BRAIDS, Q-BINOMIALS AND QUANTUM GROUPS

MARCELO AGUIAR

Abstract. The classical identities between the q-binomial coefficients and factorials can be generalized to a context where numbers are replaced by braids. More precisely, for every pair i, n of natural numbers, there is defined an element $b_i^{(n)}$ of the braid group algebra kB_n, and these satisfy analogs of the classical identities for the binomial coefficients. By choosing representations of the braid groups, one obtains numerical or matrix realizations of these identities, in particular one recovers the q-identities in this way. These binomial braids $b_i^{(n)}$ play a crucial role in a simple definition of a family of quantum groups, including the quantum groups $U_q^+(C)$ of Drinfeld and Jimbo.

1. Introduction

The classical identities between the q-binomial coefficients and factorials can be generalized to a context where numbers are replaced by braids, or more precisely, elements of the braid group algebras kB_n. Thus, for every pair i, n of natural numbers there is defined an element $b_i^{(n)} \in kB_n$ (section 3), and these satisfy analogs of the classical identities for the binomial coefficients (sections 4 through 8). Moreover, by choosing representations of the braid groups one obtains concrete realizations of these identities; the simplest such choices yield the identities for the classical and q-binomial coefficients, other choices yield new identities that involve matrices rather than numbers.

The following chart describes the action of the braids introduced in this paper when X is certain one-dimensional representation defined by $q \in k^*$ (section 2.5). The definition of the q-analogs will be reviewed before each corresponding braid analog is introduced.

Braid	name	defined in section	action
$s_i^{(n)}$	generator	2.1	q
$s^{(n)}(i,j)$		2.1	q^{j-i}
$c^{(n)}$	twistor	2.2	$q^{\binom{n}{2}}$
$\beta_{m,n}$	braiding	2.4	q^{mn}
$s_I^{(n)}$		3	$q^{\|I\|}$
$b_i^{(n)}$	binomial	3	$\begin{bmatrix} n \\ i \end{bmatrix}$
$[n]$	natural	5	$[n]$
$s_\sigma^{(n)}$		5	$q^{\mathsf{inv}(\sigma)}$
$f^{(n)}$	factorial	5	$[n]!$
$s_f^{(n)}$		7.1	$q^{\mathsf{inv}(f)}$
$m^{(\eta)}$	multinomial	7.1	$\begin{bmatrix} n \\ \eta \end{bmatrix}$
$\mu^{(n)}$	Möbius	6.2	$(-1)^n q^{\binom{n}{2}}$
$C^{(n)}$	Catalan	8	C_n
$G^{(n)}$	Galois	8	G_n
$F^{(n)}$	Fibonacci	8	F_n

Date: January 1997.
1991 *Mathematics Subject Classification.* Primary:05A30 ; Secondary:17B37.
Key words and phrases. Braids, q-analogues, quantum groups.

These *binomial* braids $b_i^{(n)}$ play a crucial role in the generalization of the definition of the quantum group $U_q^+(C)$ of Drinfeld [Dr] and Jimbo [J] presented in [A] and briefly discussed here in section 9.

At the level of braids, the proofs of the combinatorial identities follow a constant pattern: first there is the set-theoretic part, which involves dealing with the same bijections that are used for the case of the classical ($q = 1$) identities, then there is the geometric part that consists in proving that two braids, labeled by corresponding elements under the bijection considered, are in fact equal.

The classical q-identities that we generalize are taken mostly from papers by Goldman and Rota [GR]; in particular these include Pascal's, Vandermonde's and Cauchy's identities, the factorial formula, Rota's binomial theorem, Möbius inversion, several identities involving multinomial braids and definitions and formulas for the Galois, Fibonacci and Catalan braids.

It is also possible to define the braid analog of a partition of a set, and then Stirling and Bell braids. These will be studied elsewhere.

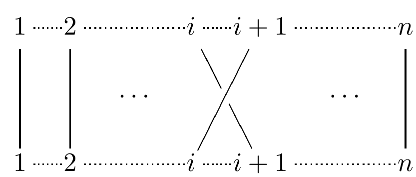
2. Braid groups and the braid category

2.1. **Basics.** The group B_n of *braids in n strands* has generators $s_1^{(n)}, \dots, s_{n-1}^{(n)}$ subject to the relations

$$s_i^{(n)} s_j^{(n)} = s_j^{(n)} s_i^{(n)} \qquad \text{if } |i-j| \geq 2, \tag{A1}$$

$$s_i^{(n)} s_{i+1}^{(n)} s_i^{(n)} = s_{i+1}^{(n)} s_i^{(n)} s_{i+1}^{(n)} \qquad \text{if } 1 \leq i \leq n-2. \tag{A2}$$

The generator $s_i^{(n)}$ is represented by the following picture, and the product st of two braids s and t in B_n is obtained by putting the picture of s on top of that of t. The identity of B_n is represented by the picture with n vertical strands; the inverse of s is obtained by reflecting its picture across a horizontal line, without leaving the plane of the picture.

$s_i^{(n)} =$ 

The collection $\mathfrak{B} = \coprod_{n \geq 0} B_n$ of all braid groups forms a category, where the objects are the natural numbers, B_n is the set of endomorphisms of n, and there are no morphisms between distinct objects. This category is monoidal; the tensor product $s \otimes t \in B_{n+m}$ of two braids $s \in B_n$ and $t \in B_m$ is obtained by putting t to the right of s, i.e. $s_i^{(n)} \otimes s_j^{(m)} = s_i^{(n+m)} s_{n+j}^{(n+m)}$. Moreover, this monoidal category is *braided*, in the sense that there is a natural map $\beta_{n,m} : n \otimes m \to m \otimes n$, i.e. a braid $\beta_{n,m} \in B_{n+m}$, satisfying some axioms (2.4 below). For more details on this, the reader is referred to [K], X.6 and XIII.2.

We develop some basic notation. For each pair (i, j) with $1 \leq i \leq j \leq n$, define

$$s^{(n)}(i,j) = \begin{cases} 1 & \text{if } i = j, \\ s_i^{(n)} s_{i+1}^{(n)} \cdots s_{j-1}^{(n)} & \text{if } i < j. \end{cases}$$

We provide a first set of lemmas.

Lemma.

(1) $s^{(n)}(i,k) = s^{(n)}(i,j)s^{(n)}(j,k)$ when $1 \le i \le j \le k \le n$

(2) $s_i^{(m+n)} = s_i^{(m)} \otimes 1^{(n)}$ when $1 \le i \le m-1,\ n \ge 0$

$s^{(m+n)}(i,j) = s^{(m)}(i,j) \otimes 1^{(n)}$ when $1 \le i \le j \le m,\ n \ge 0$

(3) $s_{i+n}^{(m+n)} = 1^{(n)} \otimes s_i^{(m)}$ when $1 \le i \le m-1,\ n \ge 0$

$s^{(m+n)}(i+n, j+n) = 1^{(n)} \otimes s^{(m)}(i,j)$ when $1 \le i \le j \le m,\ n \ge 0$

(4) $s_{i+l}^{(m+n)} = 1^{(l)} \otimes s_i^{(m)} \otimes 1^{(n-l)}$ when $1 \le i \le m-1,\ 0 \le l \le n$

$s^{(m+n)}(i+l, j+l) = 1^{(l)} \otimes s^{(m)}(i,j) \otimes 1^{(n-l)}$ when $1 \le i \le j \le m,\ 0 \le l \le n$

(5) $s^{(n)}(i,j)s_h^{(n)} = s_{h+1}^{(n)} s^{(n)}(i,j)$ when $1 \le i \le h \le j-2$

$s^{(n)}(i,j)s^{(n)}(h,k) = s^{(n)}(h+1,k+1)s^{(n)}(i,j)$ when $1 \le i \le h \le k \le j-1$.

Proof. Equation (1) is a direct consequence of the notation, the first parts of (2) and (3) hold simply by definition of the tensor product, and the second parts follow by repeated use of the first ones. Now,

$$s_{i+l}^{(m+n)} \overset{(2)}{=} s_{i+l}^{(m+l)} \otimes 1^{(n-l)} \overset{(3)}{=} 1^{(l)} \otimes s_i^{(m)} \otimes 1^{(n-l)} \ ,$$

proving the first part of (4). Similarly the second part follows from the second parts of (2) and (3).

Finally, if $1 \le i \le h \le j-2$, we can write

$$s^{(n)}(i,j)s_h^{(n)} \overset{(1)}{=} s^{(n)}(i,h)s_h^{(n)}s_{h+1}^{(n)}s^{(n)}(h+2,j)s_h^{(n)} \overset{(A1)}{=} s^{(n)}(i,h)s_h^{(n)}s_{h+1}^{(n)}s_h^{(n)}s^{(n)}(h+2,j) \overset{(A2)}{=}$$

$$s^{(n)}(i,h)s_{h+1}^{(n)}s_h^{(n)}s_{h+1}^{(n)}s^{(n)}(h+2,j) \overset{(A1)}{=} s_{h+1}^{(n)}s^{(n)}(i,h)s_h^{(n)}s_{h+1}^{(n)}s^{(n)}(h+2,j) \overset{(1)}{=} s_{h+1}^{(n)}s^{(n)}(i,j) \ ,$$

which proves the first part of (5); now for the second notice that if $k = h$ then there is nothing to prove; otherwise $j > k > h$ so it follows by repeated use of the first.

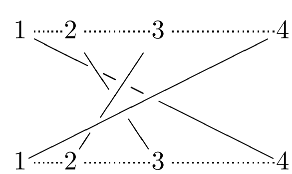
□

2.2. **Vertical symmetry.** There is an involution $\widetilde{\ }: B_n \to B_n$ defined by $\widetilde{s_i^{(n)}} = s_{n-i}^{(n)}$. The picture for $\tilde{s}$ is obtained by rotating in 3-space that of s 180 degrees around a vertical line. Consider the *twistor* braid,

$$c^{(n)} = s^{(n)}(1,n)s^{(n)}(1,n-1)\cdots s^{(n)}(1,2)s^{(n)}(1,1) \ .$$

For instance

$c^{(4)} \ =$



Repeated use of (A1) and (5) shows that $c^{(n)}s_i^{(n)} = s_{n-i}^{(n)}c^{(n)}$, hence $\widetilde{\ }$ is the inner automorphism defined by conjugation by $c^{(n)}$. It follows that ${c^{(n)}}^2$ is in the center of the braid group, since $\tilde{\tilde{s}} = s$ for any s. Moreover, it can be shown that ${c^{(n)}}^2$ generates $Z(B_n)$; we won't make use of this fact.

Let us prove that, for any $s \in B_n$ and $t \in B_m$,

(6) $$\widetilde{s \otimes t} = \tilde{t} \otimes \tilde{s} \ .$$

Proof. Notice that if the statement holds for $s\otimes t$ and $s'\otimes t'$, then so it does for $ss'\otimes tt'$. Hence it suffices to prove it for $s = s_i^{(n)}$ and $t = s_j^{(m)}$. Now,

$$s_i^{(n)}\otimes s_j^{(m)} \overset{(2),(3)}{=} s_i^{(n+m)} s_{n+j}^{(n+m)} \Rightarrow$$

$$\widetilde{s_i^{(n)}\otimes s_j^{(m)}} = s^{(n+m)}_{n+m-i}s^{(n+m)}_{n+m-(n+j)} = s^{(n+m)}_{n+m-i}s^{(n+m)}_{m-j} \overset{(A1)}{=} \text{ (since } n+m-i \geq m+1)$$

$$s^{(n+m)}_{m-j}s^{(n+m)}_{n+m-i} \overset{(2),(3)}{=} s^{(m)}_{m-j}\otimes s^{(n)}_{n-i} = \widetilde{s_j^{(m)}}\otimes\widetilde{s_i^{(n)}}\ .$$

$\square$

2.3. **Horizontal symmetry.** There is a map $*: B_n \to B_n$ defined by the conditions that ${s_i^{(n)}}^* = s_i^{(n)}$ and $(st)^* = t^*s^*$. The picture for s^* is obtained by rotating that of s in 3-space 180 degrees around a horizontal line.

It is clear that the three operators ${}^*, \widetilde{\ }, {}^{-1} : B_n \to B_n$ commute pairwise, and also that

$$s^{**} = s \qquad \forall\ s \in B_n,$$

$$(s\otimes t)^* = s^*\otimes t^* \qquad \forall\ s \in B_n,\ t \in B_m,$$

(7) $$\widetilde{s^{(n)}(i,j)}^* = s^{(n)}(n+1-j, n+1-i) \qquad \forall\ i, j, n,$$

(8) $${c^{(n)}}^* = c^{(n)} \qquad \forall\ n.$$

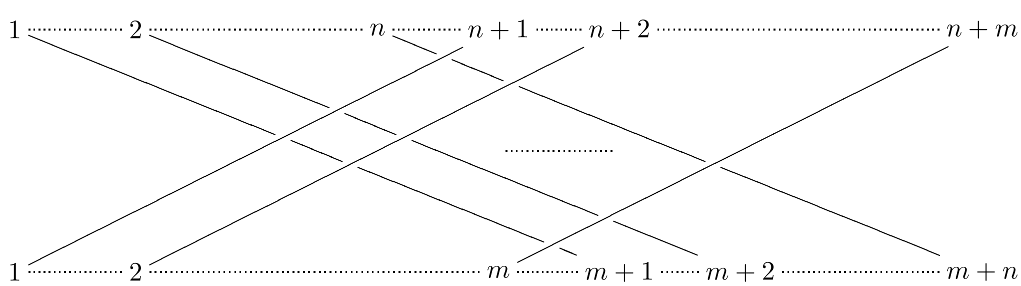
From (8) it follows easily that

$$c^{(n)} = s^{(n)}(n,n)s^{(n)}(n-1,n)\dots s^{(n)}(2,n)s^{(n)}(1,n)$$

and from here that

$${c^{(n)}}^2 = s^{(n)}(1,n)^n.$$

2.4. **Properties of the braiding.** The *braiding* $\beta_{m,n}$ is most easily defined in terms of its picture:

$\beta_{m,n} =$ 

It is viewed as a natural map $\beta_{m,n} : m\otimes n \to n\otimes m$ in the category $\mathfrak{B}$ of braids, and as such it satisfies some important properties. We will list some of them below without proof, since we won't use them, although they are very easily obtained through the use of pictures, see [K] XIII.2. However, it will be convenient for us to have an explicit description of $\beta_{m,n}$ in terms of the canonical generators. For this, we first define some special "powers" for braids as follows.

Let $m \geq 1$. For $s \in B_m$ and $n \geq 0$, define

(9) $$s^{\langle n\rangle} = \begin{cases} 1 & \text{if } n = 0, \\ s & \text{if } n = 1, \\ 1^{(n-1)}\otimes s \cdot 1^{(n-2)}\otimes s\otimes 1 \cdot \ldots \cdot 1\otimes s\otimes 1^{(n-2)} \cdot s\otimes 1^{(n-1)} & \text{if } n \geq 2. \end{cases}$$

Thus $s^{\langle n\rangle} \in B_{m+n-1}\ \forall\ m \geq 1,\ n \geq 0$ (and it is not defined if $m = 0$). Notice that $s^{\langle n+1\rangle} = 1 \otimes s^{\langle n\rangle} \cdot s \otimes 1$, from here it follows easily by induction that

$$
\begin{aligned}
& s^{\langle p+q\rangle} = 1^{(q)} \otimes s^{\langle p\rangle} \cdot s^{\langle q\rangle} \otimes 1^{(p)} && \forall\ p, q \geq 0, \\
(10)\qquad & 1^{(k)} \otimes s^{\langle n\rangle} \otimes 1^{(h)} = \left[1^{(k)} \otimes s \otimes 1^{(h)}\right]^{\langle n\rangle} && \forall\ n, k, h \geq 0\ .
\end{aligned}
$$

We then define

$$
\beta_{m,n} = s^{(m+1)}(1, m+1)^{\langle n\rangle}\ \in\ B_{m+n}\ . \tag{11}
$$

It is easy to see that this corresponds to the picture above. These are some of the properties that β satisfies:

$$
\begin{aligned}
& \beta_{m,n} \cdot s \otimes t = t \otimes s \cdot \beta_{m,n} && \forall\ s \in B_m,\ t \in B_n,\ (\textit{naturality} \text{ of the braiding}), \\
& c^{(n+m)} = c^{(n)} \otimes c^{(m)} \cdot \beta_{m,n} && \forall\ m, n \geq 0, \\
& \widetilde{\beta_{m,n}} = \beta_{n,m} = \beta^*_{m,n} && \forall\ m, n \geq 0, \\
& \beta_{p,q+r} = 1^{(q)} \otimes \beta_{p,r} \cdot \beta_{p,q} \otimes 1^{(r)} && \forall\ p, q, r \geq 0, \\
& \beta_{p+q,r} = \beta_{p+q,r} \otimes 1^{(q)} \cdot 1^{(p)} \otimes \beta_{q,r} && \forall\ p, q, r \geq 0.
\end{aligned}
$$

2.5. **Representations.** Throughout the paper k will denote a fixed field (although any commutative ring would do just as well).

The identities we will obtain between elements of the braid group algebras kB_n can be converted into matrix or numerical identities by choosing k-linear representations of the braid groups B_n.

More precisely we will be interested in *monoidal representations* of the braid category $\mathfrak{B}$, that is a vector space X, such that the braid group B_n acts on the tensor power $X^{\otimes n}$, with the property that

$$
s \otimes t \cdot x \otimes y = (s \cdot x) \otimes (t \cdot y)\ \forall\ s \in B_n,\ t \in B_m,\ x \in X^{\otimes n},\ y \in X^{\otimes m}\ .
$$

Since $s_i^{(n)} = 1^{(i-1)} \otimes s_1^{(2)} \otimes 1^{(n-i+1)}$, this condition implies that the action of B_n on $X^{\otimes n}$ is uniquely determined by the action of $s_1^{(2)}$ on $X \otimes X$. Moreover, a linear operator $R : X \otimes X \to X \otimes X$ defines a monoidal representation of $\mathfrak{B}$ if and only if it is invertible and satisfies the *Yang-Baxter equation*:

$$
(R \otimes \mathsf{id}_X) \circ (\mathsf{id}_X \otimes R) \circ (R \otimes \mathsf{id}_X) = (\mathsf{id}_X \otimes R) \circ (R \otimes \mathsf{id}_X) \circ (\mathsf{id}_X \otimes R)\ .
$$

This is a consequence of (A2).

If X is one-dimensional, then any invertible operator $R : X \to X$ satisfies this equation. R is necessarily given by multiplication by some non-zero scalar $q \in k$. Hence, in this case, $s_i^{(n)}$ acts by multiplication by q for every $n \geq 2,\ 1 \leq i \leq n-1$. It is this simplest choice that will produce the classical q-identities from the identities for braids. In particular the trivial one-dimensional representation yields the case $q = 1$. Higher dimensional representations will be discussed in sections 9 and 10.

The chart in section 1 describes the action of the braids introduced in this paper when X is the one-dimensional representation defined by $q \in k^*$ as above.

Let us also remark that since the non-commutativity of the braid groups necessarily disappears when acting on a one-dimensional representation, the actions of s, $\widetilde{s}$ and s^* coincide for any braid s in this case.

3. Binomial braids

For each pair (n, i) with $i \leq n$ let $\mathcal{S}_i(n)$ denote the set of subsets of $\{1, 2, \ldots, n\}$ with cardinality i. Recall that the q-binomial coefficients can be defined as

$$
\begin{bmatrix} n \\ i \end{bmatrix} = \sum_{I \in \mathcal{S}_i(n)} q^{\|I\|} \text{ where } \|I\| = \sum_{j \in I} j - \sum_{j=1}^{i} j\ .
$$

The braid analog of this definition is as follows.

First, for each $I \in \mathcal{S}_i(n)$, write $I = \{j_1, j_2, \ldots, j_i\}$ with $j_1 < j_2 < \ldots < j_i$, then define $s_I^{(n)} \in B_n$ as

$$s_I^{(n)} = s^{(n)}(i, j_i) \cdots s^{(n)}(2, j_2)s^{(n)}(1, j_1) \ ;$$

if $i = 0$ we let $s_{\emptyset}^{(n)} = 1$.

For instance if $I = \{m+1, m+2, \ldots, m+n\} \in \mathcal{S}_n(m+n)$ then $s_I^{(m+n)} = \beta_{m,n}$.

Then the binomial braid $b_i^{(n)} \in kB_n$ is defined as

$$b_i^{(n)} = \sum_{I \in \mathcal{S}_i(n)} s_I^{(n)} \ .$$

Thus $b_0^{(n)} = b_n^{(n)} = 1 \ \forall \ n$, while for instance

$$b_1^{(2)} = 1 + s_1^{(2)}, \ b_1^{(3)} = 1 + s_1^{(3)} + s_1^{(3)}s_2^{(3)}, \ b_2^{(3)} = 1 + s_2^{(3)} + s_2^{(3)}s_1^{(3)} \ .$$

We see that $b_i^{(n)} \neq b_{n-i}^{(n)}$ in general. However:

Proposition. For all $n \geq i \geq 0$,

$$\widetilde{b_i^{(n)}} = b_{n-i}^{(n)}. \tag{12}$$

Proof. Consider the bijection $\mathcal{S}_i(n) \to \mathcal{S}_{n-i}(n)$ that sends I to $\tilde{I}^c$, where $\tilde{I} = \{n+1-i \ / \ i \in I\}$. It is enough to show that, for every $I \in \mathcal{S}_i(n)$,

$$\widetilde{s_I^{(n)}} = s_{\tilde{I}^c}^{(n)} \ . \tag{*}$$

First, we show that if (*) holds when $n \in I$, then it holds for every I. In fact, given $I \in \mathcal{S}_i(n)$, let $m = \max I$, and let I' be the same set I but viewed as en element of $\mathcal{S}_i(m)$. Then we have that

$$s_I^{(n)} \overset{(2)}{=} s_{I'}^{(m)} \otimes 1^{(n-m)},$$

hence, by (6), and assuming (*) for I',

$$\widetilde{s_I^{(n)}} = 1^{(n-m)} \otimes \widetilde{s_{I'}^{(m)}} \overset{(*)}{=} 1^{(n-m)} \otimes s_{\tilde{I'}^c}^{(m)} = 1^{(n-m)} \otimes s_{m+1-I'^c}^{(m)} \overset{(3)}{=} s_{n+1-I^c}^{(n)} = s_{\tilde{I}^c}^{(n)},$$

so (*) holds for I as well.

To finish the proof we show (*) by induction on i. For $i = 0$ it is clear. Assume $i \geq 1$. As just explained, we can also assume that $n \in I$. Therefore, we can decompose $I = I_1 \cup \{n\}$ with $I_1 \in \mathcal{S}_{i-1}(n-1)$; then we have $\tilde{I} = \widetilde{I_1} \cup \{1\}$ and $\widetilde{I_1}^c = \tilde{I}^c \cup \{1\}$.

Write $\tilde{I}^c = \{h_1 < h_2 < \ldots < h_{n-i}\}$, so that $\widetilde{I_1}^c = \{1 < h_1 < h_2 < \ldots < h_{n-i}\}$. We have

$$s_I^{(n)} = s^{(n)}(i, n)s_{I_1}^{(n)} = s_i^{(n)}s_{i+1}^{(n)} \ldots s_{n-1}^{(n)}s_{I_1}^{(n)},$$

hence, by induction hypothesis,

$$\begin{aligned}\widetilde{s_I^{(n)}} &= s_{n-i}^{(n)}s_{n-i-1}^{(n)} \ldots s_1^{(n)}s_{\widetilde{I_1}^c}^{(n)} \\ &= s_{n-i}^{(n)}s_{n-i-1}^{(n)} \ldots s_1^{(n)}s^{(n)}(n-i+1, h_{n-i}) \ldots s^{(n)}(3, h_2)s^{(n)}(2, h_1)s^{(n)}(1, 1) \ .\end{aligned}$$

Now using (A1), $s^{(n)}(n-i+1, h_{n-i})$, can be moved to the left past all the factors $s_1^{(n)}, \dots, s_{n-i-1}^{(n)}$. Then, it combines with $s_{n-i}^{(n)}$ to form $s^{(n)}(n-i, h_{n-i})$. Similarly the other factors of the form $s^{(n)}(k+1, h_k)$ can be moved to the left until they reach $s_k^{(n)}$ to form $s^{(n)}(k, h_k)$. At the end of the process we have

$$\widetilde{s_I^{(n)}} = s^{(n)}(n-i, h_{n-i}) \dots s^{(n)}(2, h_2) s^{(n)}(1, h_1) = s_{\tilde{I}^c}^{(n)} \ .$$

This finishes the induction and the proof. □

4. Identities of Pascal and Vandermonde

For the q-binomial coefficients Pascal's identity says that

$$\begin{bmatrix} n \\ i \end{bmatrix} = q^{n-i} \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ i \end{bmatrix} = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} + q^i \begin{bmatrix} n-1 \\ i \end{bmatrix} .$$

Its generalization to braids is as follows.

Proposition. For any $i = 1, \dots, n-1$,

$$b_i^{(n)} = s^{(n)}(i,n) \cdot b_{i-1}^{(n-1)} \otimes 1 + b_i^{(n-1)} \otimes 1 = 1 \otimes b_{i-1}^{(n-1)} + s^{(n)} \widetilde{(n-i, n)} \cdot 1 \otimes b_i^{(n-1)} \ . \tag{13}$$

Proof. Consider the bijection $\mathcal{S}_{i-1}(n-1) \cup \mathcal{S}_i(n-1) \to \mathcal{S}_i(n)$ that sends $I \in \mathcal{S}_{i-1}(n-1)$ to $I \cup \{n\} \in \mathcal{S}_i(n)$ and $J \in \mathcal{S}_i(n-1)$ to $J \in \mathcal{S}_i(n)$. From (2) and the definition of s_I we see that

$$s_J^{(n)} = s_J^{(n-1)} \otimes 1 \text{ and } s_{I \cup \{n\}}^{(n)} = s^{(n)}(i,n) \cdot s_I^{(n-1)} \otimes 1 \ ;$$

summing over all such I and J we obtain the first equality. The other one follows by applying $\widetilde{\ }$, using (6) and replacing $n-i$ by i.

□

Vandermonde's identity says that

$$\begin{bmatrix} m+n \\ p \end{bmatrix} = \sum_{k=0}^{p} q^{(m-k)(p-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ p-k \end{bmatrix} .$$

Its generalization to braids reads:

Proposition. For any m, n, p with $0 \le p \le m, n$,

$$b_p^{(m+n)} = \sum_{k=0}^{p} 1^{(k)} \otimes \beta_{m-k, p-k} \otimes 1^{(n-p+k)} \ \cdot \ b_k^{(m)} \otimes b_{p-k}^{(n)} \ . \tag{14}$$

Proof. Consider the bijection

$$\bigcup_{k=0}^{p} \mathcal{S}_k(m) \times \mathcal{S}_{p-k}(n) \to \mathcal{S}_p(m+n) \ , \ (I, J) \mapsto I \cup (m+J) \ .$$

It suffices to show that, for each $I \in \mathcal{S}_k(m)$ and $J \in \mathcal{S}_{p-k}(n)$,

$$s_{I \cup (m+J)}^{(m+n)} = 1^{(k)} \otimes \beta_{m-k, p-k} \otimes 1^{(n-p+k)} \ \cdot \ s_I^{(m)} \otimes s_J^{(n)} \ . \tag{*}$$

Let $h = p-k$. If $h = 0$ then (*) reduces to $s_I^{(m+n)} = s_I^{(m)} \otimes 1^{(n)}$, which holds by (2).

Assume $h \geq 1$. Write $I = \{i_1 < \ldots < i_k\}$ and $J = \{j_1 < \ldots < j_h\}$ so that $I \cup (m+J) = \{i_1 < \ldots < i_k < m+j_1 < \ldots < m+j_h\}$. Then

$$\begin{aligned}
s^{(m+n)}_{I\cup(m+J)} &= s^{(m+n)}(k+h,m+j_h)s^{(m+n)}(k+h-1,m+j_{h-1})\ldots s^{(m+n)}(k+1,m+j_1)\,\cdot\\
&\qquad \cdot\, s^{(m+n)}(k,i_k)s^{(m+n)}(k-1,i_{k-1})\ldots s^{(m+n)}(1,i_1)\\
&\overset{(1),(2)}{=}\Big[s^{(m+n)}(k+h,m+h)s^{(m+n)}(m+h,m+j_h)\Big]\,\cdot\\
&\qquad \cdot\,\Big[s^{(m+n)}(k+h-1,m+h-1)s^{(m+n)}(m+h-1,m+j_{h-1})\Big]\cdots\\
&\qquad \cdots\Big[s^{(m+n)}(k+1,m+1)s^{(m+n)}(m+1,m+j_1)\Big]\,\cdot\\
&\qquad \cdot\,\Big[s^{(m)}(k,i_k)\otimes 1^{(n)}\Big]\Big[s^{(m)}(k-1,i_{k-1})\otimes 1^{(n)}\Big]\cdots\Big[s^{(m)}(1,i_1)\otimes 1^{(n)}\Big]
\end{aligned}$$

Now notice that each of the factors

$$s^{(m+n)}(k+h-1,m+h-1),\ s^{(m+n)}(k+h-2,m+h-2),\ldots,s^{(m+n)}(k+1,m+1)$$

can be moved to the left past all the factors

$$s^{(m+n)}(m+h,m+j_h),\ s^{(m+n)}(m+h-1,m+j_{h-1}),\ldots,s^{(m+n)}(m+2,m+j_2),$$

simply because of (A1): $s^{(m+n)}(k+h-1,m+h-1)$ only involves strands $m+h-1$ and lower, while $s^{(m+n)}(m+h,m+j_h)$ only involves strands $m+h$ and higher; similarly for the others. After performing this commutation we get that

$$\begin{aligned}
s^{(m+n)}_{I\cup(m+J)} &= s^{(m+n)}(k+h,m+h)s^{(m+n)}(k+h-1,m+h-1)\cdot\ldots\cdot s^{(m+n)}(k+1,m+1)\,\cdot\\
&\quad \cdot\, s^{(m+n)}(m+h,m+j_h)s^{(m+n)}(m+h-1,m+j_{h-1})\cdot\ldots\cdot s^{(m+n)}(m+1,m+j_1)\,\cdot\\
&\quad \cdot\, s^{(m)}_I\otimes 1^{(n)}\\
&\overset{(2),(3)}{=}\Big[1^{(h-1)}\otimes s^{(m+n-h+1)}(k+1,m+1)\Big]\Big[1^{(h-2)}\otimes s^{(m+n-h+1)}(k+1,m+1)\otimes 1\Big]\cdots\\
&\quad \cdots\Big[s^{(m+n-h+1)}(k+1,m+1)\otimes 1^{(h-1)}\Big]\cdot\Big[1^{(m)}\otimes s^{(n)}(h,j_h)\Big]\Big[1^{(m)}\otimes s^{(n)}(h-1,j_{h-1})\Big]\cdots\Big[1^{(m)}\otimes s^{(n)}(1,j_1)\Big]\,\cdot\\
&\quad \cdot\, s^{(m)}_I\otimes 1^{(n)}\\
&\overset{(9)}{=} s^{(m+n-h+1)}(k+1,m+1)^{\langle h\rangle}\cdot 1^{(m)}\otimes s^{(n)}_J\cdot s^{(m)}_I\otimes 1^{(n)}\\
&\overset{(2),(3)}{=}\Big[1^{(k)}\otimes s^{(m-k+1)}(1,m-k+1)\otimes 1^{(n-h)}\Big]^{\langle h\rangle}\cdot s^{(m)}_I\otimes s^{(n)}_J\\
&\overset{(10)}{=} 1^{(k)}\otimes s^{(m-k+1)}(1,m-k+1)^{\langle h\rangle}\otimes 1^{(n-h)}\cdot s^{(m)}_I\otimes s^{(n)}_J\\
&\overset{(11)}{=} 1^{(k)}\otimes\beta_{m-k,h}\otimes 1^{(n-h)}\cdot s^{(m)}_I\otimes s^{(n)}_J\ .
\end{aligned}$$

Thus (*) holds and the proof is complete.

□

5. Natural and factorial braids

5.1. **Definition.** The q-analog of a natural number n is

$$[n] = 1+q+q^2+\ldots+q^{n-1}\ .$$

For $n \geq 1$, the natural braid $[n] \in kB_n$ is defined as

$$[n] = \sum_{i=1}^{n} s^{(n)}(1,i) = 1 + s_1^{(n)} + s_1^{(n)} s_2^{(n)} + \ldots + s_1^{(n)} s_2^{(n)} \ldots s_{n-1}^{(n)} \ ;$$

we also set $[0] = 0 \in kB_0$.

Notice that $[n] = b_1^{(n)}$. Hence, as a particular case of Vandermonde's formula (14) we have:

$$[m+n] = [m] \otimes 1^{(n)} + s^{(m+n)}(1, m+1) \cdot 1^{(m)} \otimes [n] \ ;$$

since $\beta_{m,1} = s^{(m+1)}(1, m+1)$.

While $[1] = \widetilde{[1]} = [1]^*$ and $[2] = \widetilde{[2]} = [2]^*$, we have

$$[3] = 1 + s_1^{(3)} + s_1^{(3)} s_2^{(3)}, \ \widetilde{[3]} = 1 + s_2^{(3)} + s_2^{(3)} s_1^{(3)} \text{ and } [3]^* = 1 + s_1^{(3)} + s_2^{(3)} s_1^{(3)} \ ;$$

thus ${b_i^{(n)}}^*$ is not another binomial braid in general. However, it will turn out (17) that the factorial braids are symmetric with respect to both $\widetilde{\ }$ and *.

The q-analog of the factorial number $n!$ is

$$[n]! = \sum_{\sigma \in S_n} q^{\mathsf{inv}(\sigma)} \ ;$$

where the *inversion index* of a permutation $\sigma \in S_n$ is defined as

$$\mathsf{inv}(\sigma) = \#\{(i,j) \ / \ i < j \text{ but } \sigma(i) > \sigma(j)\}.$$

The braid analog of this definition is as follows. First, for any $\sigma \in S_n$ and $i = 1, \ldots, n$ let

$$r_i(\sigma) = \#\{j > i \ / \ \sigma(j) < \sigma(i)\}.$$

Thus,

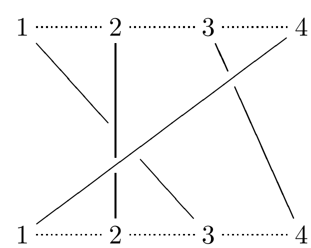
$$\mathsf{inv}(\sigma) = \sum_{i=1}^{n} r_i(\sigma) \ .$$

Notice that $\sigma(i) - i \leq r_i(\sigma) \leq \sigma(i) - 1 \ \forall \ i$, hence it makes sense to define a braid $s_\sigma^{(n)} \in B_n$ as

$$s_\sigma^{(n)} = s^{(n)}(\sigma(n) - r_n(\sigma), n) \cdot \ldots \cdot s^{(n)}(\sigma(2) - r_2(\sigma), 2) \cdot s^{(n)}(\sigma(1) - r_1(\sigma), 1) \ .$$

For instance if $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{smallmatrix}\right)$ then

$s_\sigma^{(4)} =$



The picture of $s_\sigma^{(n)}$ is obtained by drawing a straight line from 1 in the bottom to $\sigma(1)$ in the top, then *under* that a straight line from 2 to $\sigma(2)$, etc.

In section 5.3, other expressions for $s_\sigma^{(n)}$ will be given.

Now, for every $n \geq 1$ we define the factorial braid $f^{(n)} \in kB_n$ as

$$f^{(n)} = \sum_{\sigma \in S_n} s_\sigma^{(n)} \ ;$$

we also set $f^{(0)} = 1 \in kB_0$.

We next show that the factorial and natural braids are related by means of a product formula, generalizing $[n]! = [n][n-1]\cdots[2][1]$ for q-numbers. Variations of this will follow after we study the effect of $\tilde{}$ and * on the $s_\sigma^{(n)}$'s.

Proposition. For every $n \geq 1$,

$$f^{(n)} = 1^{(n-1)}\otimes[1] \cdot 1^{(n-2)}\otimes[2] \cdot \ldots \cdot 1\otimes[n-1] \cdot [n] \ . \tag{15}$$

Proof. We need to show that $f^{(n)} = 1\otimes f^{(n-1)} \cdot [n] \ \forall \ n \geq 1$.

Consider the bijection $S_{n-1}\times\{1,2,\ldots,n\} \to S_n$, $(\sigma,i) \mapsto (1\otimes\sigma)(1,2,\ldots,i)$. (From $\tau := (1\otimes\sigma)(1,2,\ldots,i)$ we recover i as $\tau^{-1}(1)$ and then $1\otimes\sigma$ as $\tau\cdot(1,2,\ldots,i)^{-1}$; here $1\otimes\sigma$ is such that $(1\otimes\sigma)(j) = \sigma(j-1)+1$.) It suffices to show that

$$s_\tau^{(n)} = 1\otimes s_\sigma^{(n-1)} \cdot s^{(n)}(1,i) \ .$$

Since $\tau = \left(\begin{smallmatrix} 1 & \ldots & i-1 & i & i+1 & \ldots & n \\ \sigma(1)+1 & \ldots & \sigma(i-1)+1 & 1 & \sigma(i)+1 & \ldots & \sigma(n-1)+1 \end{smallmatrix}\right)$, we have that $r_j(\tau) = \begin{cases} r_{j-1}(\sigma) & \text{if } j = i+1,\ldots,n, \\ 0 & \text{if } j = i, \\ r_j(\sigma)+1 & \text{if } j = 1,\ldots,i-1. \end{cases}$

Hence

$$\begin{aligned} s_\tau^{(n)} &= s^{(n)}(\tau(n)-r_n(\tau),n)\cdot\ldots\cdot s^{(n)}(\tau(i+1)-r_{i+1}(\tau),i+1)\cdot s^{(n)}(\tau(i)-r_i(\tau),i)\cdot \\ &\quad \cdot s^{(n)}(\tau(i-1)-r_{i-1}(\tau),i-1)\cdot\ldots\cdot s^{(n)}(\tau(1)-r_1(\tau),1) \\ &= s^{(n)}(\sigma(n-1)+1-r_{n-1}(\sigma),n)\cdot\ldots\cdot s^{(n)}(\sigma(i)+1-r_i(\sigma),i+1)\cdot s^{(n)}(1,i)\cdot \\ &\quad \cdot s^{(n)}(\sigma(i-1)+1-r_{i-1}(\sigma)-1,i-1)\cdot\ldots\cdot s^{(n)}(\sigma(1)+1-r_1(\sigma)-1,1) \\ &\overset{(5)}{=} s^{(n)}(\sigma(n-1)-r_{n-1}(\sigma)+1,n-1+1)\cdot\ldots\cdot s^{(n)}(\sigma(i)-r_i(\sigma)+1,i+1)\cdot \\ &\quad \cdot s^{(n)}(\sigma(i-1)-r_{i-1}(\sigma)+1,i-1+1)\cdot\ldots\cdot s^{(n)}(\sigma(1)-r_1(\sigma)+1,1+1)\cdot s^{(n)}(1,i) \\ &\overset{(3)}{=} 1\otimes s_\sigma^{(n-1)}\cdot s^{(n)}(1,i) \end{aligned}$$

and the proof is complete. □

5.2. **Symmetries of the factorial braids.** To obtain the announced symmetry of the $f^{(n)}$'s, we first describe a multiplicativity property of the map $\xi : S_n \to B_n$, $\sigma \mapsto s_\sigma^{(n)}$. From its definition it is clear that ξ is a section of the canonical projection $B_n \to S_n$, and that $\xi((i,i+1)) = s_i^{(n)}$.[1]

Lemma. Let $\sigma = \sigma_{i_1}\cdot\ldots\cdot\sigma_{i_l} \in S_n$ be a reduced expression for σ as a product of elementary transpositions $\sigma_{i_j} = (i_j, i_j+1)$. Then $s_\sigma^{(n)} = s_{i_1}^{(n)}\cdot\ldots\cdot s_{i_l}^{(n)}$.

Proof. We are given that $\mathsf{length}(\sigma) = l$, where the length of a permutation is the minimum number of elementary transpositions required to write it as a product of such. We will make use of the well-known fact that $\mathsf{inv} = \mathsf{length}$.

Clearly, it suffices to show that if $\sigma = \tau\cdot(i,i+1)$ and $\mathsf{length}(\sigma) = \mathsf{length}(\tau)+1$ then $s_\sigma^{(n)} = s_\tau^{(n)}\cdot s_i^{(n)}$. In this case, $\sigma = \left(\begin{smallmatrix} 1 & \ldots & i-1 & i & i+1 & i+2 & \ldots & n \\ \tau(1) & \ldots & \tau(i-1) & \tau(i+1) & \tau(i) & \tau(i+2) & \ldots & \tau(n) \end{smallmatrix}\right)$. Hence $r_j(\sigma) = r_j(\tau) \ \forall \ j \neq i, i+1$. We claim that $\tau(i) < \tau(i+1)$. For if not, we would have $r_i(\sigma) = r_{i+1}(\tau)$ and $r_{i+1}(\sigma) = r_i(\tau)-1$, from where $\mathsf{length}(\sigma) = \mathsf{inv}(\sigma) = \sum_{j=1}^n r_j(\sigma) = \mathsf{length}(\tau)-1$, against our hypothesis. Thus $\tau(i) < \tau(i+1)$, and

[1] Lusztig [L,2.1.2] has considered sections of this sort for arbitrary Weyl groups W. From lemma (5.2) it follows that ξ coincides with Lusztig's section for $W = S_n$.

then $r_i(\sigma) = r_{i+1}(\tau) + 1$ and $r_{i+1}(\sigma) = r_i(\tau)$. Hence,

$$\begin{aligned} s^{(n)}_\sigma &= s^{(n)}(\sigma(n) - r_n(\sigma), n) \cdot \ldots \cdot s^{(n)}(\sigma(1) - r_1(\sigma), 1) \\ &= s^{(n)}(\tau(n) - r_n(\tau), n) \cdot \ldots \cdot s^{(n)}(\tau(i+2) - r_{i+2}(\tau), i+2) \cdot s^{(n)}(\tau(i) - r_i(\tau), i+1) \cdot \\ &\quad \cdot s^{(n)}(\tau(i+1) - r_{i+1}(\tau) - 1, i) \cdot s^{(n)}(\tau(i-1) - r_{i-1}(\tau), i-1) \cdot \ldots \cdot s^{(n)}(\tau(1) - r_1(\tau), 1) \\ &\overset{(5)}{=} s^{(n)}(\tau(n) - r_n(\tau), n) \cdot \ldots \cdot s^{(n)}(\tau(i+2) - r_{i+2}(\tau), i+2) \cdot s^{(n)}(\tau(i+1) - r_{i+1}(\tau), i+1) \cdot \\ &\quad \cdot s^{(n)}(\tau(i) - r_i(\tau), i) \cdot s^{(n)}_i \cdot s^{(n)}(\tau(i-1) - r_{i-1}(\tau), i-1) \cdot \ldots \cdot s^{(n)}(\tau(1) - r_1(\tau), 1) \\ &\overset{(A1)}{=} s^{(n)}(\tau(n) - r_n(\tau), n) \cdot \ldots \cdot s^{(n)}(\tau(1) - r_1(\tau), 1) \cdot s^{(n)}_i = s^{(n)}_\tau \cdot s^{(n)}_i \end{aligned}$$

and the proof is complete. □

Corollary.

$$\widetilde{s^{(n)}_\sigma} = s^{(n)}_{\widetilde{\sigma}}, \; {s^{(n)}_\sigma}^* = s^{(n)}_{\sigma^{-1}}, \quad \text{where } \widetilde{\sigma}(j) = n+1-\sigma(n+1-j) \tag{16}$$

$$f^{(n)} = \widetilde{f^{(n)}} = {f^{(n)}}^* \tag{17}$$

$$\begin{aligned} f^{(n)} &= 1^{(n-1)}\otimes[1] \cdot 1^{(n-2)}\otimes[2] \cdot \ldots \cdot 1\otimes[n-1] \cdot [n] \\ &= \widetilde{[1]}\otimes 1^{(n-1)} \cdot \widetilde{[2]}\otimes 1^{(n-2)} \cdot \ldots \cdot \widetilde{[n-1]}\otimes 1 \cdot \widetilde{[n]} \\ &= [n]^* \cdot 1\otimes[n-1]^* \cdot \ldots \cdot 1^{(n-2)}\otimes[2]^* \cdot 1^{(n-1)}\otimes[1]^* \\ &= \widetilde{[n]}^* \cdot \widetilde{[n-1]}^*\otimes 1 \cdot \ldots \cdot \widetilde{[2]}^*\otimes 1^{(n-2)} \cdot \widetilde{[1]}^*\otimes 1^{(n-1)} \; . \end{aligned} \tag{18}$$

Proof. To prove (16), it suffices by the lemma to check these equalities on the elementary transpositions, since both $^\sim$ and $^{-1}$ preserve the length of a permutation. But in this case they hold by definition of $^\sim$ and * for braids. Then (17) follows by summing over all $\sigma \in S_n$, and the product formulas (18) follow from (15) and (17). □

5.3. **Other expressions for $s^{(n)}_\sigma$.** For any $\sigma \in S_n$ and $i = 1, \ldots, n$ let

$$e_i(\sigma) = \#\{j \le i \; / \; \sigma(j) \le \sigma(i)\}.$$

There is a simpler expression for $s^{(n)}_\sigma$ in terms of the e_i's.

Proposition. For any $\sigma \in S_n$ and $i = 1, \ldots, n$, $\sigma(i) = r_i(\sigma) + e_i(\sigma)$. Hence

$$s^{(n)}_\sigma = s^{(n)}(e_n(\sigma), n) \cdot \ldots \cdot s^{(n)}(e_2(\sigma), 2) \cdot s^{(n)}(e_1(\sigma), 1) \; . \tag{19}$$

Proof.

$$\begin{aligned} r_i(\sigma) + e_i(\sigma) &= \#\{j > i \; / \; \sigma(j) < \sigma(i)\} + \#\{j \le i \; / \; \sigma(j) \le \sigma(i)\} \\ &= \#\{j > i \; / \; \sigma(j) \le \sigma(i)\} + \#\{j \le i \; / \; \sigma(j) \le \sigma(i)\} \\ &= \#\{j / \; \sigma(j) \le \sigma(i)\} = \#\{j \; / \; \sigma(j) \in \{1, 2, \ldots, \sigma(i)\} \; \} \\ &= \sigma(i) \; . \end{aligned}$$

□

For completeness, we provide another expression for $s^{(n)}_\sigma$, this time in terms of some partial inversion indices that are obtained by reading σ from right to left. For any $i = 1, \ldots, n$ let

$$l_i(\sigma) = \#\{j < i \; / \; \sigma(j) > \sigma(i)\}.$$

Proposition. For any $\sigma \in S_n$,

$$s^{(n)}_\sigma = s^{(n)}(n, \sigma^{-1}(n) + l_n(\sigma^{-1})) \cdot \ldots \cdot s^{(n)}(2, \sigma^{-1}(2) + l_2(\sigma^{-1})) \cdot s^{(n)}(1, \sigma^{-1}(1) + l_1(\sigma^{-1})).$$

Proof. Notice that $r_i(\sigma) = l_{n+1-i}(\widetilde{\sigma})\ \forall\ i = 1, \ldots, n$. Hence

$$s^{(n)}_\sigma = s^{(n)}(\sigma(n) - r_n(\sigma), n) \cdot \ldots \cdot s^{(n)}(\sigma(1) - r_1(\sigma), 1) = s^{(n)}(\sigma(n) - l_1(\widetilde{\sigma}), n) \cdot \ldots \cdot s^{(n)}(\sigma(1) - l_n(\widetilde{\sigma}), 1)$$

$$\begin{aligned}
\Rightarrow \widetilde{s^{(n)}_\sigma}^{\,*} &= s^{(n)}(\widetilde{\sigma(1) - l_n}(\widetilde{\sigma}), 1)^* \cdot \ldots \cdot s^{(n)}(\widetilde{\sigma(n) - l_1}(\widetilde{\sigma}), n)^* \\
&\overset{(7)}{=} s^{(n)}(n, n+1-\sigma(1) + l_n(\widetilde{\sigma})) \cdot \ldots \cdot s^{(n)}(1, n+1-\sigma(n) + l_1(\widetilde{\sigma})) \\
&= s^{(n)}(n, \widetilde{\sigma}(n) + l_n(\widetilde{\sigma})) \cdot \ldots \cdot s^{(n)}(1, \widetilde{\sigma}(1) + l_1(\widetilde{\sigma})) \\
\overset{(16)}{\Rightarrow} s^{(n)}_{\widetilde{\sigma}^{-1}} &= s^{(n)}(n, \widetilde{\sigma}(n) + l_n(\widetilde{\sigma})) \cdot \ldots \cdot s^{(n)}(1, \widetilde{\sigma}(1) + l_1(\widetilde{\sigma}))\ .
\end{aligned}$$

Replacing σ by $\widetilde{\sigma}^{-1}$ yields the result. $\square$

5.4. **Factorial formulas for the binomial coefficients.** Next, we present the analog of the well-known formula $\begin{bmatrix} n-i \\ j-i \end{bmatrix}\begin{bmatrix} n \\ i \end{bmatrix} = \begin{bmatrix} j \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix}$ for q-binomials, from which the factorial formula will be deduced. We choose to provide a bijective proof, even though a proof based on Pascal's identity is possible and shorter, in particular because it yields the stronger result (*) below.

Proposition. Whenever $0 \le i \le j \le n$,

$$1^{(i)} \otimes b^{(n-i)}_{j-i} \cdot b^{(n)}_i = b^{(j)}_i \otimes 1^{(n-j)} \cdot b^{(n)}_j \tag{20}$$

Proof. Consider the map $\mathcal{S}_j(n) \times \mathcal{S}_i(j) \to \mathcal{S}_{j-i}(n-i) \times \mathcal{S}_i(n)$, $(A, B) \mapsto (X, Y)$, defined as follows. First consider the unique order-preserving bijection $k : \{1, \ldots, j\} \to A$ and let $Y = k(B) \in \mathcal{S}_i(n)$, then consider the unique order-preserving bijection $f : \{1, \ldots, n\} \setminus Y \to \{1, \ldots, n-i\}$ and let $X := f(A \setminus Y) \in \mathcal{S}_{j-i}(n-i)$.

Given $(X, Y) \in \mathcal{S}_{j-i}(n-i) \times \mathcal{S}_i(n)$ one recovers $A = Y \cup f^{-1}(X)$ and $B = k^{-1}(Y)$; thus, $(A, B) \to (X, Y)$ is a bijection, so to obtain the result it suffices to prove that

$$1^{(i)} \otimes s^{(n-i)}_X \cdot s^{(n)}_Y = s^{(j)}_B \otimes 1^{(n-j)} \cdot s^{(n)}_A\ . \tag{*}$$

We start by examining the right hand side. Write $A = \{k_1 < \ldots < k_j\} \subseteq \{1, \ldots, n\}$ and $B = \{h_1 < \ldots < h_j\} \subseteq \{1, \ldots, j\}$. Notice that then $Y := \{k_{h_1}, \ldots, k_{h_i}\} \subseteq \{1, \ldots, n\}$.

For each $r = 0, \ldots, i$ let $s^{(n)}_{A_r} := \prod_{h_r < z < h_{r+1}} s^{(n)}(z, k_z)$. (This and all products below are taken in the *decreasing* order: the index z decreases from left to right. If the interval (h_r, h_{r+1}) is empty then we take $s^{(n)}_{A_r} = 1$; also, we set $h_0 = 0$ and $h_{i+1} = j + 1$.) Then, by definition,

$$\begin{aligned}
s^{(n)}_A &= \prod_{0<z<j+1} s^{(n)}(z, k_z) = \\
&= s^{(n)}_{A_i} \cdot s^{(n)}(h_i, k_{h_i}) \cdot \ldots \cdot s^{(n)}_{A_2} \cdot s^{(n)}(h_2, k_{h_2}) \cdot s^{(n)}_{A_1} \cdot s^{(n)}(h_1, k_{h_1}) \cdot s^{(n)}_{A_0}\ .
\end{aligned}$$

Hence

$$\begin{aligned}
s^{(j)}_B \otimes 1^{(n-j)} \cdot s^{(n)}_A = s^{(n)}_B s^{(n)}_A &= s^{(n)}(i, h_i) \cdot \ldots \cdot s^{(n)}(2, h_2) \cdot s^{(n)}(1, h_1) \cdot \\
&\cdot s^{(n)}_{A_i} \cdot s^{(n)}(h_i, k_{h_i}) \cdot \ldots \cdot s^{(n)}_{A_2} \cdot s^{(n)}(h_2, k_{h_2}) \cdot s^{(n)}_{A_1} \cdot s^{(n)}(h_1, k_{h_1}) \cdot s^{(n)}_{A_0}\ .
\end{aligned}$$

In this expression, $s^{(n)}(1, h_1)$ commutes with all the factors to its right until $s^{(n)}_{A_1}$, including it, since these only involve strands $h_1 + 1$ and higher. When placed there, it joins $s^{(n)}(h_1, k_{h_1})$ to form $s^{(n)}(1, k_{h_1})$, by (1). Similarly $s^{(n)}(2,h_2)$ commutes past $s^{(n)}_{A_2}$ where it joins $s^{(n)}(h_2,k_{h_2})$ to become $s^{(n)}(2,k_{h_2})$, and finally $s^{(n)}(i,h_i)$ and $s^{(n)}(h_i,k_{h_i})$ become $s^{(n)}(i,k_{h_i})$. After this transformation we get

$$s^{(j)}_B \otimes 1^{(n-j)} \cdot s^{(n)}_A = s^{(n)}_{A_i} \cdot s^{(n)}(i,k_{h_i}) \cdot \ldots \cdot s^{(n)}_{A_2} \cdot s^{(n)}(2,k_{h_2}) \cdot s^{(n)}_{A_1} \cdot s^{(n)}(1,k_{h_1}) \cdot s^{(n)}_{A_0} \ .$$

Now notice that each factor in $s^{(n)}_{A_0}$ is of the form $s^{(n)}(z,k_z)$ with $1 \le z < h_1$, hence by (5) and (3)

$$s^{(n)}(1,k_{h_1}) \cdot s^{(n)}_{A_0} = 1 \otimes s^{(n-1)}_{A_0} \cdot s^{(n)}(1,k_{h_1}) \ .$$

Similarly we can now commute $s^{(n)}_{A_1} \cdot 1 \otimes s^{(n-1)}_{A_0}$ past $s^{(n)}(2,k_{h_2})$, using (5) and (3); this factor becomes $1 \otimes s^{(n)}_{A_1} \cdot 1^{(2)} \otimes s^{(n-1)}_{A_0}$ when placed to the left of $s^{(n)}(2,k_{h_2})$. After doing this for each $r = 0,\ldots,i-1$ we get

$$\begin{aligned} s^{(j)}_B \otimes 1^{(n-j)} \cdot s^{(n)}_A &= s^{(n)}_{A_i} \cdot 1 \otimes s^{(n-1)}_{A_{i-1}} \cdot \ldots \cdot 1^{(i-2)} \otimes s^{(n)}_{A_2} \cdot 1^{(i-1)} \otimes s^{(n)}_{A_1} \cdot 1^{(i)} \otimes s^{(n)}_{A_0} \cdot \\ &\quad \cdot s^{(n)}(i,k_{h_i}) \cdot \ldots \cdot s^{(n)}(2,k_{h_2}) \cdot s^{(n)}(1,k_{h_1}) \\ &= \prod_{r=0}^{i} 1^{(i-r)} \otimes s^{(n-i+r)}_{A_r} \cdot s^{(n)}_Y \ . \end{aligned}$$

Thus, to obtain (*), we need to show that

$$(**) \qquad 1^{(i)} \otimes s^{(n-i)}_X = \prod_{r=0}^{i} 1^{(i-r)} \otimes s^{(n-i+r)}_{A_r}$$

To this end, we describe f and X explicitly. By definition, $f : \{1,\ldots,n\} \setminus \{k_{h_1},\ldots k_{h_i}\} \to \{1,\ldots,n-i\}$ is translation by $-r$ on each open interval $(k_{h_r},k_{h_{r+1}})$, for $r = 0,\ldots,i$ (where we set $k_0 = 0$ and $k_{j+1} = n+1$). Then, since

$$A \setminus Y = k(\{1,\ldots,j\} \setminus \{h_1,\ldots,h_i\}) = \bigcup_{r=0}^{i} k((h_r,h_{r+1})) \ ,$$

we have that

$$X = f(A \setminus Y) = \bigcup_{r=0}^{i} k((h_r,h_{r+1})) - r \ .$$

Thus, letting $s^{(n-i)}_{X_r} := \prod_{h_r<z<h_{r+1}} s^{(n-i)}(z-r,k_z-r)$, we have that $s^{(n-i)}_X = \prod_{r=0}^{i} s^{(n-i)}_{X_r}$. But notice that

$$\begin{aligned} 1^{(i)} \otimes s^{(n-i)}_{X_r} &= \prod_{h_r<z<h_{r+1}} 1^{(i)} \otimes s^{(n-i)}(z-r,k_z-r) \overset{(3)}{=} \prod_{h_r<z<h_{r+1}} s^{(n)}(z+i-r,k_z+i-r) \\ &\overset{(3)}{=} 1^{(i-r)} \otimes \prod_{h_r<z<h_{r+1}} s^{(n-i+r)}(z,k_z) = 1^{(i-r)} \otimes s^{(n-i+r)}_{A_r} \ , \end{aligned}$$

hence

$$1^{(i)} \otimes s^{(n-i)}_X = \prod_{r=0}^{i} 1^{(i)} \otimes s^{(n-i)}_{X_r} = \prod_{r=0}^{i} 1^{(i-r)} \otimes s^{(n-i+r)}_{A_r}$$

so (**) holds and the proof is complete. □

We can now derive the braid analog of the usual expression for the binomial coefficients in terms of factorials.

Corollary. Whenever $0 \leq j \leq n$,

$$f^{(j)} \otimes f^{(n-j)} \cdot b_j^{(n)} = f^{(n)} \ . \tag{21}$$

Proof. Formula (20) with $i = 1$ says

$$1 \otimes b_{j-1}^{(n-1)} \cdot [n] = [j] \otimes 1^{(n-j)} \cdot b_j^{(n)} \ .$$

Repeated use of this yields

$$\begin{aligned} &1^{(j-1)} \otimes [n-j+1] \cdot 1^{(j-2)} \otimes [n-j+2] \cdot \ldots \cdot 1 \otimes [n-1] \cdot [n] \\ &\quad = 1^{(j-1)} \otimes [1] \otimes 1^{(n-j)} \cdot 1^{(j-2)} \otimes [2] \otimes 1^{(n-j)} \cdot \ldots \cdot 1 \otimes [j-1] \otimes 1^{(n-j)} \cdot [j] \otimes 1^{(n-j)} \cdot b_j^{(n)} \\ &\quad \overset{(15)}{=} f^{(j)} \otimes 1^{(n-j)} \cdot b_j^{(n)} \end{aligned}$$

Multiplying both sides by $1^{(j)} \otimes f^{(n-j)}$ and using (15) we get the result. $\square$

It seems that in the course of the proof of (21) we obtained a stronger "simplified" formula; in fact this is equivalent to (21) since the braid group algebras do not possess zero divisors[2].

Recall that the natural braids $[j]$ are not $\widetilde{\ }$-symmetric. However, an amusing consequence of (20) is this (choosing $n = j+1$, $i = 1$):

$$1 \otimes \widetilde{[j]} \cdot [j+1] = [j] \otimes 1 \cdot \widetilde{[j+1]} \ .$$

Thus this element is fixed by $\widetilde{\ }$.

6. Rota's binomial theorem, Cauchy's identities and Möbius inversion

6.1. **The binomial theorem.** The following remarkable q-binomial theorem is proven in [GR1]: if $P_k(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})(\mathbf{x} - q\mathbf{y}) \ldots (\mathbf{x} - q^{k-1}\mathbf{y})$ then

$$P_n(\mathbf{x}, \mathbf{z}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} P_k(\mathbf{x}, \mathbf{y}) P_{n-k}(\mathbf{y}, \mathbf{z}) \ ,$$

this is an identity in the ordinary polynomial ring $k[\mathbf{x}, \mathbf{y}, \mathbf{z}]$. When $q = 1$ this reduces to the familiar

$$(\mathbf{x} - \mathbf{z})^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\mathbf{x} - \mathbf{y})^k (\mathbf{y} - \mathbf{z})^{n-k} \ .$$

We will generalize this result to the context of braids, and derive from it the other results of the section.

We consider ordinary polynomial rings $kB_n[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ over the non-commutative ring kB_n; thus, the variables commute among themselves and with the coefficients. The embeddings

$$B_k \to B_n, \ s \mapsto s \otimes 1^{(n-k)} \quad \text{and} \quad B_{n-k} \to B_n, \ t \mapsto 1^{(k)} \otimes t$$

extend to embeddings

$$kB_k[\mathbf{x}_1, \ldots, \mathbf{x}_r] \to kB_n[\mathbf{x}_1, \ldots, \mathbf{x}_r], \ p \mapsto p \otimes 1^{(n-k)} \quad \text{and} \quad kB_{n-k}[\mathbf{x}_1, \ldots, \mathbf{x}_r] \to kB_n[\mathbf{x}_1, \ldots, \mathbf{x}_r], \ q \mapsto 1^{(k)} \otimes q$$

where $\mathbf{x}_i$ is sent to $\mathbf{x}_i$ in both cases. The images of $kB_k[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ and $kB_{n-k}[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ commute elementwise inside $kB_n[\mathbf{x}_1, \ldots, \mathbf{x}_r]$, so there is an induced map

$$kB_k[\mathbf{x}_1, \ldots, \mathbf{x}_r] \otimes kB_{n-k}[\mathbf{x}_1, \ldots, \mathbf{x}_r] \to kB_n[\mathbf{x}_1, \ldots, \mathbf{x}_r], \ p \otimes q \mapsto p \otimes 1^{(n-k)} \cdot 1^{(k)} \otimes q \ .$$

We will write $p \otimes q$ for $p \otimes 1^{(n-k)} \cdot 1^{(k)} \otimes q$.

[2] In fact, B_n is *right-ordered* by a recent result of Dehornoy [Deh], hence kB_n does not possess zero divisors nor non-trivial units by the results in chapter 13.1 of Passman's book [Pas]. We thank Dale Rolfsen for making us aware of this.

For any $k \geq 1$ let

$$P_k(\mathbf{x},\mathbf{y}) = [\mathbf{x} - s^{(k)}(1,k)\mathbf{y}] \cdot [\mathbf{x} - s^{(k)}(1,k-1)\mathbf{y}] \cdot \ldots \cdot [\mathbf{x} - s^{(k)}(1,1)\mathbf{y}] \ \in kB_k[\mathbf{x},\mathbf{y}] \ ;$$

and set $P_0(\mathbf{x},\mathbf{y}) = 1 \in kB_0$.

Then, with the above convention, the binomial theorem is the following identity in $kB_n[\mathbf{x},\mathbf{y},\mathbf{z}]$:

Proposition. For any $n \geq 0$,

$$P_n(\mathbf{x},\mathbf{z}) = \sum_{k=0}^{n} P_k(\mathbf{y},\mathbf{z}) \otimes P_{n-k}(\mathbf{x},\mathbf{y}) \cdot b_k^{(n)} \ . \tag{22}$$

Proof. We do induction on n. For $n = 0,1$ the statement is trivial. Assuming it true for $n-1$ with $n \geq 2$, we have

$$\begin{aligned}
P_n(\mathbf{x},\mathbf{z}) &= [\mathbf{x} - s^{(n)}(1,n)\mathbf{z}] \cdot [P_{n-1}(\mathbf{x},\mathbf{z}) \otimes 1] = [\mathbf{x} - s^{(n)}(1,n)\mathbf{z}] \cdot \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) \otimes P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1] \cdot [b_k^{(n-1)} \otimes 1] \\
&= \sum_{k=0}^{n-1} [\mathbf{x} - s^{(n)}(k+1,n)\mathbf{y} + s^{(n)}(k+1,n)\mathbf{y} - s^{(n)}(1,n)\mathbf{z}] \cdot [P_k(\mathbf{y},\mathbf{z}) \otimes P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1] \cdot [b_k^{(n-1)} \otimes 1] \\
&\overset{(A1),\,(1)}{=} \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) \otimes 1^{(n-k)}] \cdot [\mathbf{x} - s^{(n)}(k+1,n)\mathbf{y}] \cdot [1^{(k)} \otimes P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1] \cdot [b_k^{(n-1)} \otimes 1] + \\
&\quad + \sum_{k=0}^{n-1} [\mathbf{y} - s^{(n)}(1,k+1)\mathbf{z}] \cdot [P_k(\mathbf{y},\mathbf{z}) \otimes 1^{(n-k)}] \cdot s^{(n)}(k+1,n) \cdot [1^{(k)} \otimes P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1] \cdot [b_k^{(n-1)} \otimes 1] \\
&\overset{(3)}{=} \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) \otimes 1^{(n-k)}] \cdot \Big[1^{(k)} \otimes [\mathbf{x} - s^{(n)}(1,n-k)\mathbf{y}][P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1]\Big] \cdot [b_k^{(n-1)} \otimes 1] + \\
&\quad + \sum_{k=0}^{n-1} \Big[[\mathbf{y} - s^{(k+1)}(1,k+1)\mathbf{z}] \otimes 1^{(n-k-1)}\Big] \cdot [P_k(\mathbf{y},\mathbf{z}) \otimes 1^{(n-k)}] \cdot s^{(n)}(k+1,n) \cdot [1^{(k)} \otimes P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1] \cdot [b_k^{(n-1)} \otimes 1] \\
&= \sum_{k=0}^{n-1} [P_k(\mathbf{y},\mathbf{z}) \otimes 1^{(n-k)}] \cdot [1^{(k)} \otimes P_{n-k}(\mathbf{x},\mathbf{y})] \cdot [b_k^{(n-1)} \otimes 1] + \\
&\quad + \sum_{k=0}^{n-1} [P_{k+1}(\mathbf{y},\mathbf{z}) \otimes 1^{(n-k-1)}] \cdot s^{(n)}(k+1,n) \cdot [1^{(k)} \otimes P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1] \cdot [b_k^{(n-1)} \otimes 1] \ .
\end{aligned}$$

Now we use (5) to commute $s^{(n)}(k+1,n)$ past $P_{n-1-k}(\mathbf{x},\mathbf{y})$ as follows:

$$\begin{aligned}
s^{(n)}(k+1,n) \cdot [1^{(k)} \otimes P_{n-1-k}(\mathbf{x},\mathbf{y}) \otimes 1] &= s^{(n)}(k+1,n) \cdot \Big[1^{(k)} \otimes [\mathbf{x} - s^{(n-1-k)}(1,n-1-k)\mathbf{y}] \cdot \ldots \cdot [\mathbf{x} - s^{(n-1-k)}(1,1)\mathbf{y}] \otimes 1\Big] \\
&\overset{(2),\,(3)}{=} s^{(n)}(k+1,n) \cdot [\mathbf{x} - s^{(n)}(k+1,n-1)\mathbf{y}] \cdot \ldots \cdot [\mathbf{x} - s^{(n)}(k+1,k+1)\mathbf{y}] \\
&\overset{(5)}{=} [\mathbf{x} - s^{(n)}(k+2,n)\mathbf{y}] \cdot \ldots \cdot [\mathbf{x} - s^{(n)}(k+2,k+2)\mathbf{y}] \cdot s^{(n)}(k+1,n) \\
&\overset{(3)}{=} \Big[1^{(k+1)} \otimes [\mathbf{x} - s^{(n-1-k)}(1,n-k-1)\mathbf{y}] \cdot \ldots \cdot [\mathbf{x} - s^{(n-k-1)}(1,1)\mathbf{y}]\Big] \cdot s^{(n)}(k+1,n) \\
&= [1^{(k+1)} \otimes P_{n-k-1}(\mathbf{x},\mathbf{y})] \cdot s^{(n)}(k+1,n) \ .
\end{aligned}$$

Substituting this in the above expression for P_n we get

$$
\begin{aligned}
P_n(\mathbf{x},\mathbf{z}) = &\sum_{k=0}^{n-1}[P_k(\mathbf{y},\mathbf{z})\otimes 1^{(n-k)}]\cdot[1^{(k)}\otimes P_{n-k}(\mathbf{x},\mathbf{y})]\cdot[b_k^{(n-1)}\otimes 1] + \\
&+\sum_{k=0}^{n-1}[P_{k+1}(\mathbf{y},\mathbf{z})\otimes 1^{(n-k-1)}]\cdot[1^{(k+1)}\otimes P_{n-1-k}(\mathbf{x},\mathbf{y})]\cdot s^{(n)}(k+1,n)\cdot[b_k^{(n-1)}\otimes 1] \\
&= \sum_{k=0}^{n-1}[P_k(\mathbf{y},\mathbf{z})\otimes P_{n-k}(\mathbf{x},\mathbf{y})]\cdot[b_k^{(n-1)}\otimes 1] + \sum_{k=1}^{n}[P_k(\mathbf{y},\mathbf{z})\otimes P_{n-k}(\mathbf{x},\mathbf{y})]\cdot s^{(n)}(k+1,n)\cdot[b_k^{(n-1)}\otimes 1] \\
&= \sum_{k=0}^{n}[P_k(\mathbf{y},\mathbf{z})\otimes P_{n-k}(\mathbf{x},\mathbf{y})]\cdot[b_k^{(n-1)}\otimes 1 + s^{(n)}(k+1,n)\cdot b_k^{(n-1)}\otimes 1] \overset{(13)}{=} \sum_{k=0}^{n}[P_k(\mathbf{y},\mathbf{z})\otimes P_{n-k}(\mathbf{x},\mathbf{y})]\cdot b_k^{(n)} \ .
\end{aligned}
$$

□

6.2. **Cauchy's identities.** These identities are attributed to Cauchy in [GR1]:

$$(\mathbf{x}-1)(\mathbf{x}-q)\dots(\mathbf{x}-q^{n-1}) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}\mathbf{x}^{n-k} \ ,$$

$$\mathbf{x}^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(\mathbf{x}-1)(\mathbf{x}-q)\dots(\mathbf{x}-q^{k-1}) \ .$$

Just as in the q-case, its generalizations to braids are easily obtained from the binomial theorem. In this context, it is natural to introduce the *Möbius* braid $\mu^{(k)} \in kB_k$ as

$$\mu^{(k)} = (-1)^k c^{(k)}$$

where $c^{(k)} = s^{(k)}(1,k)s^{(k)}(1,k-1)\dots s^{(k)}(1,1) \in kB_k$ is the twistor braid of section 2.2.

Corollary. For any $n \geq 0$,

$$[\mathbf{x}-s^{(n)}(1,n)]\cdot[\mathbf{x}-s^{(n)}(1,n-1)]\cdot\ldots\cdot[\mathbf{x}-s^{(n)}(1,1)] = \sum_{k=0}^{n}\mu^{(k)}\otimes 1^{(n-k)}\cdot b_k^{(n)}\cdot\mathbf{x}^{n-k} \tag{23}$$

$$\mathbf{x}^n = \sum_{k=0}^{n}[\mathbf{x}-s^{(n)}(k+1,n)]\cdot[\mathbf{x}-s^{(n)}(k+1,n-1)]\cdot\ldots\cdot[\mathbf{x}-s^{(n)}(k+1,k+1)]\cdot b_k^{(n)} \ . \tag{24}$$

Proof. Setting $\mathbf{y}=0$ and $\mathbf{z}=1$ in (22) we obtain (23); setting $\mathbf{y}=1$ and $\mathbf{z}=0$ we obtain (24). These evaluations are well-defined morphisms of algebras because the evaluating points commute with the coefficients. □

Möbius inversion formula will we deduced from the following two consequences of Cauchy's identities. Setting $\mathbf{x}=1$ in (23) we obtain

$$\sum_{k=0}^{n}\mu^{(k)}\otimes 1^{(n-k)}\cdot b_k^{(n)} = 0 \ \forall\ n>0, \tag{25}$$

and setting $\mathbf{x}=0$ in (24) (or applying $\tilde{}$ to (25))

$$\sum_{k=0}^{n}1^{(k)}\otimes\mu^{(n-k)}\cdot b_k^{(n)} = 0 \ \forall\ n>0. \tag{26}$$

Both of these reduce in the q-case to the well-known

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix} = 0 \ \forall\ n > 0.$$

Some other interesting consequences of Cauchy's identities are obtained through other evaluations; these all reduce to the same identity in the q-case, but are distinct at the level of braids. To briefly discuss this situation, consider the polynomial ring $B[\mathbf{x}]$ over a non-commutative ring B. For each $b \in B$ there are two natural *evaluation* maps $B[\mathbf{x}] \to B$, according to whether we write the variable to the right or left of the coefficients. More precisely, these are defined as

$$\begin{array}{cc} \epsilon_b^r : B[\mathbf{x}] \to B & \epsilon_b^l : B[\mathbf{x}] \to B \\ a_n\mathbf{x}^n + \ldots + a_1\mathbf{x} + a_0 \mapsto a_n b^n + \ldots + a_1 b + a_0 & a_n\mathbf{x}^n + \ldots + a_1\mathbf{x} + a_0 \mapsto b^n a_n + \ldots + b a_1 + a_0\ . \end{array}$$

These maps are not multiplicative in general; however, if h, f and g are polynomials such that $h = fg$ and b commutes with the coefficients of g, then $\epsilon_b^r(h) = \epsilon_b^r(f)\epsilon_b^r(g)$. Similarly, if b commutes with the coefficients of f then $\epsilon_b^l(h) = \epsilon_b^l(f)\epsilon_b^l(g)$.

Consider $B = kB_n$, $f(\mathbf{x}) = [\mathbf{x} - s^{(n)}(1,n)] \cdot [\mathbf{x} - s^{(n)}(1,n-1)] \cdot \ldots \cdot [\mathbf{x} - s^{(n)}(1,3)]$ and $g(\mathbf{x}) = [\mathbf{x} - s^{(n)}(1,2)][\mathbf{x} - s^{(n)}(1,1)]$. Writing $\mathbf{x}$ to the right of the coefficients and evaluating (23) at $b = s^{(n)}(1,2) = s_1^{(n)}$ we obtain

$$\sum_{k=0}^{n} \mu^{(k)} \otimes 1^{(n-k)} \cdot b_k^{(n)} \cdot (s_1^{(n)})^{n-k} = 0\ .$$

Similarly, letting $f(\mathbf{x}) = x - s^{(n)}(1,n)$, $g(\mathbf{x}) = [\mathbf{x} - s^{(n)}(1,n-1)] \cdot \ldots \cdot [\mathbf{x} - s^{(n)}(1,1)]$, writing $\mathbf{x}$ to the left and evaluating (23) at $b = s^{(n)}(1,n)$ we obtain

$$\sum_{k=0}^{n} \Big[s^{(n)}(1,n)\Big]^{n-k} \cdot \mu^{(k)} \otimes 1^{(n-k)} \cdot b_k^{(n)} = 0\ .$$

6.3. **Möbius inversion.** A particular case of the general theory of Möbius inversion [R] is the following q-numerical inversion formula: for any scalars $a_0, \ldots, a_m$, $b_0, \ldots, b_m$,

$$b_i = \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} a_{i-j} \ \forall\ i = 0, \ldots, m \iff a_i = \sum_{j=0}^{i} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} i \\ j \end{bmatrix} b_{i-j} \ \forall\ i = 0, \ldots, m\ .$$

Its generalization is:

Proposition. Let $x^{(i)}$ and $y^{(i)} \in kB_i$ be given braids for $i = 0, \ldots, m$. Then

$$x^{(i)} = \sum_{j=0}^{i} 1^{(j)} \otimes y^{(i-j)} \cdot b_j^{(i)} \ \forall\ i = 0, \ldots, m \iff y^{(i)} = \sum_{j=0}^{i} \mu^{(j)} \otimes x^{(i-j)} \cdot b_j^{(i)} \ \forall\ i = 0, \ldots, m\ . \tag{27}$$

Proof. $(\Rightarrow)$

$$\begin{aligned}
\sum_{j=0}^{i}\mu^{(j)}\otimes x^{(i-j)}\cdot b_j^{(i)} &\overset{(hyp.)}{=} \sum_{j=0}^{i}\mu^{(j)}\otimes\Big[\sum_{h=0}^{i-j}1^{(h)}\otimes y^{(i-j-h)}\cdot b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\mu^{(j)}\otimes 1^{(h)}\otimes y^{(i-j-h)}\Big]\cdot\Big[1^{(j)}\otimes b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&\overset{(20)}{=} \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\mu^{(j)}\otimes 1^{(h)}\otimes y^{(i-j-h)}\Big]\cdot\Big[b_j^{(h+j)}\otimes 1^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\mu^{(j)}\otimes 1^{(i-j)}\Big]\cdot\Big[b_j^{(h+j)}\otimes y^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&\overset{(k:=h+j)}{=} \sum_{k=0}^{i}\sum_{j=0}^{k}\Big[\mu^{(j)}\otimes 1^{(i-j)}\Big]\cdot\Big[b_j^{(k)}\otimes y^{(i-k)}\Big]\cdot b_k^{(i)} = y^{(i)},
\end{aligned}$$

since by (23) all terms corresponding to $k\neq 0$ in the above sum vanish.

$(\Leftarrow)$

$$\begin{aligned}
\sum_{j=0}^{i}1^{(j)}\otimes y^{(i-j)}\cdot b_j^{(i)} &\overset{(hyp.)}{=} \sum_{j=0}^{i}1^{(j)}\otimes\Big[\sum_{h=0}^{i-j}\mu^{(h)}\otimes x^{(i-j-h)}\cdot b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[1^{(j)}\otimes\mu^{(h)}\otimes x^{(i-j-h)}\Big]\cdot\Big[1^{(j)}\otimes b_h^{(i-j)}\Big]\cdot b_j^{(i)}\\
&\overset{(20)}{=} \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[1^{(j)}\otimes\mu^{(h)}\otimes x^{(i-j-h)}\Big]\cdot\Big[b_j^{(h+j)}\otimes 1^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&= \sum_{j=0}^{i}\sum_{h=0}^{i-j}\Big[\Big[1^{(j)}\otimes\mu^{(h)}\Big]\cdot b_j^{(h+j)}\otimes x^{(i-j-h)}\Big]\cdot b_{h+j}^{(i)}\\
&\overset{(k:=h+j)}{=} \sum_{k=0}^{i}\Big[\sum_{j=0}^{k}\Big[1^{(j)}\otimes\mu^{(k-j)}\cdot b_j^{(k)}\Big]\otimes x^{(i-k)}\Big]\cdot b_k^{(i)} = x^{(i)},
\end{aligned}$$

since by (24) all terms corresponding to $k\neq 0$ in the above sum vanish. □

7. Multinomial braids

7.1. **Definition.** For each n and $r\in\mathbb{N}$ let $\mathcal{F}(n,r)$ denote the set of all functions $\{1,\ldots,n\}\to\{1,\ldots,r\}$, and $\mathcal{C}(n,r)=\{(\eta_1,\ldots,\eta_r)\in\mathbb{N}^r \;/\; \eta_1+\ldots+\eta_r=n\}$. A sequence $\eta\in\mathcal{C}(n,r)$ is sometimes called a *weak composition of* n *into* r *parts*. For any $\eta\in\mathcal{C}(n,r)$ let

$$\mathcal{S}(\eta)=\{f\in\mathcal{F}(n,r)\;/\;\#f^{-1}(1)=\eta_1,\#f^{-1}(2)=\eta_2,\ldots,\#f^{-1}(r)=\eta_r\}.$$

We usually write $f=\left(\begin{smallmatrix}1&2&3&4&5\\2&1&1&3&2\end{smallmatrix}\right)$ to abbreviate that $f:\{1,2,3,4,5\}\to\{1,2,3\}$ is $f(1)=f(5)=2$, $f(2)=f(3)=1$, $f(4)=3$. One may think of the elements of $\mathcal{S}(\eta)$ as permutations of the elements of $\{1,2,\ldots,r\}$ with repetitions as specified by η. For this reason the elements of $\mathcal{S}(\eta)$ are called *permutations of the multiset* $\{1^{\eta_1},2^{\eta_2},\ldots,r^{\eta_r}\}$.

There are canonical identifications $\mathcal{S}(1,1,\ldots,1)=S_r$ (r ones) and (when $r=2$) $\mathcal{S}(i,n-i)=\mathcal{S}_i(n)$, $f\mapsto\{j\in\{1,2,\ldots,n\}\;/\;f(j)=1\}$.

Given $\eta \in \mathcal{C}(n,r)$, the corresponding q-multinomial coefficient is defined as

$$\begin{bmatrix} n \\ \eta \end{bmatrix} = \sum_{f \in \mathcal{S}(\eta)} q^{\mathsf{inv}(f)} \ ,$$

where the *inversion index* $\mathsf{inv}(f)$ is

$$\mathsf{inv}(f) = \#\{(i,j) \ / \ 1 \leq i < j \leq n, \ f(i) > f(j)\} \ .$$

To define its braid analog we proceed as follows. First, for any $f \in \mathcal{F}(n,r)$ and $i \in \{1, 2, \dots, n\}$, set

$$e_i(f) = \#\{j \leq i \ / \ f(j) \leq f(i)\} \ .$$

Next, define $s_f^{(n)} \in B_n$ as

$$s_f^{(n)} = s^{(n)}(e_n(f), n) \cdot \ldots \cdot s^{(n)}(e_2(f), 2) \cdot s^{(n)}(e_1(f), 1) \ .$$

Then, for any $\eta \in \mathcal{C}(n,r)$, define the multinomial braid $m^{(\eta)} \in kB_n$ as

$$m^{(\eta)} = \sum_{f \in \mathcal{S}(\eta)} s_f^{(n)} \ ;$$

and $m^{(0,\dots,0)} = 1 \in kB_0$.

A few remarks are in order. First, notice that for $\sigma \in S_r = \mathcal{S}(1,1,\dots,1)$ (r ones), the definition of $s_\sigma^{(n)}$ given here coincides with that of section 5, because of equation (19). Hence $m^{(1,1,\dots,1)} = f^{(r)}$, the factorial braid.

Second, suppose $r = 2$, and let $I \in \mathcal{S}_i(n)$ correspond to $f \in \mathcal{S}(i, n-i)$ under the bijection described above: if $I = \{j_1 < j_2 < \ldots < j_i\}$ then $f = \left(\begin{smallmatrix} 1 & \cdots & & j_1 & & \cdots & & j_2 & & \cdots & & j_i & & \cdots & n \\ 2 & \ldots & 2 & 1 & 2 & \ldots & 2 & 1 & 2 & \ldots & 2 & 1 & 2 & \ldots & 2 \end{smallmatrix}\right)$.
Thus, $e_j(f) = \begin{cases} j & \text{if } j \notin I, \\ h & \text{if } j = j_h \in I \end{cases}$, from where $s_f^{(n)} = s^{(n)}(i, j_i) \cdot \ldots \cdot s^{(n)}(2, j_2) \cdot s^{(n)}(1, j_1) = s_I^{(n)}$, and hence $m^{(i,n-i)} = b_i^{(n)}$. Thus multinomial braids reduce to binomial braids when $r = 2$.

Finally, let us check that in the one-dimensional representation defined by q (section 2.5), $s_f^{(n)}$ acts as multiplication by $q^{\mathsf{inv}(f)}$, and hence $m^{(\eta)}$ as $\begin{bmatrix} n \\ \eta \end{bmatrix}$.

To this end, we introduce the η-*shuffle* $\sigma_f \in S_n$ corresponding to $f \in \mathcal{S}(\eta)$ as follows: on $f^{-1}(1)$, σ_f is the unique increasing bijection onto $\{1, \dots, \eta_1\}$, similarly on $f^{-1}(2)$ onto $\{\eta_1 + 1, \dots, \eta_1 + \eta_2\}$, $\dots$, and on $f^{-1}(r)$ onto $\{\eta_1 + \ldots + \eta_{r-1} + 1, \dots, \eta_1 + \ldots + \eta_r\}$.

We also introduce the partial inversion index $r_i(f) = \#\{j > i \ / \ f(j) < f(i)\}$, extending the one already defined for permutations in section 5. Notice that $\mathsf{inv}(f) = \sum_{i=1}^n r_i(f)$.

Lemma. For any $f \in \mathcal{S}(\eta)$ and $i \in \{1, 2, \dots, n\}$, $e_i(f) = e_i(\sigma_f)$ and $r_i(f) = r_i(\sigma_f)$.

Proof. From the definition of σ_f we see that:
For $j \leq i$, $\sigma_f(j) \leq \sigma_f(i) \Leftrightarrow f(j) \leq f(i)$. From here, $e_i(f) = e_i(\sigma_f)$.
For $j > i$, $\sigma_f(j) < \sigma_f(i) \Leftrightarrow f(j) < f(i)$. From here, $r_i(f) = r_i(\sigma_f)$. □

Now we can show that $s_f^{(n)}$ acts as $q^{\mathsf{inv}(f)}$, i.e. that the number of elementary generators in $s^{(n)}(e_n(f), n) \cdot \ldots \cdot s^{(n)}(e_2(f), 2) \cdot s^{(n)}(e_1(f), 1)$ is $\mathsf{inv}(f)$. Recall (section 5.3) that for any $\sigma \in S_n$ we have $\sigma(i) = r_i(\sigma) + e_i(\sigma)$. Hence, $\sigma_f(i) = r_i(\sigma_f) + e_i(\sigma_f) = r_i(f) + e_i(f)$, from where

$$\#\text{generators in } s_f^{(n)} = \sum_{i=1}^n i - e_i(f) = \sum_{i=1}^n \sigma_f(i) - e_i(f) = \sum_{i=1}^n r_i(f) = \mathsf{inv}(f),$$

as needed.

From the lemma we also deduce that $s_f^{(n)} = s_{\sigma_f}^{(n)}$, just comparing their definitions. This shows that our multinomial braids coincide with those braids already considered by Schauenburg in [S, definition 2.6]. Some of the identities we prove here ((13), (21), and a particular case of (29)) are stated in that paper, altough the connection to combinatorics is not pointed out.

7.2. **Symmetry of the multinomial braids.** Here we generalize the facts (12) and (17) that $\widetilde{b_i^{(n)}} = b_{n-i}^{(n)}$ and $\widetilde{f^{(n)}} = f^{(n)}$. For any $\eta = (\eta_1, \eta_2, \dots, \eta_r)$, let $\tilde{\eta} = (\eta_r, \dots, \eta_2, \eta_1)$.

Proposition. For any $\eta \in \mathcal{C}(n,r)$, $\widetilde{m^{(\eta)}} = m^{(\tilde{\eta})}$.

Proof. Consider the bijection $\mathcal{F}(n,r) \to \mathcal{F}(n,r)$, $f \to \tilde{f}$, where $\tilde{f}(i) = r+1-f(n+1-i)$. This clearly restricts to a bijection $\mathcal{S}(\eta) \to \mathcal{S}(\tilde{\eta})$, so it is enough to show that

$$\widetilde{s_f^{(n)}} = s_{\tilde{f}}^{(n)} \ \forall\ f \in \mathcal{S}(\eta)$$

to obtain the result.

We have that

$$\tilde{f}^{-1}(h) = n+1-f^{-1}(r+1-h),\ \forall\ h = 1, \dots, r,$$

from where

$$\sigma_{\tilde{f}}(i) = n+1-\sigma_f(n+1-i) = \widetilde{\sigma_f}(i)\ \forall\ i = 1, \dots, n,$$

and thus

$$\widetilde{s_f^{(n)}} = \widetilde{s_{\sigma_f}^{(n)}} \overset{(16)}{=} s_{\widetilde{\sigma_f}}^{(n)} = s_{\sigma_{\tilde{f}}}^{(n)} = s_{\tilde{f}}^{(n)}$$

as needed.

$\square$

7.3. **Pascal's identity for multinomial braids.** Let $\mathcal{C}^+(n,r)$ denote the set of *strict* compositions of n into r parts, i.e. those sequences $(\eta_1, \dots, \eta_r)$ such that $\eta_1 + \ldots + \eta_r = n$ *and* $\eta_i \in \mathbb{Z}^+\ \forall\ i = 1, \dots, r$.

Pascal's identity (13) is actually a particular case of the following identity for multinomial braids.

Proposition. For any $\eta \in \mathcal{C}^+(n,r)$,

$$\begin{aligned}(28)\quad m^{(\eta_1,\eta_2,\dots,\eta_r)} = s^{(n)}(\eta_1,n)\cdot m^{(\eta_1-1,\eta_2,\dots,\eta_r)}\otimes 1 + s^{(n)}(\eta_1+\eta_2,n)\cdot m^{(\eta_1,\eta_2-1,\dots,\eta_r)}\otimes 1 + \ldots \\ \ldots + s^{(n)}(\eta_1+\eta_2+\ldots+\eta_r,n)\cdot m^{(\eta_1,\eta_2,\dots,\eta_r-1)}\otimes 1.\end{aligned}$$

Proof. Consider the bijection

$$\coprod_{i=1}^{r} \mathcal{S}(\eta_1, \dots, \eta_i - 1, \dots, \eta_r) \to \mathcal{S}(\eta_1, \eta_2, \dots, \eta_r)$$

that sends $f \in \mathcal{S}(\eta_1, \dots, \eta_i - 1, \dots, \eta_r)$ to $g \in \mathcal{S}(\eta_1, \eta_2, \dots, \eta_r)$ defined by

$$g(j) = \begin{cases} f(j) & \text{if } j \in \{1, 2, \dots, n-1\}, \\ i & \text{if } j = n. \end{cases}$$

Clearly,

$$e_j(g) = \begin{cases} e_j(f) & \text{if } j \in \{1,2,\dots,n-1\}, \\ \eta_1 + \ldots + \eta_i & \text{if } j = n. \end{cases}$$

Hence $s_g^{(n)} = s^{(n)}(\eta_1+\eta_2+\ldots+\eta_i, n) \cdot s_{n-1}^{(f)} \otimes 1$. The result follows by summing over all such $f's$. $\square$

7.4. **Multinomials in terms of binomials and factorials.** In this section we relate the multinomial braids to the binomials and factorials, obtaining identities that generalize (20) and (21).

Proposition. Let $(\eta_1,\dots,\eta_r) \in \mathcal{C}(n,r)$, $s \le r$, and $n_1 = \eta_1+\ldots+\eta_s$, $n_2 = \eta_{s+1}+\ldots+\eta_r$. Then

$$m^{(\eta_1,\dots,\eta_r)} = m^{(\eta_1,\dots,\eta_s)} \otimes m^{(\eta_{s+1},\dots,\eta_r)} \cdot m^{(n_1,n_2)} \ . \tag{29}$$

Proof. Consider the bijection

$$\begin{aligned} \mathcal{S}(\eta_1,\dots,\eta_r) &\to \mathcal{S}(\eta_1,\dots,\eta_s) \times \mathcal{S}(\eta_{s+1},\dots,\eta_r) \times \mathcal{S}(n_1,n_2) \\ f &\mapsto (f_1,f_2,I) \end{aligned}$$

defined as follows:

$$\begin{aligned} I &= \{j \in \{1,\dots,n\} \ / \ f(j) \le s\} = \{j_1 < j_2 < \ldots < j_{n_1}\} \in \mathcal{S}_{n_1}(n), \\ I^c &= \{k \in \{1,\dots,n\} \ / \ f(k) > s\} = \{k_1 < k_2 < \ldots < k_{n_2}\} \in \mathcal{S}_{n_2}(n), \\ f_1 &= \left(\begin{smallmatrix} 1 & 2 & \dots & n_1 \\ f(j_1) & f(j_2) & \dots & f(j_{n_1}) \end{smallmatrix} \right) \in \mathcal{S}(\eta_1,\dots,\eta_s), \\ f_2 &= \left(\begin{smallmatrix} 1 & 2 & \dots & n_2 \\ f(k_1)-s & f(k_2)-s & \dots & f(k_{n_2})-s \end{smallmatrix} \right) \in \mathcal{S}(\eta_{s+1},\dots,\eta_r). \end{aligned}$$

(Informally, $f_1 = f|_I$, $f_2 = f|_{I^c}$.)

It is enough to show that

$$s_f^{(n)} = s_{n_1}^{(f_1)} \otimes s_{n_2}^{(f_2)} \cdot s_I^{(n)} \ .$$

We start by noting that for any $j \in \{1,\dots,n\}$,

$$\begin{aligned} e_j(f) &= \#\{h \in \{1,\dots,n\} \ / \ h \le j \text{ and } f(h) \le f(j)\} \\ &= \#\{h \in I \ / \ h \le j \text{ and } f(h) \le f(j)\} + \#\{h \in I^c \ / \ h \le j \text{ and } f(h) \le f(j)\}. \end{aligned}$$

Thus, if $j = j_i \in I$,

$$e_j(f) = \#\{h \in I \ / \ h \le j \text{ and } f(h) \le f(j)\} = e_i(f_1), \tag{*}$$

while if $j = k_i \in I^c$,

$$\begin{aligned} e_j(f) &= \#\{h \in I \ / \ h \le j\} + \#\{h \in I^c \ / \ h \le j \text{ and } f(h) \le f(j)\} \\ &= \#\{1,2,\dots,k_i\} - \#\{h \in I^c \ / \ h \le k_i\} + e_i(f_2) \\ &= k_i - i + e_i(f_2). \end{aligned} \tag{**}$$

Now,

$$\begin{aligned} s_{n_1}^{(f_1)} \otimes s_{n_2}^{(f_2)} \cdot s_I^{(n)} &= 1^{(n_1)} \otimes s_{n_2}^{(f_2)} \cdot s_{n_1}^{(f_1)} \otimes 1^{(n_2)} \cdot s_I^{(n)} \\ &\overset{(2)}{=} 1^{(n_1)} \otimes s_{n_2}^{(f_2)} \cdot s^{(n)}(e_{n_1}(f_1), n_1) \cdot \ldots \cdot s^{(n)}(e_2(f_1),2) \cdot s^{(n)}(e_1(f_1),1) s^{(n)}(n_1, j_{n_1}) \cdot \ldots \cdot s^{(n)}(2,j_2) \cdot s^{(n)}(1,j_1) \\ &\overset{(A1),(1)}{=} 1^{(n_1)} \otimes s_{n_2}^{(f_2)} \cdot s^{(n)}(e_{n_1}(f_1), j_{n_1}) \cdot \ldots \cdot s^{(n)}(e_2(f_1), j_2) \cdot s^{(n)}(e_1(f_1), j_1) \\ &\overset{(3)}{=} s^{(n)}(n_1 + e_{n_2}(f_2), n_1+n_2) \cdot s^{(n)}(n_1 + e_2(f_2), n_1+2) \cdot s^{(n)}(n_1+e_1(f_2), n_1+1) \cdot \\ &\quad \cdot s^{(n)}(e_{n_1}(f_1), j_{n_1}) \cdot \ldots \cdot s^{(n)}(e_2(f_1), j_2) \cdot s^{(n)}(e_1(f_1), j_1). \end{aligned}$$

At this point there are two cases to distinguish, according to whether $k_1 = n_1 + 1$ or $k_1 \leq n_1$ (notice that necessarily $k_1 \leq n_1 + 1$, since k_1 is the first element of I^c).

If $k_1 = n_1 + 1$ then necessarily $j_i = i$ and $k_i = n_1 + i \;\forall\, i$, so

$$s^{(n)}(n_1 + e_i(f_2), n_1 + i) = s^{(n)}(k_i - i + e_i(f_2), n_1 + i) \overset{(**)}{=} s^{(n)}(e_{k_i}(f), n_1 + i) = s^{(n)}(e_{n_1+i}(f), n_1 + i)$$

and

$$s^{(n)}(e_i(f_1), j_i) \overset{(*)}{=} s^{(n)}(e_{j_i}(f_1), j_i) = s^{(n)}(e_i(f), i)\ .$$

Thus, in this case, all the factors in the above expression for $s_{n_1}^{(f_1)} \otimes s_{n_2}^{(f_2)} \cdot s_I^{(n)}$ are already in the "right order":

$$\begin{aligned} s_{n_1}^{(f_1)} \otimes s_{n_2}^{(f_2)} \cdot s_I^{(n)} &= s^{(n)}(e_{n_1+n_2}(f), n_1 + n_2) \cdot \ldots \cdot s^{(n)}(e_{n_1+1}(f), n_1 + 1) \cdot s^{(n)}(e_{n_1}(f), n_1) \cdot \ldots \cdot s^{(n)}(e_1(f), 1) \\ &= s_f^{(n)}, \end{aligned}$$

as needed.

The other case occurs when $k_1 \leq n_1$. In this case j_{k_1} is well-defined. We will move $s^{(n)}(n_1 + e_1(f_2), n_1 + 1)$ to its right past the factors $x_i := s^{(n)}(e_i(f_1), j_i)$ from $i = n_1$ down to $i = k_1$, using (5). We illustrate this process as follows:

$$\begin{aligned} s^{(n)}(n_1 + e_1(f_2), n_1 + 1) &\xrightarrow{\text{past } x_{n_1}} s^{(n)}(n_1 - 1 + e_1(f_2), n_1) \xrightarrow{\text{past } x_{n_1-1}} \ldots \xrightarrow{\text{past } x_{i+1}} s^{(n)}(i + e_1(f_2), i + 1) \\ &\xrightarrow{\text{past } x_i} s^{(n)}(i - 1 + e_1(f_2), i) \xrightarrow{\text{past } x_{i-1}} \ldots \\ \ldots &\xrightarrow{\text{past } x_{k_1}} s^{(n)}(k_1 - 1 + e_1(f_2), k_1) \overset{(**)}{=} s^{(n)}(e_{k_1}(f), k_1). \end{aligned}$$

Before proceeding, we must check that the hypothesis of (5) hold, in order to validate this commutation. In this situation those hypothesis are

$$e_i(f_1) \leq i - 1 + e_1(f_2) \text{ and } i \leq j_i - 1,\ \forall\, i \in \{k_1, \ldots, n_1\}.$$

The first inequality holds because, for any f and g, $e_i(f) \leq i$ and $e_1(g) \geq 1$. And the second one does too, for if not, we would have that $j_i \leq i$ and hence $\{j_1, j_2, \ldots, j_i\} = \{1, 2, \ldots, i\}$. But since $k_1 \leq i$, this would imply that $k_1 \in I$, a contradiction. Thus the commutation process described above is valid.

Returning to the main argument, we next proceed similarly with the remaining factors $s^{(n)}(n_1 + e_2(f_2), n_1{+}2), \ldots, s^{(n)}(n_1{+}e_{n_2}(f_2), n_1{+}n_2)$, moving them to the right until they become $s^{(n)}(e_{k_2}(f), k_2)$, $\ldots$,$s^{(n)}(e_{k_{n_2}}(f), k_{n_2})$. After this has been done we are left with all the factors in the "right order":

$$\begin{aligned} s_{n_1}^{(f_1)} \otimes s_{n_2}^{(f_2)} \cdot s_I^{(n)} &= s^{(n)}(e_{n_1+n_2}(f), n_1 + n_2) \cdot \ldots \cdot s^{(n)}(e_{n_1+1}(f), n_1 + 1) \cdot s^{(n)}(e_{n_1}(f), n_1) \cdot \ldots \cdot s^{(n)}(e_1(f), 1) \\ &= s_f^{(n)}. \end{aligned}$$

This completes the proof. □

From (29) we can easily deduce expressions for the multinomial braids in terms of binomials or factorials, that generalize well-known q-formulas.

Corollary.

(30)
$$m^{(\eta_1,\dots,\eta_r)} = 1^{(\eta_1+\dots+\eta_{r-1})}\otimes b_{\eta_r}^{(\eta_r)} \cdot 1^{(\eta_1+\dots+\eta_{r-2})}\otimes b_{\eta_{r-1}}^{(\eta_{r-1}+\eta_r)} \cdot \dots \cdot 1^{(\eta_1+\eta_2)}\otimes b_{\eta_3}^{(\eta_3+\dots+\eta_r)} \cdot 1^{(\eta_1)}\otimes b_{\eta_2}^{(\eta_2+\dots+\eta_r)} \cdot b_{\eta_1}^{(\eta_1+\dots+\eta_r)}$$

(31)
$$m^{(\eta_1,\dots,\eta_r)} = b_0^{(\eta_1)}\otimes 1^{(\eta_2+\dots+\eta_r)} \cdot b_{\eta_1}^{(\eta_1+\eta_2)}\otimes 1^{(\eta_3+\dots+\eta_r)} \cdot b_{\eta_1+\eta_2}^{(\eta_1+\eta_2+\eta_3)}\otimes 1^{(\eta_4+\dots+\eta_r)} \cdot \dots \cdot b_{\eta_1+\dots+\eta_{r-2}}^{(\eta_1+\dots+\eta_{r-1})}\otimes 1^{(\eta_r)} \cdot b_{\eta_1+\dots+\eta_{r-1}}^{(\eta_1+\dots+\eta_r)}$$

(32)
$$f^{(\eta_1)}\otimes\dots\otimes f^{(\eta_r)} \cdot m^{(\eta_1,\dots,\eta_r)} = f^{(\eta_1+\dots+\eta_r)}$$

Proof. Choosing $s = 1$ in equation (29) we get
$$m^{(\eta_1,\dots,\eta_r)} = 1^{(\eta_1)}\otimes m^{(\eta_2,\dots,\eta_r)} \cdot b_{\eta_1}^{(\eta_1+\dots+\eta_r)}.$$
From here (30) follows immediately by induction on r.

Similarly, (31) follows by induction on r from
$$m^{(\eta_1,\dots,\eta_r)} = m^{(\eta_1,\dots,\eta_{r-1})}\otimes 1^{(\eta_r)} \cdot b_{\eta_1+\dots+\eta_{r-1}}^{(\eta_1+\dots+\eta_r)},$$
which is the case $s = r-1$ of (29).

The remaining identity can also be obtained by induction on r, as follows:
$$f^{(\eta_1)}\otimes\dots\otimes f^{(\eta_r)} \cdot m^{(\eta_1,\dots,\eta_r)} \overset{(29)}{=} \Big[f^{(\eta_1)}\otimes\dots\otimes f^{(\eta_s)} \cdot m^{(\eta_1,\dots,\eta_s)}\Big]\otimes\Big[f^{(\eta_{s+1})}\otimes\dots\otimes f^{(\eta_r)} \cdot m^{(\eta_{s+1},\dots,\eta_r)}\Big] \cdot m^{(n_1,n_2)}$$
$$\overset{\text{(ind.hyp.)}}{=} f^{(n_1)}\otimes f^{(n_2)} \cdot m^{(n_1,n_2)} \overset{(21)}{=} f^{(n)}.$$
∎

7.5. **Witt's identity.** The following identity for q-multinomials is a particular case of an identity that holds for all finite reflection groups, sometimes known as Witt's identity:
$$\sum_{r=0}^{n}(-1)^r \sum_{\eta\in\mathcal{C}^+(n,r)} \begin{bmatrix} n \\ \eta \end{bmatrix} = (-1)^n q^{\binom{n}{2}}$$
(this is [H, proposition 1.11] for the case of the reflection group S_n).

Recall that $\mathcal{C}^+(n,r)$ denotes the set of strict compositions of n into r parts. We should agree that $\mathcal{C}^+(n,0) = \begin{cases} \emptyset & \text{if } n > 0, \\ \{0\} & \text{if } n = 0 \end{cases}$, and that $m^{(0)} = 1 \in B_0$.

Witt's identity can be generalized to braids as follows.

Proposition. For every $n \geq 0$,

(33)
$$\sum_{r=0}^{n}(-1)^r \sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} = \mu^{(n)} .$$

Proof. We do induction on n. For $n = 0$ the statement is obvious. Assume $n \geq 1$. Consider the decomposition
$$\coprod_{k=0}^{n-1} \mathcal{C}^+(k, r-1) \xrightarrow{\cong} \mathcal{C}^+(n,r),\ (\eta_1,\dots,\eta_{r-1}) \mapsto (\eta_1,\dots,\eta_{r-1}, n-k).$$
Recall that, by (31), for any $\eta \in \mathcal{C}^+(k, r-1)$ we have
$$m^{(\eta,n-k)} = m^{(\eta)}\otimes 1^{(n-k)} \cdot b_k^{(n)} .$$

Hence

$$(*)\qquad \sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} = \sum_{k=0}^{n-1}\sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta,n-k)} = \sum_{k=0}^{n-1}\sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta)}\otimes 1^{(n-k)}\cdot b_k^{(n)} \ .$$

Thus,

$$\begin{aligned}\sum_{r=0}^{n}(-1)^r\sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} &\overset{(n\geq 0)}{=} \sum_{r=1}^{n}(-1)^r\sum_{\eta\in\mathcal{C}^+(n,r)} m^{(\eta)} \overset{(*)}{=} \sum_{k=0}^{n-1}\left[\sum_{r=1}^{n}(-1)^r\sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta)}\right]\otimes 1^{(n-k)}\cdot b_k^{(n)}\\ &= -\sum_{k=0}^{n-1}\left[\sum_{s=0}^{n-1}(-1)^s\sum_{\eta\in\mathcal{C}^+(k,r-1)} m^{(\eta)}\right]\otimes 1^{(n-k)}\cdot b_k^{(n)} \overset{\text{(ind.hyp.)}}{=} -\sum_{k=0}^{n-1}\mu^{(k)}\otimes 1^{(n-k)}\cdot b_k^{(n)}\\ &\overset{(25)}{=} \mu^{(n)} \ .\end{aligned}$$

□

8. Galois, Fibonacci and Catalan braids

The q-numbers

$$G_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}$$

are studied in [GR2], where they are called the Galois numbers. They satisfy the following recurrence, that when $q = 1$ simply says that $G_n = 2^n$:

$$G_{n+1} = 2G_n + (q^n - 1)G_{n-1} \ .$$

One may define Galois braids $G^{(n)} \in kB_n$ as

$$G^{(n)} = \sum_{k=0}^{n} b_k^{(n)};$$

then one easily obtains the following generalization of the recurrence above:

$$G^{(n+1)} = G^{(n)}\otimes 1 + 1\otimes G^{(n)} + \sum_{k=0}^{n} s^{(n+1)}(1,k+1)^* s^{(n+1)}(k+1,n+1)\cdot 1\otimes b_{n-1}^{(k-1)}\otimes 1 - 1\otimes G^{(n-1)}\otimes 1 \ .$$

Alternatively, one can define Galois braids $g^{(n)} \in kB_n$ as follows:

$$g^{(n)} = \sum_{k=0}^{n} c^{(k)}\otimes 1^{(n-k)}\cdot b_k^{(n)};$$

these braids satisfy the simpler formula:

$$g^{(n)} = [1 + s^{(n)}(1,n)]\cdot[1 + s^{(n)}(1,n-1)]\cdot\ldots\cdot[1 + s^{(n)}(1,2)]\cdot[1 + s^{(n)}(1,1)] \ ,$$

in fact, this is just the binomial theorem (22) at $\mathbf{x} = 1$, $\mathbf{y} = 0$, $\mathbf{z} = -1$.

These Galois braids $g^{(n)}$ specialize to Galois numbers

$$g_n = \sum_{k=0}^{n} q^{\binom{k}{2}}\begin{bmatrix} n \\ k\end{bmatrix}$$

and the formula above becomes

$$g_n = (1 + q^{n-1})\cdot(1 + q^{n-2})\cdot\ldots\cdot(1 + q)\cdot(1 + 1) \ .$$

The Fibonacci numbers F_n count the number of subsets of $\{1, 2, \ldots, n\}$ without consecutive elements; one has $F_n = F_{n-1} + F_{n-2}$. It is easy to obtain q-versions of these numbers. More general braid analogs can be defined as follows. Let $\mathcal{F}(n, k)$ denote the set of subsets of $\{1, 2, \ldots, n\}$ with k elements no two of which are consecutive, and set

$$F_k^{(n)} = \sum_{I \in \mathcal{F}(n,k)} s_I^{(n)} \ \in \ kB_n.$$

As for the Galois braids, we have two options for defining the Fibonacci braids in terms of the $F_k^{(n)}$, according to whether we weight by the twistors $c^{(k)}$ or not. As before, weighting leads to simpler identities. So we define the Fibonacci braids $F^{(n)} \in kB_n$ as

$$F^{(n)} = \sum_{k=0}^{n} c^{(k)} \otimes 1^{(n-k)} \cdot F_k^{(n)} \ .$$

The same bijection considered in the proof of Pascal's identity (13) shows that

$$F_k^{(n)} = F_k^{(n-1)} \otimes 1 + s^{(n)}(k, n) \cdot F_{k-1}^{(n-2)} \otimes 1^{(2)};$$

from here it follows easily that

$$F^{(n)} = F^{(n-1)} \otimes 1 + s^{(n)}(1, n) \cdot F^{(n-2)} \otimes 1^{(2)} \ .$$

Thus these braids specialize to q-numbers F_n that satisfy

$$F_n = F_{n-1} + q^{n-1} F_{n-2} \ .$$

The Catalan numbers C_n count the number of subsets I of $\{1, 2, \ldots, 2n\}$ satisfying the following two conditions:

$$\#I = n \text{ and for every } j = 1, 2, \ldots, 2n, \ \#I \cap \{1, 2, \ldots, j\} \geq \#I^c \cap \{1, 2, \ldots, j\} \ .$$

Let $\mathcal{C}(n)$ denote the family of those subsets, and set

$$C^{(n)} = \sum_{I \in \mathcal{C}(n)} s_I^{(2n)} \ \in \ kB_{2n}.$$

It is easy to see from (*) in the proof of (12) that

$$C^{(n)} = \widetilde{C^{(n)}} \ .$$

Similarly, from (*) in the proof of (14) one deduces that

$$C^{(n+1)} = \sum_{k=0}^{n} 1^{(k+1)} \otimes \beta_{k+1, n-k} \otimes 1^{(n-k)} \cdot 1 \otimes C^{(k)} \otimes 1 \otimes C^{(n-k)} \ .$$

Thus these braids specialize to q-numbers C_n that satisfy

$$C_{n+1} = \sum_{k=0}^{n} q^{(k+1)(n-k)} C_k C_{n-k} \ .$$

These are the q-Catalan numbers of Carlitz and Riordan [CR].

9. Binomial braids and quantum groups

In section 2.5 we explained how Yang-Baxter operators yield monoidal representations of the braid category $\mathfrak{B}$. In this regard we should add that Majid began the study of combinatorial identities between operators on tensor powers of a vector space X corresponding to a Yang-Baxter operator on $X\otimes X$: in thm. 10.4.12 of [Ma] the case $i = 1$ of (20) is obtained.

So far in this paper we have considered only one-dimensional representations, corresponding to the Yang-Baxter operator that simply multiplies by q. Other Yang-Baxter operators are obtained through the theory of quantum groups. Every module X over a *quasitriangular* Hopf algebra comes equipped with a canonical Yang-Baxter operator on $X\otimes X$. The converse essentially holds: every Yang-Baxter operator on a vector space X gives rise to a coquasitriangular bialgebra that coacts on X (if the bialgebra is finite-dimensional then X can be seen as a module over the dual bialgebra, which is quasitriangular).

An equivalent way to describe monoidal representations of the braid category is by means of the following fact: $\mathfrak{B}$ is the free braided monoidal strict category on one object (the object $1 \in \mathbb{N}$). This says that given any object X of a braided monoidal category $\mathfrak{C}$, there is a unique functor $F : \mathfrak{B} \to \mathfrak{C}$ that preserves the monoidal structures and the braidings and such that $F(1) = X$. If $\mathfrak{C}$ carries in addition a k-linear structure (compatible with the rest of the structure), then F extends to $F : k\mathfrak{B} \to \mathfrak{C}$. Usually $\mathfrak{C}$ consists of vector k-spaces with some additional structure, and thus $F : k\mathfrak{B} \to \mathfrak{C}$ yields linear representations of the various braid groups. This is the case for instance when $\mathfrak{C}$ is the category of modules over a quasitriangular Hopf algebra as above. Another family of examples arises from the category $\mathfrak{D}_G$ of *crossed G-modules*, for any group G. An object of $\mathfrak{D}_G$ is a k-space X equipped with a linear action of G and a linear G-grading, i.e. a decomposition $X = \oplus_{g\in G}X_g$ into subspaces, such that the action of $h \in G$ carries X_g to $X_{hgh^{-1}}$. In this context, one usually writes $|x| = g$ when $x \in X_g$, so that the condition just mentioned becomes $|h \cdot x| = h|x|h^{-1}$. This category is braided monoidal under the usual tensor product of k-spaces, where $X\otimes Y$ is equipped with the G-action $g \cdot (x, y) = (g \cdot x, g \cdot y)$ and the G-grading $|(x, y)| = |x||y|$, and the braiding is

$$\beta_{X,Y} : X\otimes Y \to Y\otimes X, \quad \beta_{X,Y}(x, y) = (|x| \cdot y)\otimes x.$$

This construction can in fact be carried out for any Hopf algebra H in place of G. If H is finite-dimensional, then $\mathfrak{D}_H$ is the category of modules over the *Drinfeld double* $D(H)$ of H, which is a quasitriangular Hopf algebra.

We have described in [A] a general procedure for constructing a quantum group out of this data (that is, a group G, or more generally a Hopf algebra, and a crossed module X). In this procedure the action of the binomial braids $b_i^{(n)}$ on the various tensor powers $X^{\otimes n}$ plays a crucial role. Drinfeld and Jimbo's quantized enveloping algebra $U_q^+(C)$ associated to a symmetrizable generalized Cartan matrix C arises from this construction with $G = \mathbb{Z}^r$, the free abelian group of rank r, where r is the size of C, and the following crossed $\mathbb{Z}^r$-module X: let $A = [a_{hk}]$ be the symmetrized matrix corresponding to C, an integer square matrix of size r, let X be the vector space with basis $\{x_1, \dots , x_r\}$ and define

$$|x_k| = (a_{1k}, \dots , a_{rk}) \in \mathbb{Z}^r, \quad (n_1, \dots , n_r) \cdot x_h = q^{n_h} x_h \ \forall\ (n_1, \dots , n_r) \in \mathbb{Z}^r,$$

where $q \in k^*$ is any fixed scalar, not a root of unity.

We will now briefly describe this procedure, without proofs. For the general case of a crossed module over a Hopf algebra, the construction involves the notions of *internal categories* and *admissible sections* developed in [A]. A small linear category [Mi] is an example of an internal category. Let us concentrate on this special case that requires less strange terminology, and that covers the main example $U_q^+(C)$.

The idea is to attach a small linear category $\mathfrak{U}_G^+(X)$ to the given group G and crossed module X, and then obtain the quantum group as the *matrix ring* of the category, as defined by Mitchell in [Mi]. The coalgebra structure on the quantum group is seen to come from a *deltacategory* structure on the category. This is the crucial point where the binomial braids enter, so we had better explain it in some detail.

For small linear categories (and more generally for internal categories) there is an alternative notion of morphisms, besides that of ordinary functors. We have called them *cofunctors*, because a different special case (Lie groupoids) has received that name in the literature [HM]. A cofunctor induces a morphism of algebras between the corresponding matrix rings (on the other hand, a functor does not, unless it is bijective on objects -this was the case considered in [Mi]). A small deltacategory $\mathfrak{U}$ is a small linear category equipped with a coassociative cofunctor $\Delta : \mathfrak{U} \to \mathfrak{U} \otimes \mathfrak{U}$. (More precisely, it is a comonoid in the monoidal category of small linear categories and cofunctors). The matrix ring of such a category is then a bialgebra. The main point of these considerations is that most quantum groups arise as matrix rings of naturally defined deltacategories.

Let us describe the deltacategory $\mathfrak{U}^+_G(X)$. First we consider the graph whose vertex set is G and whose set of arrows is $\coprod_{g\in G} X_g \times G$, where each (x, h) is an arrow from h to $h|x|$. Then we pass to the free linear category $\mathfrak{T}_G(X)$ on this graph. It turns out that $\mathfrak{T}_G(X)$ possesses a deltacategory structure, defined on the generating arrows as follows:

$$\Delta(x, gh) = e_g \otimes (x, h) + (h \cdot x, g) \otimes e_h \ .$$

Here e_g denotes the identity arrow of the object $g \in G$.

It follows from coassociativity that the general expression for Δ is

$$\Delta(x, gh) = \sum_{i=0}^{n} \Big(h \cdot (b_i^{(n)} x)_{(i)}, g \Big) \otimes \Big((b_i^{(n)} x)_{(i)'}, h \Big) \ \text{ for } x \in X^{\otimes n}.$$

Here $b_i^{(n)}$ is of course the binomial braid, acting on $X^{\otimes n}$ as explained above; we have also used the notation $y = y_{(i)} \otimes y_{(i)'}$ for the canonical identification $X^{\otimes n} \cong X^{\otimes i} \otimes X^{\otimes (n-i)}$. This is in fact how we ran into these braids in the first place. One immediately sees from this expression for Δ that there is a natural set of relations on $\mathfrak{T}_G(X)$ that are preserved by Δ. Namely, if for each $n \geq 2$ we let

$$I^{(n)} := \bigcap_{i=1}^{n-1} \mathsf{Ker}\Big(b_i^{(n)} : X^{\otimes n} \to X^{\otimes n} \Big) \ \text{ and } \ I := \bigoplus_{n=2}^{\infty} I^{(n)},$$

then the ideal of $\mathfrak{T}_G(X)$ spanned by $I \times G$ is preserved by Δ. Hence, the quotient category $\mathfrak{U}^+_G(X) := \mathfrak{T}_G(X)/I \times G$ inherits a deltacategory structure.

One can show that when $G = \mathbb{Z}^r$ and X is defined from a symmetrizable generalized Cartan matrix C as above, then the matrix ring of $\mathfrak{U}^+_G(X)$ is the quantum group $U_q^+(C)$.

Some other simple choices of G and X yield well-known quantum groups, for instance $G = \mathbb{Z}_2$ and X=the non-trivial one dimensional representation of G yield Sweedler's Hopf algebra H_4; $G = \mathbb{Z}_n$ and X a one dimensional representation of G yield Taft's Hopf algebras.

But notice that this construction is more general: one can use any integral matrix for C, and of course other groups or even Hopf algebras, and obtain other (new) quantum groups.

These assertions will be complemented with more details and proofs in [A].

10. Additional remarks

Further interesting combinatorial phenomena arises from the study of the behavior of the various braid analogs on higher dimensional representations X of the braid groups. In particular, the determinants of $b_i^{(n)}$ and $f^{(n)}$ on $X^{\otimes n}$ seem to factor in some rather remarkable ways, intimately related to the combinatorics of the braid arrangement $\mathcal{A}_{r-1} = \{H_{hk} \ / \ 1 \leq h < k \leq r\}$, where $H_{hk} = \{(x_1, \dots, x_r) \in \mathbb{R}^r \ / \ x_h = x_k\}$.

For instance, consider the representation constructed from a symmetric matrix $A = [a_{hk}]$ of size r as in section 9. Thus, B_n acts on $X^{\otimes n}$ $\forall \ n \geq 0$, where X is a vector space with basis $\{x_1, \dots, x_r\}$.

The subspace X_r of $X^{\otimes r}$ spanned by those tensors of the form $x_{\sigma(1)}\otimes x_{\sigma(2)}\otimes \ldots \otimes x_{\sigma(r)}$, where σ runs over S_r, is invariant under the action of B_r. The matrix of $f^{(r)} : X_r \to X_r$ with respect to this basis turns out to be the same matrix that Varchenko associates to the weighted hyperplane arrangement $\mathcal{A}_{r-1}$ (weighted by the a_{hk}'s) [V]. A factorization formula for the determinant of the matrix of an arbitrary weighted real hyperplane arrangement is obtained in that work. For the special case of the braid arrangement, further factorization formulas seem to hold, not only for the determinant of the factorial braid, but also for the binomials, and on other invariant subspaces of $X^{\otimes n}$ as well.

In particular, on the subspace $X_{h,k}$ of $X^{\otimes(n+1)}$ spanned by $x_h\otimes x_k^{\otimes n}$ and its permutations, one can show that

$$\det\Big(b_1^{(n+1)}|_{X_{h,k}}\Big) = (1-q^{a[n]})(1-q^{a[n-1]})\ldots(1-q^{a[1]})[n]!_{q^{a_{kk}}}\,,$$

where

$$a[i] = (i-1)a_{kk} + a_{hk} + a_{kh}\ .$$

These questions will be the subject of further work.

Bibliography

[A]: M.Aguiar, *Internal Categories and Quantum Groups,* Cornell University Ph.D. thesis.

[CR]: L.Carlitz, J.Riordan *Two element lattice permutation numbers and their q-generalization,* Duke J. Math. 31 (1964), 371-388.

[De]: P.Dehornoy, *From large cardinals to braids via ditributive algebra,* in Jour. Knot Theory 4 n1 (1995), 33-79.

[Dr]: V.G.Drinfeld, *Quantum Groups,* in Proc. Int. Cong. Math. (Berkeley, 1986), 798-820.

[H]: J.E.Humphreys, *Reflection Groups and Coxeter Groups,* Cambridge Stud. Adv. Math. 29, Cambridge U. Press (1990).

[HM]: P. Higgins, K.Mackenzie, *Duality for base-changing morphisms of vector bundles, modules, Lie algebroids and Poisson structures,* preprint.

[J]: M.Jimbo, *A q-difference analogue of $U(g)$ and the Yang-Baxter equation,* Lett. Math. Phys. 10 (1985), 63-69.

[K]: C.Kassel, *Quantum Groups,* GTM 155, Springer-Verlag (1995).

[L]: G.Lusztig, *Introduction to Quantum Groups,* Progr. Math. 110, Birkhäuser (1993).

[Ma]: S.Majid, *Foundations of Quantum Group Theory,* Cambridge U. Press (1995).

[Mi]: B.Mitchell, *Rings with several objects,* Advances in Math. 8 (1972), 1-161.

[P]: D.S.Passman, *The algebraic structure of group rings,* Wiley&Sons (1977).

[S]: P.Schauenburg, *A Characterization of the Borel-like Subalgebras of Quantum Enveloping Algebras,* Comm. in Algebra 24 (1996), 2811-2823.

[V]: A.Varchenko, *Bilinear form of a real configuration of hyperplanes,* Adv. Math. 97 (1993), 110-144.

The following articles can be found in *Gian-Carlo Rota on Combinatorics: Introductory papers and commentaries,* edited by Joseph P.S. Kung, Birkhäuser (1995).

[R]: G.C.Rota, *On the foundations of Combinatorial Theory I. Theory of Möbius functions,* Z. Wahrscheinlichkertstheorie und Verw. Gebiete 2 (1964), 340-368.

[GR1]: J.Goldman, G.C.Rota, *On the foundations of Combinatorial Theory IV. Finite Vector Spaces and Eulerian Generating Functions,* Studies in Applied Mathematics vXLIX n3 (1970), 239-258.

[GR2]: J.Goldman, G.C.Rota, *The number of subspaces of a vector space,* Recent Progress in Combinatorics, Academic Press (1969), 75-83.

Department of Mathematics, Cornell University, Ithaca, NY 14853
E-mail address: maguiar@math.cornell.edu