Unitary Spherical Dual, Split Groups

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The Unitarity Problem

Main Results

Petite K-types

1   Notation

G a real reductive group, the real points of a
linear reductive connected group $\mathbb{G}$,

$G = KS$ a Cartan decomposition corresponding to the
Cartan involution θ,

$\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the corresponding Lie algebras
and Cartan decomposition

$P = MAN$ a parabolic subgroup with $Lie(A) = \mathfrak{a} \subset \mathfrak{s}$.
$P_0 = M_0 A_0 N_0$ minimal parabolic subgroup

$\delta \in \widehat{M},\ \nu \in \widehat{A}$ characters, we identify characters of A
with elements of $\mathfrak{a}_{\mathbb{C}}^*$

$Ind_P^G[\delta \otimes \nu]$ normalized Harish-Chandra induction,
$(\delta \otimes \nu$ trivial on N, multiplied with $(\det \mathrm{Ad} a\ |_{\mathfrak{n}})^{-1/2}$).

2 Hermitian and Unitary Modules

Definition 2.0.1. *A representation $(\pi, \mathcal{X})$ on a Hilbert space $\mathcal{X}$ is called unitary if $\mathcal{X}$ admits a (nondegenerate) inner product $\langle \, , \, \rangle$ such that G acts by unitary transformations.*

More general, let $\mathcal{H}$ be an algebra. In order to talk about unitary representations of $\mathcal{H}$, we need a star operation on $\mathcal{H}$.

Definition 2.0.2. *A star operation for an (associative) algebra $\mathcal{H}$ is a complex conjugate linear anti-involution $\star : \mathcal{H} \longrightarrow \mathcal{H}$, i.e. satisfying $(ab)^* = b^* a^\star$, and $\star^2 = Id$.*

Example 2.0.3. *Let $\mathfrak{g}$ be a complex Lie algebra, $U(\mathfrak{g})$ its enveloping algebra. If $\mathfrak{g}_0$ is a real form, $\mathfrak{g}$ inherits a complex conjugate involution $^- : \mathfrak{g} \longrightarrow \mathfrak{g}$, which extends to a star operation on $U(\mathfrak{g})$. All star operations on $U(\mathfrak{g})$ which preserve the natural filtration are of this form.*

Definition 2.0.4. *A representation $(\pi, \mathcal{X})$ of $\mathcal{H}$ is called hermitian if $\mathcal{X}$ has a hermitian form $\langle \, , \, \rangle$ (usually nondegenerate) such that*
$$\langle \pi(h)v_1, v_2 \rangle = \langle v_1, \pi(h^*)v_2 \rangle.$$

Unitarity Problem

Classify the unitary irreducible representations of G.

In general, in order to classify the unitary dual of a group, one proceeds in three steps:

STEP 0. Classify admissible irreducible $(\mathfrak{g}, K)$ modules.

A module $(\pi, \mathcal{X})$ of $(\mathfrak{g}, K)$ satisfying

$$\pi(k)\pi(X)v = \pi(\mathrm{Ad}kX)\pi(k)v$$

is called admissible, if $\pi(K)v$ is finite dimensional for any v, and any $K-$isotypic component of $\mathcal{X}$ is finite dimensional.

This is due to Harish-Chandra.

STEP 1. Classify the irreducible admissible $(\mathfrak{g}, K)$ modules.

STEP 2. Classify the irreducible admissible $(\mathfrak{g}, K)$ modules that admit invariant hermitian forms.

STEP 3. Classify those that admit positive definite invariant forms.

The rest of the talk will be about the spherical case.

3 Spherical Modules

Definition 3.0.5. *A(n admissible) representation $(\pi, \mathcal{X})$ is called spherical if $\mathcal{X}^K \neq (0)$.*

3.1 Principal Series

G is split. In this case P_0 is a Borel subgroup, and we denote it by $B = MAN$. Write $X(B, \delta, \nu) := Ind_B^G[\delta \otimes \nu]$. This is not only normalized induction, but we consider the subspace of $K-$finite functions, an admissible $(\mathfrak{g}, K)-$module.

Theorem 3.1.1.

a) *Assume $\langle Re\nu, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta(\mathfrak{n})$. Then $X(B, \delta, \nu)$ has a unique quotient $\overline{X}(\delta, \nu)$ which is a direct sum of irreducible modules.*

b) *If $\langle Re\nu, \alpha \rangle \leq 0$, then $X(\delta, \nu)$ has a unique submodule $\overline{X}(\delta, \nu)$ which is a direct sum of irreducible modules.*

c) *$\overline{X}(\delta, \nu) \cong \overline{X}(\delta', \nu')$ if and only if there exists $w \in W := N_K(\mathfrak{a})/M$ such that $w\delta \cong \delta'$, and $w\nu = \nu'$.*

d) *Let $\overline{B} := MA\overline{N}$ be the opposite parabolic subgroup. Then there is an intertwining operator*

$$A(B, \overline{B}, \delta, \nu) : X(B, \delta, \nu) \longrightarrow X(\overline{B}, \delta, \nu)$$

whose image is $\overline{X}(\delta, \nu)$.

e) *Any spherical irreducible module is the spherical subquotient of an $X(triv, \nu)$.*

f) *$\overline{X}(\delta, \nu)$ is hermitian if and only if there is a $w \in W$ such that $w\delta \cong \delta$, and $w\nu = -\overline{\nu}$.*

In the case of spherical modules, $\delta = triv$, so we simplify the notation to $X(\nu)$ and $\overline{X}(\nu)$.

Remark 3.1.2 ([Kn], Chapter 16). *Write $\nu = Re\nu + iIm\nu$. Let $P = MN$ be the parabolic subgroup such that*

$$\Delta(\mathfrak{m}) = \{\alpha \mid (\alpha^\vee, Im\nu) = 0\},$$
$$\Delta(\mathfrak{n}) = \{\alpha \mid (\alpha^\vee, Im\nu) > 0\}.$$

Then

$$\overline{X}(\nu) = Ind_P^G[\overline{X}_M(Re\nu) \otimes iIm\nu].$$

$\overline{X}(\nu)$ is unitary if and only if $\overline{X}(Re\nu)$ is unitary.

So assume from now on that ν is real.

4   The Shape of the Unitary Dual

The following pictures describe the spherical unitary dual of the two classical rank two split groups, $SO(2,3)$ and $Sp(4,\mathbb{R})$. The reducibility hyperplanes divide the dominant cone in regions of constant signature; the unitary dual is the union of the simplicial complexes with positive signature.

$\boxed{SO(2,3)_0}$
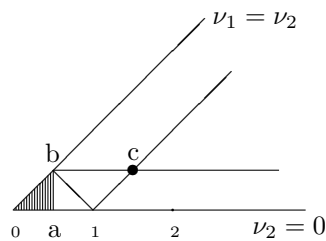
$\boxed{Sp(4)}$
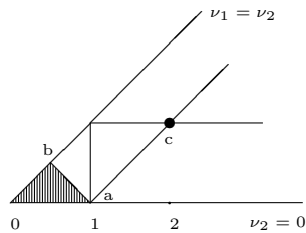
4.1 Complementary Series

The set of spherical irreducible representations of G with real infinitesimal character can be partitioned into *series* attached to the nilpotent orbits in $\mathfrak{g}^\vee$. Connected reductive groups are classified by root data, and they come in pairs, groups and their *duals*. Let $G^\vee$ be the (complex) dual group, and let $A^\vee$ be the torus dual to A. Then $\mathfrak{a}^* \otimes_\mathbb{R} \mathbb{C}$ is canonically isomorphic to $\mathfrak{a}^\vee$, the Lie algebra of $A^\vee$. So we can regard χ as an element of $\mathfrak{a}^\vee$. We attach to each χ a nilpotent orbit $\mathcal{O}^\vee(\chi)$ as follows. By the Jacobson-Morozov theorem, there is a 1-1 correspondence between nilpotent orbits $\mathcal{O}^\vee$ and $G^\vee-$conjugacy classes of Lie triples $\{e^\vee, h^\vee, f^\vee\}$; the correspondence satisfies $e^\vee \in \mathcal{O}^\vee$. Choose the Lie triple such that $h^\vee \in \mathfrak{a}^\vee$. Then there are many $\mathcal{O}^\vee$ such that χ can be written as $w\chi = h^\vee/2 + \nu$ with $\nu \in \mathfrak{z}(e^\vee, h^\vee, f^\vee)$, the centralizer in $\mathfrak{g}^\vee$ of the triple. For example this is always possible with $\mathcal{O}^\vee = (0)$. Results of B-Moy guarantee that for any χ there is a unique $\mathcal{O}^\vee(\chi)$ satisfying
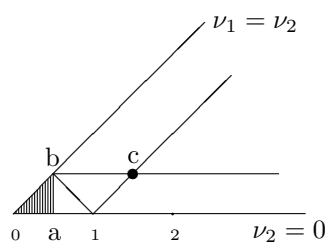
1. there exists $w \in W$ such that $w\chi = \frac{1}{2}h^\vee + \nu$ with
   $\nu \in \mathfrak{z}(e^\vee, h^\vee, f^\vee)$,

2. if χ satisfies property (1) for any other $\mathcal{O}^{\vee\prime}$, then
   $\mathcal{O}^{\vee\prime} \subset \overline{\mathcal{O}^\vee}(\chi)$.

Definition 4.1.1. *Let $\mathcal{O}^\vee$ be a nilpotent orbit and let $\chi = \frac{1}{2}h^\vee + \nu$ be a spherical parameter attached to $\mathcal{O}^\vee$. Then χ is in the complementary series of $\mathcal{O}^\vee$ if and only if $\overline{X}(\chi)$ is unitary.*

If the nilpotent orbit $\mathcal{O}^\vee$ is *distinguished* in $\mathfrak{g}^\vee$ (*i.e.* $\mathcal{O}^\vee$ does not meet any proper Levi subalgebra of $\check{\mathfrak{g}}$), then the only spherical parameter attached to $\mathcal{O}^\vee$ is $\chi = \frac{1}{2}\check{h}$.

Here are the deconstructed pictures for $SO(2,3)_o$ and $Sp(4,\mathbb{R})$..

$\boxed{SO(2,3)_0}$

c ●	$\check{\mathcal{O}} = (4)$
b ●	$\check{\mathcal{O}} = (2^2)$
b ○ ┃ ● a	$\check{\mathcal{O}} = (21^2)$
b ○ ┆ ○ 0 a	$\check{\mathcal{O}} = (1^4)$

$\boxed{Sp(4)}$

c ●	$\check{\mathcal{O}} = (5)$
a ●	$\check{\mathcal{O}} = (3,1,1)$
b ●＼ ○ a	$\check{\mathcal{O}} = (2,2,1)$
b ○ ○ 0 a	$\check{\mathcal{O}} = (1,1,1,1,1)$

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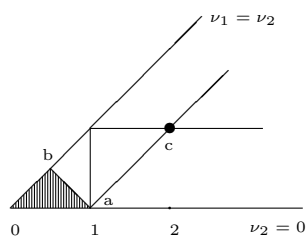
4.2 Main Result

Definition 4.2.1. *The complementary series attached to the trivial nilpotent orbit is called the 0-complementary series.*

Theorem 4.2.2. *The parameter $\chi = \frac{1}{2}h^\vee + \nu$ is in the complementary series of $\mathcal{O}^\vee$ if and only if ν is in the 0-complementary series of $\mathfrak{z}(\mathcal{O}^\vee)$, with the following exceptions. For*

$F_4\colon \ A_1 + \widetilde{A}_1$

$E_7\colon \ A_2 + 3A_1$

$E_8\colon \ A_4 + A_2 + A_1, \ A_4 + A_2, \ D_4(a_1) + A_2, \ A_3 + 2A_1, \ A_2 + 3A_1$

the complementary series of $\mathcal{O}^\vee$ is smaller than 0-complementary series of $\mathfrak{z}(\mathcal{O}^\vee)$, and for

$E_8\colon \ 4A_1$

the complementary series of $\mathcal{O}^\vee$ is bigger than 0-complementary series of $\mathfrak{z}(\mathcal{O}^\vee)$.

The labeling of the nilpotent orbits is from the Bala-Carter classification.

This theorem is due to [B] and [B-Ciubotaru], and holds as stated for p-adic split groups. It holds due to [B] for split classical real groups. For exceptional split real groups, [B-Ciubotaru] prove that the unitary dual is contained in this set.

The 0-complementary series are known, and have a simple description in terms of hyperplane arrangements (given by the roots). They are unions of simplices.

4.3   Complementary Series for the Classical Cases

Theorem 4.3.1. *The complementary series attached to $\mathcal{O}^\vee$ coincides with the one attached to the trivial orbit in $\mathfrak{z}(\mathcal{O}^\vee)$. For the trivial orbit (0) in each of the classical cases, the complementary series are*

G of type B

$$0 \le \nu_1 \le \cdots \le \nu_k < 1/2.$$

G of type C, D

$$0 \le \nu_1 \le \cdots \le \nu_k \le 1/2 < \nu_{k+1} < \cdots < \nu_{k+l} < 1$$

so that $\nu_i + \nu_j \le 1$. There are

1. *an even number of ν_i such that $1 - \nu_{k+1} < \nu_i \le 1/2$,*

2. *for every $1 \le j \le l$, there is an odd number of ν_i such that $1 - \nu_{k+j+1} < \nu_i < 1 - \nu_{k+j}$.*

3. *In type D of odd rank, $\nu_1 = 0$ or else the parameter is not hermitian.*

4.4   Exceptional Cases

The centralizers are listed in [Car]. In most cases the centralizers are products of classical groups. Here are some pictures for the exceptions (courtesy of Dan Ciubotaru).
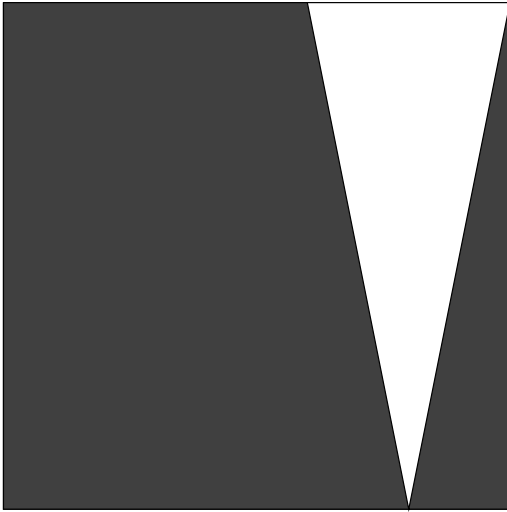

Figure 1: Spherical unitary parameters for the nilpotent orbit A_4A_2 in E_8

5 Intertwining Operators

5.1

Formally the intertwining operator $A(B, \overline{B}, \nu)$ is given by

$$A(B, \overline{B}, \nu)f(x) = \int_{\overline{N}} f(x\overline{n}) \, d\overline{n}. \tag{1}$$

It is more convenient for our purpose to modify the definition to $I(w_0, \nu) : X(B, \nu) \longrightarrow X(B, w_0\nu)$,

$$I(w_0, \nu)f(x) = \int_N f(xnw_0) \, dn,$$

where w_0 is the long Weyl group element. This definition immediately extends to any $w \in W$,

$$I(w, \nu)f(x) = \int_{N/(N \cap w^{-1}Nw)} f(xnw) \, dn. \tag{2}$$

The convergence of the integrals is a nontrivial matter.
If (μ, V) is a K-type, then I induces a map

$$I_V(w, \nu) : \operatorname{Hom}_K[V, X(\nu)] \longrightarrow \operatorname{Hom}_K[V, X(w\nu)]. \tag{3}$$

By Frobenius reciprocity, we get a map

$$R_V(w, \nu) : (V^*)^M \longrightarrow (V^*)^M. \tag{4}$$

In case (μ, V) is trivial the spaces are 1-dimensional and $I_V(w, \nu)$ is a scalar. We normalize $I(w, \nu)$ so that this scalar is 1. The $R_V(w, \nu)$ are meromorphic functions in ν, and the $I(w, \nu)$ have the following additional properties.

1. If $w = w_1 \cdot w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$, then $I(w, \nu) = I(w_1, w_2\nu) \circ I(w_2, \nu)$. In particular if $w = s_{\alpha_1} \cdots s_{\alpha_k}$ is a

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reduced decomposition, then $I(w)$ factors into a product of intertwining operators I_j, one for each s_{α_j}. These operators are

$$I_j \ : \ X(s_{\alpha_{j+1}} \ldots s_{\alpha_k} \cdot \nu) \longrightarrow X(s_{\alpha_j} \ldots s_{\alpha_k} \cdot \nu) \qquad (5)$$

2. Let $P = MN$ be a standard parabolic subgroup (so $A \subset M$) and $w \in W(M, A)$. Write $X(\nu) := Ind_P^G[X_M(\nu)]$. The intertwining operator

$$I(w, \nu) \ : \ X(\nu) \longrightarrow X(w\nu)$$

is of the form $I(w, \nu) = Ind_M^G[I_M(w, \nu)]$.

3. If $Re\langle \nu, \alpha \rangle \geq 0$ for all positive roots α, then $R_V(w_0, \nu)$ has no poles, and the image of $I(w_0, \nu)$ ($w_0 \in W$ is the long element) is $\overline{X}(\nu)$.

4. If there is w such that $w\nu = -\nu$, (ν is assumed real), then the hermitian dual of $X(\nu)$ is $X(-\nu)$. Letting w be the shortest element such that $w\nu = -\nu$, the hermitian form

$$\langle v_1, v_2 \rangle := \big(v_1, I(w, \nu)v_2\big) \qquad (6)$$

is nonzero and has radical the maximal proper submodule of $X(\nu)$, so that it descends to a nondegenerate hermitian form on $\overline{X}(\nu)$.

Let (μ, V) be a K-type, and fix a positive definite inner product. The map $R_V(w, \nu)$ of equation (4) depends meromorphically on ν, and is normalized to be the identity on the trivial K-type. It has no poles when ν is dominant with respect to the positive roots for B. Via the fixed positive inner product on V, this map induces a hermitian form $r(\mu, \nu)$ on $(V^*)^M$, and $\overline{X}(\nu)$ is unitary if and only if $r(\mu, \nu)$ is positive semidefinite for all μ.

Example 5.1.1. *In the case of $G = Sl(2, \mathbb{R})$, the K-types of the spherical principal series are parametrized by even integers. Since $K-$types are $1-$dimensional, the $r(\mu, \nu)$ can be viewed as scalars:*

$$r(2m, \nu) = \prod_{0 \leq j < m} \frac{2j + 1 - (\nu, \check{\alpha})}{2j + 1 + (\nu, \check{\alpha})}. \tag{7}$$

They completely determine which $\overline{X}(\nu)$ are unitary. The scalars are simplest when $m \leq 1$, and they determine which representations are **not unitary**. This is the motivation for the definition of *petite K$-$types* in the next section. They were called single petaled in [Oda], and studied independently and for different reasons.

6   Petite K-types

Let α be a simple root and $P_\alpha = M_\alpha N$ be the standard parabolic subgroup so that the Lie algebra of M_α is isomorphic to the $sl(2, \mathbb{R})$ generated by the root vectors $E_{\pm\alpha}$. We assume that $\theta E_\alpha = -E_{-\alpha}$. Let $D_\alpha = \sqrt{-1}(E_\alpha - E_{-\alpha})$ and $s_\alpha = e^{\sqrt{-1}\pi D_\alpha/2}$. Then $s_\alpha^2 = m_\alpha$ is in $M \cap M_\alpha$. Since the square of any element in M is in the center and M normalizes the the root vectors, $\mathrm{Ad}m(D_\alpha) = \pm D_\alpha$. Grade $V^* = \oplus V_i^*$ according to the absolute values of the eigenvalues of D_α (which are integers). Then M preserves this grading and

$$(V^*)^M = \bigoplus_{i \ even} (V_i^*)^M.$$

The map $\psi_\alpha : sl(2, \mathbb{R}) \longrightarrow \mathfrak{g}$ determined by

$$\psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_\alpha, \qquad \psi_\alpha \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = E_{-\alpha}$$

determines a map

$$\Psi_\alpha \; : \; SL(2, \mathbb{R}) \longrightarrow G \tag{8}$$

with image G_α, a connected group with Lie algebra isomorphic to $sl(2, \mathbb{R})$. Let R_α be the maps (4) for G_α.

Proposition 6.0.2. *On $(V_{2m}^*)^M$,*

$$R_V(s_\alpha, \nu) = \begin{cases} Id & \text{if } m = 0, \\ \prod_{0 \le j < m} \frac{2j+1-\langle \nu, \check{\alpha} \rangle}{2j+1+\langle \nu, \check{\alpha} \rangle} \; Id & \text{if } m \neq 0. \end{cases}$$

In particular, $I(w, \nu)$ is an isomorphism unless $\langle \nu, \check{\alpha} \rangle \in -\mathbb{N}$.

Proof. The formula is well known for $SL(2, \mathbb{R})$. The second assertion follows from this and the listed properties of intertwining operators. $\square$

Definition 6.0.3. *A K-type μ is called petite or single petaled if all the $I(s_\alpha)$ are as in Proposition 6.0.2 with $0 \le m \le 1$.*

This is true whenever the eigenvalues $\mu(iZ_\alpha)$ are $0, \pm 1, \pm 2$ or ± 3.

Corollary 6.0.4. *For petite K-types the formula is*

$$R_V(s_\alpha, \nu) = \begin{cases} Id & \text{on the +1 eigenspace of } s_\alpha, \\ \frac{1-\langle \nu, \check{\alpha} \rangle}{1+\langle \nu, \check{\alpha} \rangle} \; Id & \text{on the -1 eigenspace of } s_\alpha. \end{cases}$$

When restricted to $(V^*)^M$, the long intertwining operator is the product of the $R_V(s_\alpha, \nu)$ corresponding to the reduced decomposition of w_0 and depends only on the Weyl group structure of $(V^*)^M$.

Proof. The element s_α acts by 1 on the zero eigenspace of D_α and by -1 on the ± 2 eigenspace. Eigenvalue ± 3 does not play any role for K-types occuring in the spherical principal series. $\qquad\square$

Definition 6.0.5. *A set of K-types $\mathcal{R}$ is called relevant, if it is formed of petite K-types, and a representation $\overline{X}(\nu)$ is unitary if and only the form is positive definite on all the K-types in $\mathcal{R}$.*

We want such a set to be rather small. The main point is for it to give necessary onditions for unitarity. These $K-$types provide the link with the unitarity problem for split p-adic groups.

7 The Graded Affine Hecke Algebra

Notation:

- $\Phi = (V, R, V^\vee, R^\vee)$ an $\mathbb{R}-$root system, reduced.

- W the Weyl group.

- $\Pi \subset R$ simple roots, R^+ positive roots.

- $k : \Pi \to \mathbb{R}$ a function such that $k_\alpha = k_{\alpha'}$ whenever $\alpha, \alpha' \in \Pi$ are W-conjugate.

Definition 7.0.6 (Graded Affine Hecke Algebra). $\mathbb{H} = \mathbb{H}(\Phi, k) \cong \mathbb{C}[W] \otimes S(V_\mathbb{C})$ *such that*

(i) $\mathbb{C}[W]$ and $S(V_\mathbb{C})$ have the usual algebra structure,

(ii) $\omega t_{s_\alpha} = t_{s_\alpha} s_\alpha(\omega) + k_\alpha \langle \omega, \check\alpha \rangle$ for all $\alpha \in \Pi$, $\omega \in V_\mathbb{C}$.

Assume $k_\alpha = 1$ for simplicity, sufficient for split groups and spherfical representations. The module $X(\nu) := \mathbb{H} \otimes_{V_\mathbb{C}} \mathbb{C}_\nu$ is called the (spherical) principal series. As a $\mathbb{C}[W]-$module it is

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isomorphic to $\mathbb{C}[W] = \sum_{\mu \in \widehat{W}} V_\mu \otimes V_\mu^*$. For every simple root α, there is an element $r_\alpha = (t_{s_\alpha}\alpha - 1)(\alpha - 1)^{-1}$. One needs to extend the definition from $S(V_\mathbb{C})$ to rational functions, but never mind. These elements satisfy the braid relations, so we can define r_w for any $w \in W$. These elements define intertwining operators

$$A_w : X(\nu) \longrightarrow X(w\nu)$$

by the formula $h \otimes \mathbb{1}_\nu \mapsto hr_w \otimes \mathbb{1}_{w\nu}$.

Proposition 7.0.7. *If ν is dominant, the operators r_w are well defined. Since they are intertwining operators, they induce operators $r_\mu(w, \nu)$ on each $(V_\mu)^*$. The formulas coincide with the ones for $R_V(w, \nu)$ fdrom before whenever μ comes from a petite $K-$type.*

The affine graded Hecke algebra also has a star,

$$*(t_w) = t_{w^{-1}}, \qquad *(v) = t_{w_0} a(v) t_{w_0},$$

where a is the isomorphism $-w_0$ on V. In view of this, one can look for the unitary dual of $\mathbb{H}$.

Remark 7.0.8. *The unitary dual formed of representations with Iwahori spherical representations of a p-adic group can be computed from the unitary dual of Hecke algebras as given above, [BM1], [BM2],[BC], [C] and others.*

The role of the relevant $K-$types is to show that the set of parameters of spherical unitary representations embed in the set of parameters of spherical unitary representations of the Hecke algebra, and via the result in the remark, embed in the spherical unitary dual of the p-adic group.

8   Explicit Relevant K-types

Classical Cases

Type A

$G = SL(n, \mathbb{R})$, and $K = SO(n)$. A set of relevant representations is

K-type   W-representation

$(\underbrace{2, \ldots, 2}_{k}, 0, \ldots, 0)$ $(n - k, k)$

Type B

$G = SO(n + 1, n)$, and $K = S[O(n + 1) \times O(n)]$. We can use $O(n + 1) \times O(n)$ or $SO(n + 1) \times SO(n)$ for this purpose.

A set of relevant representations is

K-type   W-representation

$(0, \ldots, 0) \otimes (\underbrace{2, \ldots, 2}_{\ell}, 0, \ldots, 0)$ $(n - \ell, \ell) \times (0)$

$(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0) \otimes (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$ $(n - k) \times (k), \quad k \leq [n/2],$

$(\underbrace{1, \ldots, 1}_{[(n+1)/2] - k}, 0, \ldots, 0) \otimes (\underbrace{1, \ldots, 1}_{[n/2] - k}, 0, \ldots, 0)$ $(n - k) \times (k), \quad k > [n/2],$

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Type C

$G = Sp(n)$, and $K = U(n)$. A set of relevant representations is

K-type	W-representation on $(V^*)^M$
$(\underbrace{2,\ldots,2}_{\ell},0,\ldots,0)$	$(n-\ell) \times (\ell)$
$(\underbrace{1,\ldots,1}_{k},0,\ldots,0)$	$(n-k,k) \times (0)$.

Type D

$G = SO(n,n)$, and $K = S[O(n) \times O(n)]$. A set of relevant representations is

K-type	W-representation on $(V^*)^M$
$(0,\ldots,0) \otimes (\underbrace{2,\ldots,2}_{\ell},0,\ldots,0)$	$(n-\ell,\ell) \times (0)$,
$(\underbrace{1,\ldots,1}_{k},0,\ldots,0) \otimes (\underbrace{1,\ldots,1}_{k},0,\ldots,0)$	$(n-k) \times (k)$.

Exceptional Cases

For these cases work of D. Ciubotaru in the case of F_4 and Ciubotaru and myself for E_6, E_7 and E_8, find Weyl group representations which form relevant sets for the p-adic group. If we can find K-types in the real cases for which the W-representations on $(V^*)^M$ are formed of relevant W-types, then we get powerful necessary conditions for unitarity.

8.1 Type F4

$G = F_4$, $K = Sp(1) \times Sp(3)/\pm I$. Let T be a maximal compact Cartan subgroup. We use the standard positive system and

roots, for K :
$$\{2\epsilon_1,\ 2\epsilon_k,\ \epsilon_k \pm \epsilon_\ell\}_{2 \leq k \leq \ell \leq 4}. \tag{9}$$
The highest weight of a $\widetilde{K}$-type will be denoted
$$(a_1 \mid a_2, a_3, a_4, a_5), \qquad a_i \in \mathbb{N}, \quad a_2 \geq a_3 \geq a_4 \geq 0. \tag{10}$$

The representations of the Weyl group of type F4 are parametrized as in [L1].

Theorem 8.1.1 ([C]). *A spherical representation of $\mathbb{H}$ of type F4 is unitary if and only if the r_σ are positive semidefinite on the W-types*
$$1_1,\ 2_3,\ 8_1,\ 4_2,\ 9_1.$$

The next result was obtained joint with D. Vogan.

Proposition 8.1.2. *The following list consists of petite K-types matching the relevant W representations.*

$K - type$	W-type on $(V^*)^M$
$(0 \mid 0, 0, 0)$	$1_1,$
$(0 \mid 1, 1, 0)$	$2_1,$
$(4 \mid 0, 0, 0)$	$2_3,$
$(1 \mid 2, 1, 0)$	$8_1,$
$(1 \mid 1, 1, 1)$	$4_2,$
$(2 \mid 2, 0, 0)$	$9_1.$

E6

$G = E6$, $K = Sp(4)/\{\pm I\}$. The W-types are parametrized as in [L1].

Theorem 8.1.3 ([BC]). *A spherical representation of $\mathbb{H}$ of type E_6 is unitary if and only if the r_σ are positive semidefinite on the W-types*

$$1_p, \ 6_p, \ 20_p, \ 30_p, \ 15_q.$$

We denote by ω_i the fundamental weights of $sp(4)$. In coordinates they are

$$\begin{aligned}
\omega_1 &= (1, 0, 0, 0), \\
\omega_2 &= (1, 1, 0, 0), \\
\omega_3 &= (1, 1, 1, 0), \\
\omega_4 &= (1, 1, 1, 1).
\end{aligned} \tag{11}$$

Proposition 8.1.4. *The following list consists of the petite K-types which have no nontrivial M-fixed vectors, and the underlying Weyl group representations.*

$K-type$	$W\text{-type on } (V^*)^M$
$(0) = (0, 0, 0, 0)$	$1_p,$
$\omega_4 = (1, 1, 1, 1)$	6_p
$2\omega_2 = (2, 2, 0, 0)$	$20_p,$
$4\omega_1 = (4, 0, 0, 0)$	$15_q,$
$2\omega_1 + \omega_4 = (3, 1, 1, 1)$	$30_p,$
$\omega_1 + \omega_2 + \omega_3 = (3, 2, 1, 0)$	$64_p,$
$3\omega_1 + \omega_3 = (4, 1, 1, 0)$	$60_p,$
$2\omega_3 = (2, 2, 2, 0)$	$15_p,$
$2\omega_1 + 2\omega_2 = (4, 2, 0, 0)$	$81_p,$
$3\omega_2 = (3, 3, 0, 0)$	$24_p,$
$6\omega_1 = (6, 0, 0, 0)$	$24'_p.$

E7

$$K = SU(8)/\{\pm Id\}. \tag{12}$$

Proposition 8.1.5. *Petite K-types with M-spherical vectors, and the corresponding Weyl group representations:*

$K - type$	$W\text{-type on } (V^*)^M$
(0)	$1_a,$
ω_4	$7'_a$
$2\omega_2,\ 2\omega_6$	$21'_b,$
$\omega_2 + \omega_6$	$27_a,$
$2\omega_1 + 2\omega_7$	$35_b,$
$4\omega_1,\ 4\omega_7$	$15'_a,$
$\omega_2 + \omega_3 + \omega_7,\ \omega_1 + \omega_5 + \omega_6$	$105'_a,$
$\omega_1 + \omega_4 + \omega_7$	$56'_a,$
$2\omega_1 + \omega_3 + \omega_7,\ \omega_1 + \omega_5 + 2\omega_7$	$189'_b,$
$\omega_1 + \omega_3 + \omega_6,\ \omega_2 + \omega_5 + \omega_7$	$168_a,$
$3\omega_1 + \omega_5,\ \omega_3 + 3\omega_7$	$105_b,$
$\omega_3 + \omega_5$	$21_a,$
$\omega_1 + \omega_2 + \omega_5,\ \omega_3 + \omega_6 + \omega_7$	$120_a,$
$\omega_1 + \omega_2 + \omega_6 + \omega_7,$	$168_a + 210_a,$
$3\omega_1 + 3\omega_7,$	$84_a + 105_c,$
$\omega_1 + 2\omega_2 + \omega_7, \omega_1 + \omega_6 + \omega_7,$	$189'_c$
$3\omega_1 + \omega_2 + \omega_7, \omega_1 + \omega_6 + 3\omega_7,$	$216'_a,$
$3\omega_1 + \omega_6 + \omega_7, \omega_1 + \omega_2 + 3\omega_7,$	$280_b,$
$5\omega_1 + \omega_7, \omega_1 + 5\omega_7,$	$84'_b.$

The Weyl group representations are parametrized as in [L1].

Theorem 8.1.6 ([BC]). *A spherical representation of $\mathbb{H}$ of type E7 is unitary if and only if r_σ is positive semidefinite for*

$$1_a, \ 7'_a, \ 27_a, \ 56'_a, \ 21'_b, \ 35_b, \ 105_b.$$

E8

The maximal compact subgroup of the split real form of the simply connected complex group of type E8 is

$$Spin(16)/\{Id, \omega\}, \tag{13}$$

for ω the appropriate element in the center (the quotient is **not** SO(16)). The group $\widetilde{M}$ is $\mathbb{Z}_2^8$.

The representations of the Weyl group are parametrized as in [L1].

Theorem 8.1.7. *A spherical representation of $\mathbb{H}$ of type E8 is unitary if and only if r_σ is positive semidefinite for*

$$1_x, \ 8_z, \ 35_x, \ 50_x, \ 84_x, \ 112_z, \ 400_z, \ 300_x, \ 210_x.$$

Proposition 8.1.8. *The following list gives petite K-types and the corresponding Weyl group representations on* $(V^*)^M$:

$K-type$	W-type on $(V^*)^M$
(0)	$1_x,$
ω_8	8_z
ω_4	$35_x,$
$2\omega_2$	$84_x,$
$\omega_2 + \omega_8$	$112_z,$
$4\omega_1$	$50_x,$
$3\omega_1 + \omega_7$	$400_z,$
$\omega_3 + \omega_7$	$160_z,$
ω_6	$28_x,$
$\omega_1 + \omega_5$	$210_x,$
$\omega_1 + \omega_2 + \omega_7$	$560_x,$
$\omega_2 + \omega_4$	$567_x,$
$2\omega_3$	$300_x,$
$2\omega_1 + \omega_4$	$700_x,$
$3\omega_1 + \omega_3$	$1050_x,$
$\omega_1 + \omega_2 + \omega_3$	$1344,$
$3\omega_2$	$525,$
$2\omega_1 + 2\omega_2$	$972,$
$4\omega_1 + \omega_2$	$700,$
$6\omega_1$	$168.$

8.2 Petite $K-$types in the Classical Cases

Let $(V, \langle\ ,\ \rangle)$ be a finite dimensional space with a nondegenerate symplectic or orthogonal form, $G(V)$ the corresponding group with maximal compact subgroup $K(V)$ and Lie algebra $\mathfrak{g}(V)$. Assume that $G(V)$ is split, the dimension of V is $2n$ or $2n+1$ in the orthogonal cases, $2n$ in the symplectic case.

Proposition 8.2.1 ([CT]). *There is a (1−dimensional) representation $\mu \in \widehat{K}$ such that*

$$\mathrm{Hom}_M[triv, V^{\otimes^{\dim V}} \otimes V_\mu] \cong \mathbb{C}[W].$$

Furthermore, the $K-$types contributing are single petaled.

Recall that $W-types$ for W of type B, C are parametrized by pairs of partitions of $n = \mathrm{rk}(G)$ and for type D they are parametrized by restriction from type B to type D.

Proposition 8.2.2 ([Gu]). *Each $W-type$ is realized as V^M of a single petaled $K-type$ as follows. In the matchup below, the $W-$types have to be tensored with sgn. This has the effect of taking the transposes of the partitions and interchanging them.*

B,D: *Let $O(a, b)$ be the group. To a pair of partitions $(\alpha' \mid \beta') = (\alpha'_1, \ldots, \alpha'_q \mid \beta'_1, \ldots, \beta'_{n-q})$ associate $(\alpha_1, \ldots, \alpha_a \mid \beta_1, \ldots, \beta_b)$ where the α_i, β_j are the α'_i, β'_j padded by zeroes to make a and b coordinates respectively. The $K-type$ has highest weight*

$$(\alpha_1 - \alpha_a, \alpha_2 - \alpha_{a-1}, \cdots \mid \beta_1 - \beta_b, \beta_2 - \beta_{b-1}, \ldots)$$

with $[a/2]$ and $[b/2]$ coordinates respectively. The procedure is that $(\alpha' \mid \beta')$ is an irreducible representation of $U(a) \times U(b)$, and we are taking the highest weight component of its restriction to $O(a) \times O(b)$.

C: *The pair of partitions (α', β') arises as V^M for the representation with highest weight*

$$(\alpha'_1, \ldots, \alpha'_q, 0, \ldots, 0, -\beta_{n-q}, \ldots, -\beta_1) + (1, \ldots, 1).$$

A sharpening of these results is the following.

Corollary 8.2.3. *The component with $M-$fixed vectors in $V^{\otimes \dim V} \otimes V_\mu]$ is*

$$\sum_\mu \dim V_{\sigma(\mu)} \otimes V_\mu$$

where $\mu \longleftrightarrow \sigma(\mu)$ is the matching between highest weights and partitions in Proposition 8.2.2.

Related to this is the following.

Theorem 8.2.4 ([CT]). *Let X be any admissible spherical $(\mathfrak{g}, K)-$module. The space $\mathrm{Hom}_K[X, V^{\otimes \dim V} \otimes V_\mu]]$ admits an action of $\mathbb{H}$. In particular if $X = X_G(\nu)$ a principal series , the resulting module is the principal series, $\mathrm{Hom}_K[X(\nu), V^{\otimes \dim V} \otimes V_\mu] = X_\mathbb{H}(\nu).$*

Some Proofs

Let $\widetilde{K}$ be a compact group, $\widetilde{M} \subset \widetilde{N}$ finite groups such that M is normal in N. Denote by W the quotient $\widetilde{N}/\widetilde{M}$. In the applications, $\widetilde{N}$ is the normalizer of $\mathfrak{a}$ in $\widetilde{K}$, which is the simply connected cover of the maximal compact subgroup of the rational points of the simply connected complex group E_8.

Let V_a, V_b be representations of K. In some cases we assume that the restrictions of V_a, V_b to M are multiples of the same representation V_δ. We also assume that V_δ extends to a representation of $\widetilde{K}$. This is the case for δ_{16} and genuine representations of $\widetilde{K}$.

Proposition 8.2.5. *There is a natural action of W on $\mathrm{Hom}_M[V_a, V_b]$.*

Proof. The action on $f \in \mathrm{Hom}[V_a, V_b]$ is given by

$$(n \cdot f)(v) = \mu_a(n^{-1}) f(\mu_b(n)v). \tag{14}$$

This is clearly an action of N. Since $nm = (nmn^{-1})n$, and $\widetilde{M}$ is normal, $nmn^{-1} \in \widetilde{M}$. Then the fact that the action does not depend on the right M-coset, follows from the fact that f is an M-homomorphism. $\qquad\square$

This action is compatible with the canonical isomorphism

$$\mathrm{Hom}_M[V_a, V_b] \cong [V_a^* \otimes V_b]^M. \tag{15}$$

It is also compatible with the isomorphism

$$\mathrm{Hom}_M[V_a, V_b]^* \cong \mathrm{Hom}[V_b^*, V_a^*]. \tag{16}$$

This action is a generalization of the usual one on V^M. We apply it to the case of *fine* K-types. A K-type (μ, V) is called fine, if

$\mu(iZ_\alpha)$ equals $0, \pm 1$ only. Its restriction to M is formed of a single orbit, under W, of characters of M. Every $\delta \in \widehat{M}$ belongs to (possibly several) fine K-types. Let W_δ be the centralizer of a δ. It has the following structure. The coroots $\check{\alpha}$ such that $\delta(m_\alpha) = 1$ form a roots system $^\vee\Delta^\delta$, and the group generated by the corresponding reflections, W_δ^0 is a subgroup of W_δ. We assume that G is simply connected. In this case, let $R_\delta := \{w \in W \mid w(^\vee\Delta^\delta) = ^\vee \Delta^\delta\}$. This is a product of $\mathbb{Z}_2'2$. Then $W_\delta = W_\delta^0 \ltimes R_\delta$. For a fixed δ, let $S(\delta)$ be the set of fine K-types containing it. Then $\widehat{R_\delta}$ acts transitively. For each W conjugacy class in $\widehat{M}$, fix a δ and a μ_δ. Then $S(\delta)$ is in 1-1 correspondence with $\widehat{R_\delta}$ in such a way that the trivial represntation corresponds to μ_δ. Suppose $\tau \in \widehat{R_\delta}$ corresponds to μ. The representation τ gives rise to an irreducible representation of W_δ by extending it trivially to the stabilizer of τ in W_δ^0, and inducing up.

Proposition 8.2.6. $\text{Hom}_M[\mu_\delta, \mu] \cong Ind_{W_\delta}^W[\tau]$.

Example 1

Consider the simply connected real form of E_8 The group $\widetilde{M}$ has size 2^9, and its quotient by the center in (13) is $\mathbb{Z}_2^8$. In this case $K = Spin(16)$, and denote by ω_i the fundamental weights. In all cases, $W_\delta = W_\delta^0$. The fine K-types are

$\widetilde{K} - type$	$\widetilde{M} - type$	
(0)	δ_1, trivial representation,	
ω_1	δ_{16}, sixteen dimensional representation,	(17)
ω_2	δ_{120}, one hundred and twenty characters,	
$2\omega_1$	δ_{135}, one hundred thirty five characters,	

Only the second representation is genuine, and the others are single orbits under the action of W.

Let (μ, V) be the fine K-type ω_2. Then the W-representation is

$$\mathrm{Hom}_M[V,V] \cong Ind_{W(E7A1)}^{W(E8)}[triv] = 1_x + 35_x + 84_x. \qquad (18)$$

Similarly, if (μ, V) is the fine K-type $2\omega_1$,

$$\mathrm{Hom}_M[V,V] \cong Ind_{W(D8)}^{W(E8)}[triv] = 1_x + 84_x + 50_x. \qquad (19)$$

These representations are self-dual, and their tensor products are

$$\omega_2 \otimes \omega_2 = (2\omega_2) + (\omega_1 + \omega_3) + (2\omega_1) + (\omega_2) + (\omega_4) + (0),$$
$$(2\omega_1) \otimes (2\omega_1) = (4\omega_1) + (2\omega_1 + \omega_2) + (2\omega_1) + (2\omega_2) + (\omega_2) + (0).$$
$$(20)$$

Similarly ω_3 restricts to $35\delta_{16}$, and

$$\omega_1 \otimes \omega_3 = (\omega_1 + \omega_3) + (\omega_2) + (\omega_4) \qquad (21)$$

Thus the multiplicity of δ_1 in $(\omega_1 + \omega_3) + (\omega_4)$ is 35. On the other hand, $\dim \omega_4 = 8020$, so the multiplicity of δ_1 in ω_4 is nonzero. From (20) it follows that the multiplicity is exactly 35, and in fact

$$\omega_4 \longleftrightarrow 35_x, \quad 2\omega_2 \longleftrightarrow 84_x. \qquad (22)$$

We also conclude that the multiplicity of δ_1 in $\omega_1 + \omega_3$ is zero.

Consider $\omega_1 + \omega_2$ which restricts to $84\delta_{16}$, so δ_1 occurs 84 times. Then

$$(\omega_1 + \omega_2) \otimes \omega_1 = (2\omega_1 + \omega_2) + (2\omega_1) + (\omega_1 + \omega_3) + (2\omega_2) + (\omega_2). \qquad (23)$$

Thus only $2\omega_2$ contains δ_1. This also implies

$$\omega_3 \longleftrightarrow 35_x. \qquad (24)$$

Combined with (20) we get

$$4\omega_1 \longleftrightarrow 50_x. \qquad (25)$$

Example 2

Let $G = Sp(n, \mathbb{R})$. The two K-types with highest weights

$$\mu_+(k) := (\underbrace{1, \ldots, 0}_{k}, 0, \ldots, 0), \qquad \mu_-(k) := (0, \ldots, 0, \underbrace{-1, \ldots, -1}_{k})$$

are fine K-types containing the same orbit of a character $\delta \in \widehat{M}$. The stabilizer $W_\delta^0 \cong W(D_k) \times W(C_{n-k})$, and $W_\delta \cong W(C_k) \times W(C_{n-k})$. The two representations corresponding to $\mu_\pm(k)$ are

$$[(k) \times (0)] \otimes [(n-k) \times (0)], \qquad [(0) \times (k)] \otimes [(n-k) \times (0)].$$

The correspnding induced modules are

$$\sum (n - \ell, \ell) \times (0), \qquad 0 \le \ell \le min(k, n-k),$$
$$(n - k) \times (k).$$

The corresponding tensor products are

$$\sum (\underbrace{1, \ldots, 1}_{a}, \underbrace{0, \ldots, 0}_{b}, -1, \ldots, -1),$$
$$\sum (\underbrace{2, \ldots, 2}_{a}, \underbrace{1, \ldots, 1}_{b}, 0, \ldots, 0)$$

These K-types are automatically petite, and in fact satisfy $\mu(iZ_\alpha) = 0, \pm 1, \pm 2$. With some extra work it is possible to derive the result stated earlier.

In the exceptional cases, we need cases when $\mu(iZ_\alpha) = \pm 3$ as well.

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