Unipotent Representations and the Dual Pairs Correspondence

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Much of the material in this talk was presented at the conference in honor of Roger Howe. The slides are available at

http://math.mit.edu/conferences/howe/program.php

The theme of the talk is to find realizations of unipotent representations in terms of the $\Theta-$correspondence as defined by Howe.
The groups are complex viewed as real groups because [AB1] provides a complete explicit description of the $\Theta-$correspondence.
A particular focus is the $K-$spectrum of these representations, and their relation to the geometry of nilpotent orbits in the Lie algebra $\mathfrak{g}$ (or rather its linear dual $\mathfrak{g}^*$). This is a different aspect than the relation to nilpotent orbits in the dual algebra $\mathfrak{g}^\vee$.

One of the reasons to study the $K-$structure of small unitary representations as in the last examples is to compute Dirac cohomology as introduced by Vogan and Huang-Pandzic. Subject for Another Time.

Unipotent Representations, Complex Classical Groups

First recall the Langlands parametrization of irreducible modules. We use the standard realizations of the classical groups, roots, positive roots and simple roots. Let

- θ Cartan involution, K the fixed points of θ, $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$,
- $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a Borel subalgebra,
- $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ a CSA, $\mathfrak{t} \subset \mathfrak{k}$, $\theta \mid_{\mathfrak{a}} = -Id$,
- $X(\mu, \nu) = \mathrm{Ind}_B^G(\mathbb{C}_\mu \otimes \mathbb{C}_\nu)$ standard module,
- $L(\mu, \nu)$, the unique subquotient containing $V_\mu \in \widehat{K}$,
- $\lambda_L = (\mu + \nu)/2$ and $\lambda_R = (-\mu + \nu)/2$.

The parameters of unipotent representations have real ν.

Theorem

1. *$L(\lambda_L, \lambda_R) \cong L(\lambda_L', \lambda_R')$ if and only if there is a $w \in W$ such that $w \cdot (\lambda_L, \lambda_R) = (\lambda_L', \lambda_R')$.*
2. *$L(\lambda_L, \lambda_R)$ is hermitian if and only if there is $w \in W$ such that $w \cdot (\mu, \nu) = (\mu, -\nu)$.*

We rely on [BV2] and [B1]. For each $\mathcal{O} \subset \mathfrak{g}$ we will give an infinitesimal character $(\lambda_{\mathcal{O}}, \lambda_{\mathcal{O}})$, and a set of parameters.

Main Properties of $\lambda_{\mathcal{O}}$:

- $\mathrm{Ann}\,\Pi \subset U(\mathfrak{g})$ is the maximal primitive ideal $I_{\lambda_{\mathcal{O}}}$ with infinitesimal character $\lambda_{\mathcal{O}}$,
- Π unitary.
- $\mid \{\Pi \;:\; \mathrm{Ann}\,\Pi = I_{\lambda_{\mathcal{O}}}\} \mid = \mid \widehat{A(\mathcal{O})} \mid$,
  where $A(\mathcal{O})$ is the component group of the centralizer of an $e \in \mathcal{O}$.

This depends on the isogeny class of the group.
In the $\Theta-$correspondence, the reductive dual pairs are
$GL(n) \times GL(m)$,

$Sp(2n) \times O(m)$,

$O(m) \times Sp(2n)$.

The notation is as in [B1]. For special orbits whose dual is even, the infinitesimal character is one half the semisimple element of the Lie triple corresponding to the dual orbit. For the other orbits we need a case-by-case analysis. The parameter will always have integer and half-integer coordinates, the coresponding set of integral (co)roots is maximal.

Special orbits in the sense of Lusztig and in particular stably trivial orbits defined below will play a special role.

Definition

A special orbit $\mathcal{O}$ is called stably trivial if Lusztig's quotient $\overline{A}(\mathcal{O}) = A(\mathcal{O})$.

Example

1) $\mathcal{O} = (2222) \subset sp(8)$ is stably trivial, $A(\mathcal{O}) = \overline{A(\mathcal{O})} \cong \mathbb{Z}_2$,
$\lambda_{\mathcal{O}} = (2, 1, 1, 0)$.
2) $\mathcal{O} = (222) \subset sp(6)$ is not, $A(\mathcal{O}) \cong \mathbb{Z}_2$, $\overline{A(\mathcal{O})} \cong 1$,
$\lambda_{\mathcal{O}} = (3/2, 1/2, 1/2)$. (222) is special, $h^\vee/2 = (1, 1, 0)$.

The partitions denote rows.

Type A, $GL(n, \mathbb{C})$

Nilpotent orbits are determined by their Jordan canonical form. An orbit is given by a partition, *i.e.* a sequence of numbers in decreasing order $(n_1, \dots, n_k)$ that add up to n. Let $(m_1, \dots, m_l)$ be the dual partition. Then the infinitesimal character is

$$\left(\frac{m_1 - 1}{2}, \dots, -\frac{m_1 - 1}{2}, \dots, \frac{m_l - 1}{2}, \dots, -\frac{m_l - 1}{2}\right)$$

The orbit is induced from the trivial orbit on the Levi component $GL(m_1) \times \cdots \times GL(m_l)$. The corresponding unipotent representation is spherical and induced irreducible from the trivial representation on the same Levi component. All orbits are *special* and *stably trivial*.

Type B, $SO(2m+1)$

A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \dots, n_k)$ of $2m+1$ such that every even entry occurs an even number of times. Let $(m'_0, \dots, m'_{2p'})$ be the dual partition (add an $m'_{2p'} = 0$ if necessary, in order to have an odd number of terms). If there are any $m'_{2j} = m'_{2j+1}$ then pair them together and remove them from the partition. Then relabel and pair up the remaining columns $(m_0)(m_1, m_2) \dots (m_{2p-1} m_{2p})$. The members of each pair have the same parity and m_0 is odd. $\lambda_{\mathcal{O}}$ is given by the coordinates

$$\begin{aligned} (m_0) &\longleftrightarrow (\frac{m_0 - 2}{2}, \dots, \frac{1}{2}), \\ (m'_{2j} = m'_{2j+1}) &\longleftrightarrow (\frac{m_{2j} - 1}{2}, \dots, -\frac{m_{2j} - 1}{2}) \\ (m_{2i-1} m_{2i}) &\longleftrightarrow (\frac{m_{2i-1}}{2}, \dots, -\frac{m_{2i} - 2}{2}) \end{aligned} \tag{1}$$

Type B, continued

In case $m'_{2j} = m'_{2j+1}$, $\mathcal{O}$ is induced from a $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m} = so(*) \times gl(m'_{2j})$ where $\mathfrak{m}$ is the Levi component of a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$. $\mathcal{O}_{\mathfrak{m}}$ is the trivial nilpotent on the $gl-$factor. The component groups satisfy $A_G(\mathcal{O}) \cong A_M(\mathcal{O}_{\mathfrak{m}})$. Each unipotent representation is unitarily induced from a unipotent representation attached to $\mathcal{O}_{\mathfrak{m}}$.
Similarly if some $m_{2i-1} = m_{2i}$, then $\mathcal{O}$ is induced from a $\mathcal{O}_{\mathfrak{m}} \subset so(*) \times gl(\frac{m_{2i-1}+m_{2i}}{2})$ with (0) on the $gl-$factor.
$A_G(\mathcal{O}) \ncong A_M(\mathcal{O}_{\mathfrak{m}})$, but each unipotent representation is (not necessarily unitarily) induced irreducible from a $\mathcal{O}_{\mathfrak{m}} \subset \mathfrak{m} \cong so(\ ) \times gl(\ )$.
The *stably trivial* orbits are the ones such that every odd sized part appears an even number of times, except for the largest size. An orbit is called triangular if it has partition

$$\mathcal{O} \longleftrightarrow (2m+1, 2m-1, 2m-1, \dots, 3, 3, 1, 1).$$

Type B, continued

We give the explicit Langlands parameters of the unipotent representations in terms of their . There are $|\, A_G(\mathcal{O})|$ distinct representations. Let

$$(\underbrace{1,\ldots,1}_{r_1},\ldots,\underbrace{k,\ldots k}_{r_k})$$

be the rows of the Jordan form of the nilpotent orbit. The numbers r_{2i} are even. The reductive part of the centralizer (when $G = O(*)$) of the nilpotent element is a product of $O(r_{2i+1})$, and $Sp(r_{2j})$.

The columns are paired as in (1). The pairs $(m'_{2j} = m'_{2j+1})$ contribute to the spherical part of the parameter,

$$(m'_{2j} = m'_{2j+1}) \longleftrightarrow \begin{pmatrix} \lambda_L \\ \lambda_R \end{pmatrix} = \begin{pmatrix} \frac{m'_{2j}-1}{2} & , & \dots & , & -\frac{m'_{2j}-1}{2} \\ \frac{m'_{2j}-1}{2} & , & \dots & , & -\frac{m'_{2j}-1}{2} \end{pmatrix}. \qquad (2)$$

The singleton (m_0) contributes to the spherical part,

$$(m_0) \longleftrightarrow \begin{pmatrix} \frac{m_0-2}{2} & , & \dots & , & \frac{1}{2} \\ \frac{m_0-2}{2} & , & \dots & , & \frac{1}{2} \end{pmatrix}. \qquad (3)$$

Let $(\eta_1, \dots, \eta_p)$ with $\eta_i = \pm 1$, one for each (m_{2i-1}, m_{2i}). An $\eta_i = 1$ contributes to the spherical part of the parameter, with coordinates as in (1). An $\eta_i = -1$ contributes

$$\begin{pmatrix} \frac{m_{2i-1}}{2} & , & \dots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}}{2} & , & \dots & , & -\frac{m_{2i}-2}{2} \\ \frac{m_{2i-1}}{2} & , & \dots & , & \frac{m_{2i}+2}{2} & \frac{m_{2i}-2}{2} & , & \dots & , & -\frac{m_{2i}}{2} \end{pmatrix}. \qquad (4)$$

If $m_{2p} = 0$, $\eta_p = 1$ only.

Explanation

1. Odd sized rows contribute a $\mathbb{Z}_2$ to $A(\mathcal{O})$, even sized rows a 1.
2. When there are no $m'_{2j} = m'_{2j+1}$, every row size occurs. $\ldots(m_{2i-1} \geq m_{2i}) > (m_{2i+1} \geq m_{2i+2})\ldots$ determines that there are $m_{2i} - m_{2i+1}$ rows of size $2i+1$. The pair $(m_{2i-1} \geq m_{2i})$ contributes exactly 2 parameters corresponding to the $\mathbb{Z}_2$ in $A(\mathcal{O})$.
3. The pairs $(m'_{2j} = m'_{2j+1})$ lengthen the sizes of the rows without changing their parity. The component group does not change, they do not affect the number of parameters.

Type C, $Sp(2n, \mathbb{C})$

A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \dots, n_k)$ of $2n$ such that every odd part occurs an even number of times. Let $(c'_0, \dots, c'_{2p'})$ be the dual partition (add a $c'_{2p'} = 0$ if necessary in order to have an odd number of terms). If there are any $c'_{2j-1} = c'_{2j}$ pair them up and remove them from the partition. Then relabel and pair up the remaining columns $(c_0 c_1) \dots (c_{2p-2} c_{2p-1})(c_{2p})$. The members of each pair have the same parity. The last one, c_{2p}, is always even. Then form a parameter

$$(c'_{2j-1} = c'_{2j}) \leftrightarrow (\frac{c_{2j} - 1}{2}, \dots, -\frac{c_{2j} - 1}{2}), \tag{5}$$

$$(c_{2i} c_{2i+1}) \leftrightarrow (\frac{c_{2i}}{2}, \dots, -\frac{c_{2i+1} - 2}{2}), \tag{6}$$

$$c_{2p} \leftrightarrow (\frac{c_{2p}}{2}, \dots, 1). \tag{7}$$

Type C, continued

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.
An orbit is called triangular if it corresponds to the partition $(2m, 2m, \ldots, 4, 4, 2, 2)$.

We give a parametrization of the unipotent representations in terms of their Langlands parameters. There are $\mid A_G(\mathcal{O}) \mid$ representations.
Let

$$(\underbrace{1, \ldots, 1}_{r_1}, \ldots, \underbrace{k, \ldots k}_{r_k})$$

be the rows of the Jordan form of the nilpotent orbit. The numbers r_{2i+1} are even.

Type C, continued

The reductive part of the centralizer of the nilpotent element is a product of $Sp(r_{2i+1})$, and $O(r_{2j})$.
The elements $(c'_{2j-1} = c'_{2j})$ and c_{2p} contribute to the spherical part of the parameter as in (2) and (3). Let $(\epsilon_1, \dots, \epsilon_p)$ be such that $\epsilon_i = \pm 1$, one for each (c_{2i}, c_{2i+1}). An $\epsilon_i = 1$ contributes to the spherical paramater according to the infinitesimal character. An $\epsilon_i = -1$ contributes

$$\begin{pmatrix} \frac{c_{2i}}{2} & , & \dots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}}{2} & \dots & , & -\frac{c_{2i+1}-2}{2} \\ \frac{c_{2i}}{2} & , & \dots & , & \frac{c_{2i+1}+2}{2} & \frac{c_{2i+1}-2}{2} & \dots & , & -\frac{c_{2i+1}}{2} \end{pmatrix}. \tag{8}$$

The explanation is similar to type B.

Type D, $SO(2m, \mathbb{C})$

A nilpotent orbit is determined by its Jordan canonical form (in the standard representation). It is parametrized by a partition $(n_1, \dots, n_k)$ of $2m$ such that every even part occurs an even number of times. Let $(m'_0, \dots, m'_{2p'-1})$ be the dual partition (add a $m'_{2p'-1} = 0$ if necessary). If there are any $m'_{2j} = m'_{2j+1}$ pair them up and remove from the partition. Then pair up the remaining columns $(m_0, m_{2p-1})(m_1, m_2)\dots(m_{2p-3}, m_{2p-2})$. The members of each pair have the same parity and m_0, m_{2p-1} are both even. The infinitesimal character is

$$
\begin{aligned}
(m'_{2j} = m'_{2j+1}) &\longleftrightarrow (\frac{m'_{2j}-1}{2} \dots, -\frac{m'_{2j}-1}{2}) \\
(m_0 m_{2p-1}) &\longleftrightarrow (\frac{m_0-2}{2}, \dots, -\frac{m_{2p-1}}{2}), \\
(m_{2i-1} m_{2i}) &\longleftrightarrow (\frac{m_{2i-1}}{2} \dots, -\frac{m_{2i}-2}{2})
\end{aligned} \tag{9}
$$

Type D, continued

The nilpotent orbits and the unipotent representations have the same properties with respect to these pairs as the corresponding ones in type B. An exception occurs for $G = SO(2m)$ when the partition is formed of pairs $(m'_{2j} = m'_{2j+1})$ only. In this case there are two nilpotent orbits corresponding to the partition. There are also two nonconjugate Levi components of the form $gl(m'_0) \times gl(m'_2) \times \dots gl(m'_{2p'-2})$ of parabolic subalgebras. There are two unipotent representations each induced irreducible from the trivial representation on the corresponding Levi component.

The *stably trivial* orbits are the ones such that every even sized part appears an even number of times.

A nilpotent orbit is triangular if it corresponds to the partition $(2m-1, 2m-1, \dots, 3, 3, 1, 1)$.

Type D, continued

The parametrization of the unipotent representations follows types B,C, with the pairs $(m'_{2j} = m'_{2j+1})$ and (m_0, m_{2p-1}) contributing to the spherical part of the parameter only. Similarly for (m_{2i-1}, m_{2i}) with $\epsilon_i = 1$ spherical only, while $\epsilon_i = -1$ contributes analogous to (4) and (8).

The explanation parallels that for types B,C.

Metaplectic Correspondence

The next results are motivated by [KP1].
Restrict attention to the cases

(B) $(m_0)(m_1, m_2)\dots(m_{2p-1}, m_{2p})$ with $m_{2k} > m_{2k+1}$,

(C) $(c_0, c_1)\dots(c_{2p-2}, c_{2p-1})(c_{2p})$ with $c_{2j-1} > c_{2j}$,

(D) $(m_0, m_{2p+1})(m_1, m_2)\dots(m_{2p-1}, m_{2p})$ with $m_{2j} > m_{2j+1}$.

Let (V_k, ϵ_k) be a symplectic space if $\epsilon_k = -1$, orthogonal if $\epsilon_k = 1$, $k = 0, \dots, 2p$. ϵ_0 is the same as the type of the Lie algebra, dim V_0 is the sum of the columns. Let (V_k, ϵ_k) be the space with dimension the sum of the lengths of the columns labelled $\geq k$, $\epsilon_{k+1} = -\epsilon_k$. Then

$$(V_k, V_{k+1})$$

gives rise to a dual pair. The main result will be that unipotent representations corresponding to $\mathcal{O}_k$ are obtained from the unipotent representations corresponding to $\mathcal{O}_{k+1}$.

Metaplectic Correspondence, continued

More precisely,

- The matching of infinitesimal characters from the $\Theta-$correspondence applies.
- $\epsilon_0 = 1$. There is a $1-1$ correspondence between unipotent representations of $Sp(V_1)$ attached to $\mathcal{O}_1$ and unipotent representations of $SO(V_1)$ attached to $\mathcal{O} = \mathcal{O}_0$.
- $\epsilon_0 = -1$. There is a $1-1$ correspondence between unipotent representations of $O(V_1)$ attached to $\mathcal{O}_1$ and unipotent representations of $Sp(V_0)$ attached to $\mathcal{O} = \mathcal{O}_0$.

In the case $\epsilon_0 = -1$, the passage between unipotent representations of $SO(V)$ and $O(V)$ is done by pulling back and tensoring with the *sign* character of $O(V)$.

I use Weyl's conventions for parametrizing representations of $O(n)$. The proof is a straightforward application of [AB1].

Sketch of Proof

1. Adding a column longer than any existing columns changes the parity of the rows, and adds a number of rows of size 1. If we pass from $sp()$ to $so()$, another $\mathbb{Z}_2$ is added to $A(\mathcal{O})$. If we pass from $so()$ to $sp()$ the component group does not change.
2. $\epsilon_0 = -1$. If the pair is from type C to type D, $c_0, \ldots, c_{2p}$ are changed to $m_1, \ldots, m_{2p+1}$ and an m_0 is added. They are paired up $(m_0, c_{2p})(c_1, c_2)\ldots(c_{2p-2}, c_{2p-1})$. A parameter corresponding to a $(\eta_1, \ldots, \eta_p)$ goes to the corresponding one for type D.
   If the pair is from type C to type B, and $c_{2p} = 0$, then $c_0, \ldots, c_{2p-1}$ go to $m_1, \ldots, m_{2p}$ and an m_0 is added. If $c_{2p} \neq 0$, then $c_0, \ldots, c_{2p}$ go to $m_1, \ldots, m_{2p+1}$, a m_0 and a $m_{2p+2} = 0$ is added. The pairs are $(m_0)(c_0, c_1)\ldots(c_{2p-2}, c_{2p-1})(c_{2p}, 0)$ and $(\eta_1, \ldots, \eta_p)$ goes to the corresponding one for type B.
3. $\epsilon_0 = 1$. We have to use the more difficult matching in [AB1].

Motivating Example

The pair is $O(m) \times Sp(2n)$. The trivial representation of $O(m)$ corresponds to the partition (m), and it matches (m, m) in $Sp(2n)$. The trivial representation of $O(m)$ has parameter

$$\begin{pmatrix} \frac{m}{2}-1 & \dots & \frac{\tau}{2} \\ \frac{m}{2}-1 & \dots & \frac{\tau}{2} \end{pmatrix}$$

with $\tau = 0, 1$ depending on the parity of m. The infinitesimal character of $\Theta(Triv)$ is obtained by adding $(\frac{m}{2} - n, \dots, 1 - \frac{\tau}{2})$ to both λ_L and λ_R, and the prameter is the spherical representation

$$\begin{pmatrix} \frac{m}{2}-1 & \dots & \frac{\tau}{2}, \frac{\tau}{2}-1, \dots, n-\frac{m}{2} \\ \frac{m}{2}-1 & \dots & \frac{\tau}{2}, \frac{\tau}{2}-1, \dots, n-\frac{m}{2} \end{pmatrix}$$

in the case of Sgn the parameter of $\Theta(Sgn)$ is

$$\begin{pmatrix} \frac{m}{2}-1 & \dots & \frac{m}{2}-n+2 & \frac{m}{2}-n+1 & \dots & n-\frac{m}{2} \\ \frac{m}{2}-1 & \dots & \frac{m}{2}-n+2 & \frac{m}{2}-n & \dots & n-\frac{m}{2}-1 \end{pmatrix}$$

Example 4

Let $\mathcal{O} \longleftrightarrow (4,2,2)$ in $sp(8)$. It corresponds to $\mathcal{O} \longleftrightarrow (2,2)$ in $so(4)$. There are two such nilpotent orbits if we use $SO(4)$, one if we use $O(4)$. We will use orbits of the orthogonal group. The infinitesimal character corresponding to $(2,2)$ is $(1/2,1/2)$. The representations corresponding to $(4,2,2)$ have infinitesimal character $(1,0,1/2,1/2)$. The Langlands parameters are spherical

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 1/2 & 1/2 \\ 1 & 0 & 1/2 & 1/2 \end{pmatrix}$$

We can go further and match $(2,2)$ in $so(8)$ with (2) in $sp(2)$. If we combine these steps we get infinitesimal characters $(1) \mapsto (0,1) \mapsto (2,1,0,1)$. There is nothing wrong with the correspondence of irreducible modules. But note that the infinitesimal character $(2,1,1,0)$ has maximal primitive ideal corresponding to the orbit $\mathcal{O} \longleftrightarrow (4,4)$, rows $(2,2,2,2)$.

Example 4, continued

This is one of the reasons for imposing the conditions on the absence of certain equalities of the columns. We want to iterate, and stay within the class of unipotent representations.

In the absence of these restrictions one obtains induced modules with interesting composition series. In this example,

$$\mathrm{Ind}_{GL(2)xSp(4)}^{Sp(8)}[\chi \otimes \mathit{Triv}] =$$
$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix}.$$

The first two parameters are unipotent, corresponding to $\mathcal{O} \longleftrightarrow (4,4)$, the last one is bigger, the annihilator gives $(4,2,2)$. All these composition factors have nice character formulas analogous to those for the special unipotent representations; their annihilators are no longer maximal. Daniel Wong has made an extensive study of these representations in his thesis.

Example 4, continued

This example is tied up with the fact that nilpotent orbits are not always normal.

A nilpotent orbit is normal if and only if $R(\mathcal{O}) = R(\overline{\mathcal{O}})$. The orbit $(4, 2, 2)$ is **not normal**.

For $\mathcal{O} \longleftrightarrow (4, 2, 2)$, $R(\mathcal{O})$ equals the full induced representation from the previous slide, $R(\overline{\mathcal{O}})$ is missing the middle representation.
These equalities are in the sense that the $K-$types of the representations match the $G-$types of the regular functions, I am using the identification $K_c \cong G$.

K-P Model

We follow [Bry]. Let $(\mathcal{G}, K) := (\mathfrak{g}_0, K_0) \times \cdots \times (\mathfrak{g}_\ell, K_\ell)$ be the algebras corresponding to removing a column at a time. Each pair $(\mathfrak{g}_i, K_i) \times (\mathfrak{g}_{i+1}, K_{i+1})$ is equipped with a metaplectic representation Ω_i. Form $\Omega := \otimes\Omega_i$. The representation we are interested in, is

$$\Pi = \Omega/(\mathfrak{g}_1 \times \ldots \mathfrak{g}_\ell)(\Omega).$$

Let $(\mathfrak{g}^1, K^1) := (\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_\ell, K_1 \times \cdots \times K_\ell)$, and let $\mathfrak{g}^1 = \mathfrak{k}^1 + \mathfrak{p}^1$ be the Cartan decomposition.

Π is an admissible $(\mathfrak{g}_0, K_0)-$ module. It has an infinitesimal character compatible with the $\Theta-$correspondence. Furthermore the K_i which are orthogonal groups are disconnected, so the component group $\mathcal{K}^1 := K^1/(K^1)^0$ still acts, and commutes with the action of $(\mathfrak{g}_0, K_0)$. Thus Π decomposes

$$\Pi = \bigoplus \Pi_\Psi$$

where $\Pi_\Psi := \mathrm{Hom}_{K^1}[\Pi, \Psi]$. The main result in [Bry] is that

$$\Pi_{Triv} \mid_{K_0} = R(\overline{\mathcal{O}}).$$

K-P Model, continued

We would like to conclude that the representations corresponding to the other Ψ are the unipotent ones, and they satisfy an analogous relation to the above. The problem is that the component group of the centralizer of an element in the orbit does not act on $\overline{\mathcal{O}}$, so $R(\overline{\mathcal{O}}, \psi)$ does not make sense.

Let $\mathcal{V} := \prod \mathrm{Hom}[V_i, V_{i+1}]$. This can be identified with a Lagrangian. Consider the variety $\mathcal{Z} = \{(A_0, \ldots, A_\ell)\} \subset \mathcal{V}$ given by the equations $A_i^\star \circ A_i - A_{i+1} \circ A_i^\star = 0, \ldots, A_{\ell-1} \circ A_\ell^* = 0$. The detailed statement in [Bry] is that

$$Gr[\Omega/\mathfrak{p}^1\Omega] = R(\mathcal{Z}).$$

This is compatible with taking (co)invariants for $\mathfrak{k}^1$. To each character ψ of the component group of the centralizer of an element $e \in \mathcal{O}$, there is attached a character $\Psi \in \widehat{\mathcal{K}^1}$. Then

$$\Pi_\Psi \mid_{K_0} = R(\mathcal{Z}, \Psi).$$

Main Result

Theorem (Types B,C,D)

Let $\mathcal{O}$ be a stably trivial orbit. There is an explicit matchup

$$Unip(\mathcal{O}) \longleftrightarrow \widehat{A(\mathcal{O})}, \qquad \psi \longleftrightarrow X_\psi,$$

such that

$$X_\psi \mid_K \cong R(\mathcal{O}, \psi).$$

The matchup satisfies a compatibility with induction.

For the rest of the orbits, the best I can do is to show that there is a matchup such that

$$X_\psi \cong R(\mathcal{O}, \psi) - Y_\psi$$

where Y_ψ is a genuine $K-$character supported on strictly smaller orbits (*cf* [V]).
As in [V], the conjecture is that $Y_\nu = 0$.

Covers $Spin(m)$

Let $\mathcal{O} \longleftrightarrow (2n, 2n-1, 1)$ in $so(4n)$. Then $\lambda_{\mathcal{O}} = (n-1/2, \dots, 1/2, n-1, \dots, 1, 0)$. There are two complex representations:

$$\begin{pmatrix} n-1/2 & \dots & 1/2 & n-1 & \dots & 0 \\ n-1/2 & \dots & 1/2 & n-1 & \dots & 0 \end{pmatrix}$$
$$\begin{pmatrix} n-1/2 & \dots & 1/2 & n-1 & \dots & 0 \\ n-1/2 & \dots & -1/2 & n-1 & \dots & 0 \end{pmatrix}$$

Their $K-$structure is

$$(\epsilon + a_1, a_2, \dots, a_n), \tag{10}$$

with $\epsilon = 0, 1$ and $(a_1, \dots, a_n)$ in the root lattice. The relations to sections on equivariant line bundles on the orbit hold. The representations are obtained by the $\Theta-$correspondence as before.

Covers, continued

However if we consider $Spin(4n)$ there are two more ([Br]),

$$\begin{pmatrix} n-1/2 & \dots & 1/2 & 0 & \dots & -n+1 \\ n-1 & \dots & 0 & -1/2 & \dots & -n+1/2 \end{pmatrix}$$
$$\begin{pmatrix} n-1/2 & \dots & -1/2 & 0 & \dots & -n+1 \\ n-1 & \dots & 0 & -1/2 & \dots & -n+1/2 \end{pmatrix}$$

They cannot come from the $\Theta-$correspondence. The $K-$structure is

$$(1/2+a_1,\dots,1/2+a_{n-1},\pm 1/2+a_n) \tag{11}$$

with $(a_1,\dots,a_n)$ in the root lattice. They have analogous relations to the corresponding $R(\mathcal{O},\psi)$.

For the real case, Wan-Yu Tsai has made an extensive study of the analogues of these representations. They cannot come from the $\Theta-$correspondence either. In work in progress we have shown that they satisfy the desired relations to sections of equivariant bundles on orbits.

Type A, $SL(n,\mathbb{C})$

The center of the group is the cyclic group of order n. For the principal nilpotent (*cf* [G]), the infinitesimal character is

$$\chi = \left(\frac{n-1}{2n}, \frac{n-3}{2n}, \dots, -\frac{n-3}{2n}, -\frac{n-1}{2n}\right)$$

and the parameters are $\begin{pmatrix}\chi \\ \sigma\chi\end{pmatrix}$ with σ a cyclic permutation.

For a general nilpotent orbit with $(\underbrace{k,\dots,k}_{n_k},\dots,\underbrace{1,\dots,1}_{n_1})$ corresponding to the columns of the tableau, the infinitesimal character is

$$\chi = \begin{matrix} \frac{k-1}{2} + \frac{n_k-1}{2n_k} & \dots & \frac{k-1}{2} - \frac{n_k-1}{2n_k} & \dots \\ \frac{k-3}{2} + \frac{n_k-1}{2n_k} & \dots & \frac{k-3}{2} - \frac{n_k-1}{2n_k} & \dots \\ \dots & \dots & \dots & \dots \end{matrix}$$

and the parameters involve cyclic permutations that have to do with $gcd(n_k,\dots,n_1)$.

$SL(n,\mathbb{C})$, Examples

Example

The nilpotent orbit $\mathcal{O} \longleftrightarrow (2,1)$ has component group $A(\mathcal{O}) = 1$. There is only one parameter the spherical representation with infinitesimal character $(1/2,-1/2,0) \times (1/2,-1/2,0)$.

Example

$\mathcal{O} \longleftrightarrow (2,2)$ has component group $A(\mathcal{O}) = \mathbb{Z}_2$. The parameters are

$$\begin{pmatrix} 1/2+1/4 & 1/2-1/4 & -1/2+1/4 & -1/2-1/4 \\ 1/2+1/4 & 1/2-1/4 & -1/2+1/4 & -1/2-1/4 \end{pmatrix}$$
$$\begin{pmatrix} 1/2+1/4 & 1/2-1/4 & -1/2+1/4 & -1/2-1/4 \\ 1/2-1/4 & 1/2+1/4 & -1/2-1/4 & -1/2+1/4 \end{pmatrix} \tag{12}$$

The second parameter has lowest $K-$type the fundamental weight $(1/2,1/2,-1/2,-1/2)$.

$SL(n,\mathbb{C})$, Examples

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Real Groups

Let $\mathcal{S}_0$ be a (real) symplectic space, $\mathcal{S}_0 = \mathcal{L}_0 + \mathcal{L}_0^{\perp}$, a decomposition into transverse Lagrangians. Let $\mathcal{R}_0$ be an orthogonal space. The real form of the orthogonal group gives a decomposition $\mathcal{R}_0 = V_0 + W_0$ where the form is positive definite on V_0, negative definite on W_0. The complexifications of the Cartan decompositions of $sp(\mathcal{S}_0)$ and $so(\mathcal{R}_0)$ are given by

$$\begin{aligned} sp(\mathcal{S}) &= \mathfrak{k} + (\mathfrak{p}^+ + \mathfrak{s}^-) = \mathrm{Hom}[\mathcal{L},\mathcal{L}] + (\mathrm{Hom}[\mathcal{L}^{\perp},\mathcal{L}] + \mathrm{Hom}[\mathcal{L},\mathcal{L}^{\perp}]),\\ so(\mathcal{R}) &= \mathfrak{k} + \mathfrak{s} = (\mathrm{Hom}[\mathcal{V},\mathcal{V}] + \mathrm{Hom}[\mathcal{W},\mathcal{W}]) + \mathrm{Hom}[\mathcal{V},\mathcal{W}]. \end{aligned} \tag{14}$$

Note that due to the presence of the nondegenerate forms, $\mathrm{Hom}[\mathcal{L},\mathcal{L}] \cong \mathrm{Hom}[\mathcal{L}^{\perp},\mathcal{L}^{\perp}]$ and $\mathrm{Hom}[\mathcal{V},\mathcal{W}] \cong \mathrm{Hom}[\mathcal{W},\mathcal{V}]$. The canonical isomorphisms are denoted by $\ ^*$.

Consider the pair $sp(\mathcal{S}) \times o(\mathcal{R}) \subset sp(\text{Hom}[\mathcal{S}, \mathcal{R}])$. The space $\mathcal{X} := \text{Hom}[\mathcal{S}, \mathcal{R}]$ has symplectic nondegenerate form

$$\langle A, B \rangle := \text{Tr}(A \circ B^*) = \text{Tr}(B^* \circ A).$$

A Lagrangian subspace is provided by

$$\mathfrak{L} := \text{Hom}[\mathcal{L}, \mathcal{V}] + \text{Hom}[\mathcal{L}^\perp, \mathcal{W}]. \tag{15}$$

The moment map

$$\begin{aligned} m = (m_{sp}, m_{so}) : \mathfrak{L} \longrightarrow \mathfrak{sp}(\mathcal{S}) \times \mathfrak{o}(\mathcal{R}) \\ m(A, B) = (A^*A - B^*B, BA^* \cong AB^*) \end{aligned} \tag{16}$$

maps $\mathfrak{L}$ to $\mathfrak{s}(sp) \times \mathfrak{s}(so)$. It is standard that $m_{sp} \circ m_{so}^{-1}$ (and symmetrically $m_{so} \circ m_{sp}^{-1}$) take nilpotent orbits to nilpotent orbits. In this special case, the moment maps take nilpotent K_c-orbits on $\mathfrak{s}$ to nilpotent K_c-orbits on $\mathfrak{s}$ of the other group.

The K-P model has a straightforward generalization, one considers the columns of K_c-orbits on $\mathfrak{s}_c$.

Theorem (Type B,C,D)

Let $\mathcal{O} \subset \mathfrak{s}_c$ be a K_c-orbit such that $\mathcal{O}_c$ is stably trivial. There is a 1-1 correspondence between $Unip(\mathcal{O})$ and $\widehat{A(\mathcal{O})}$.
$X \in Unip(\mathcal{O}) \longleftrightarrow \psi \in \widehat{A(\mathcal{O})}$.

This uses the seesaw pairs determined by the columns. There is an explicit matchup $\psi \longleftrightarrow \Psi \in K^1/K^1_0$ such that onjecturally

$$X_\psi \mid_{K_1} \cong (R\mathcal{Z}, \psi).$$

Some unipotent representations won't have support just a single orbit. For them one has to consider seesaw pairs in the $\Theta-$correspondence other than those coming from the columns.

Example

Let $\mathcal{O} \longleftrightarrow (4,4,2,2) \subset sp(12,\mathbb{C})$. There are 9 K_c nilpotent orbits in $\mathfrak{s}_c$ for $sp(12,\mathbb{R})$. For example one of them can be encoded as

$$\begin{matrix} + & - & + & - \\ + & - & + & - \\ + & - & & \\ - & + & & \end{matrix} \tag{17}$$

The seesaw pairs are $Sp(12,\mathbb{R}) \times O(3,5) \times Sp(4,\mathbb{R}) \times O(0,2)$. There are $8 = 4 \times 2$ parameters, 4 coming from $O(3,5)$ and 2 from $O(0,2)$.

I don't know the Langlands parameters as explictly as in the complex case.

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