n, write

$$\ell(\sigma) = \min\{\sigma(1), \dots, \sigma(n)\}$$

and

$$r(\sigma) = \#\{v \in [n] : \sigma(v) \geq n\}.$$

Write

$$|\sigma| = \sum_{v=1}^{n} \sigma(v)$$

for the total number of chips in the system.

Theorem 23. *Every chip configuration σ on K_n with $n^2 - n < |\sigma| < n^2$ has eventual period* 2.

Proof. If $|\sigma| < n^2$, then $a(\sigma) < 1$, so by Lemma 12 we may assume σ is confined. Moreover, by relabeling the vertices we may assume that $\sigma(1) \geq \sigma(2) \geq \ldots \geq \sigma(n)$. In particular, for any vertex v, since $\sigma(w) \geq \sigma(v) - n + 1$ for $w = v + 1, \dots, n$, we have

$$\begin{aligned} |\sigma| &\geq v\sigma(v) + (n - v)(\sigma(v) - n + 1) \\ &= n\sigma(v) - (n - v)(n - 1). \end{aligned}$$

Since $|\sigma| < n^2$, we obtain

$$\sigma(v) < \frac{n^2 + (n - v)(n - 1)}{n} \leq 2n - v. \tag{29}$$

For a fixed vertex v, as j ranges over $[n]$, the quantity $c^j\sigma(v)$ takes on each of the values

$$\sigma(v) + v - n, \dots, \sigma(v) + v - 1$$

exactly once. By (29), at least $\sigma(v) + v - n$ of these values are $\geq n$, hence

$$\sum_{j=1}^{n} r(c^j\sigma) \geq \sum_{v=1}^{n} (\sigma(v) + v - n) = |\sigma| - \frac{n(n-1)}{2}. \tag{30}$$

Since $\ell(c^j\sigma) = \sigma(j) + j - n$, we have

$$\sum_{j=1}^{n} \ell(c^j\sigma) = |\sigma| - \frac{n(n-1)}{2}. \tag{31}$$

Summing equations (30) and (31), and using $|\sigma| > n^2 - n$, we obtain

$$\sum_{j=1}^{n} (\ell(c^j\sigma) + r(c^j\sigma)) > n^2 - n.$$

Since each term in the sum on the left is a nonnegative integer, we must have

$$\ell(c^j\sigma) + r(c^j\sigma) \geq n.$$

for some $j \in [n]$. Thus every vertex v fires at least once during the first two updates of $c^j\sigma$. From Corollary 21 and Lemma 22 we obtain

$$a(\sigma) = a(c^j\sigma) \geq \frac{1}{2}.$$

Finally, the chip configuration $\tau(v) := 2n - 1 - \sigma(v)$ also satisfies $n^2 - n < |\tau| < n^2$, so $a(\tau) \geq \frac{1}{2}$. By Lemma 19 we have $a(\sigma) + a(\tau) = 1$, so $a(\sigma) = a(\tau) = \frac{1}{2}$. From Lemma 17 we conclude that $m(\sigma) = 2$. $\square$

Acknowledgements

The author thanks Anne Fey for many helpful conversations, and an anonymous referee for suggesting the formulation of generalized chip configurations in terms of measures on $[0, 2)$.

References

[1] F. Bagnoli, F. Cecconi, A. Flammini, and A. Vespignani, Short-period attractors and non-ergodic behavior in the deterministic fixed-energy sandpile model, *Europhys. Lett.* **63** (2003), 512–518.

[2] P. Bak, C. Tang and K. Wiesenfeld, Self-organized criticality: an explanation of the $1/f$ noise, *Phys. Rev. Lett.* **59**, no. 4 (1987), 381–384.

[3] J. Bitar, Juegos Combinatoriales en Redes de Autómatas, Tesis Ingeniero Matemático, Dept. Ingeniería Matemática, U. de Chile, 1989.

[4] J. Bitar and E. Goles, Parallel chip firing games on graphs, *Theoretical Comp. Sci.* **92**, no. 2 (1992), 291–300.

[5] A. Björner, L. Lovász and P. Shor, Chip-firing games on graphs, *European J. Combin.* **12** (1991), no. 4, 283–291.

[6] M. Casartelli, L. Dall'Asta, A. Vezzani and P. Vivo, Dynamical invariants in the deterministic fixed-energy sandpile, *European Phys. J. B* **52** (2006), no. 1, 91–105.

[7] L. Dall'Asta, Exact solution of the one-dimensional deterministic fixed-energy sandpile, *Phys. Rev. Lett.* **96** (2006).

[8] D. Dhar, Self-organized critical state of sandpile automaton models, *Phys. Rev. Lett.* **64** (1990), 1613–1616.

[9] B. Hasselblatt and A. Katok, *A First Course in Dynamics, With a Panorama of Recent Developments*, Cambridge University Press, 2003.

[10] S. A. Janowsky and C. A. Laberge, Exact solutions for a mean-field abelian sandpile, *J. Phys. A* **26** (1993) L973–L980.

[11] M. A. Kiwi, R. Ndoundam, M. Tchuente and E. Goles, No polynomial bound for the period of the parallel chip firing game on graphs, *Theoretical Comp. Sci.* **136**, no. 2 (1994), 527–532.
[12] J. C. Lagarias, Number theory and dynamical systems, *Proc. Symp. Applied Math.* **46** (1992), 35–72.
[13] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer-Verlag, 1991.
[14] E. Prisner, Parallel chip-firing on digraphs, *Complex Systems* **8** (1994), 367–383.
[15] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Absorbing-state phase transitions in fixed-energy sandpiles, *Phys. Rev. E* **62** (2000), 4564–4582.

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Ave., Cambridge, MA 02139; http://math.mit.edu/~levine