A G_δ IDEAL OF COMPACT SETS STRICTLY ABOVE THE NOWHERE DENSE IDEAL IN THE TUKEY ORDER

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ABSTRACT. We prove that there is a G_δ σ-ideal of compact sets which is strictly above NWD in the Tukey order. Here NWD is the collection of all compact nowhere dense subsets of the Cantor set. This answers a question of Louveau and Veličković asked in [4].

1. INTRODUCTION

Given two directed partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$, we say that P is *Tukey reducible* to Q, in symbols $P \leq_T Q$, if there exists a function $f : P \to Q$ such that for each $q \in Q$, $\{p \in P : f(p) \leq_Q q\}$ is bounded in P. Tukey reducibility is used to compare the cofinal structure of directed partial orders. The reader is referred to the literature cited in [8] for a glimpse of the work done so far on the subject. Each directed partial order is easily seen to be Tukey bi-reducible with the ideal of its bounded subsets ordered by inclusion. Thus the study of Tukey reducibility is equivalent to the study of Tukey reducibility of ideals of sets ordered by inclusion.

In order to rule out pathologies and make the study more tractable, it is natural to impose additional structural and definability requirements on the ideals under consideration. Recently in [8] the class of *basic orders* was introduced as a class which both included many of the motivating examples and for which a broad theory of Tukey reduction can be developed. The class of basic orders includes two subclasses of ideals that have emerged as playing a fundamental role. They are the

analytic P-ideals of subsets of ω and the *analytic σ-ideals of compact sets* in a fixed compact metric space. These two classes are referred to in [8], as the *measure leaf* and the *category leaf*, respectively. Within each class, one obtains a further automatic reduction in descriptive complexity: the analytic P-ideals are all $F_{\sigma\delta}$ (i.e. $\mathbf{\Pi}^0_3$) [6] and the analytic σ-ideals of compact sets are all G_δ (i.e. $\mathbf{\Pi}^0_2$) [3].

It was proved in [4] that the ideal

$$\{x \subseteq \omega : \sum_{n \in x} \frac{1}{n+1} < \infty\}$$

is the largest element with respect to the Tukey reduction in the measure leaf. Louveau and Veličković asked in [4] if the ideal of nowhere dense compact subsets of 2^ω is the largest element with respect to the Tukey reduction in the category leaf. In Theorem 3.1, we answer this question in the negative. In fact, we show that there exists a G_δ, σ-ideal of compact subsets of 2^ω that is Tukey strictly above the nowhere dense ideal. It should be noted, however, that in [7] a condition was isolated which is fulfilled by all "naturally occurring" analytic σ-ideals of compact sets and which insures that the ideal is Tukey reducible to the nowhere dense ideal.

In this note the letters i, j, k, l, m, and n will always represent elements of the set ω of natural numbers. Hence "$j < 2$" should be interpreted as meaning that j comes from the set $\{0,1\}$. We will use FF to denote the collection of all functions whose domain is a finite, non-empty subinterval of ω and whose range is contained in $\{0,1\}$. We will use $\bar{0}$ and $\bar{1}$ to denote the constant sequences of unspecified (possibly infinite) length. If σ is in FF, $[\sigma]$ will be used to denote the clopen set $\{x \in 2^\omega : \sigma \subseteq x\}$. If X is a topological space, $\mathrm{NWD}(X)$ will be used to denote the collection of all closed nowhere dense subsets of X. For brevity, we will write NWD for $\mathrm{NWD}(2^\omega)$.

2. The definition of $\mathscr{I}_0$ and its basic properties

The first examples of "exotic" G_δ σ-ideals of compact sets were constructed by Mátrai in [5] in order to answer a question of Kechris from [2]. The G_δ σ-ideal of compact sets, which will be used to prove our theorem, was defined in [7, Section 6] and was called there $\mathscr{I}_0$. We will retain this notation here. We will recall the definition of $\mathscr{I}_0$ and, for completeness, re-prove that it is a G_δ. This proof also draws out some

of the important features of the ideal. (After the present paper was completed, Mátrai showed that also his ideal from [5] does not Tukey reduce to NWD. It is not clear if it is Tukey strictly above NWD.)

In order to define the ideal $\mathscr{I}_0$, it will be useful to introduce some notation. If x is in 2^ω and Π is a partition of ω into finite intervals, let $R(x, \Pi)$ be the set of all y in 2^ω such that for every $i < \omega$ there is a $j < 2$ such that

$$x \upharpoonright \Pi(2i + j) = y \upharpoonright \Pi(2i + j).$$

Here $\Pi(i)$ is the element of Π with the ith least minimum.

We will let $\mathscr{R}$ denote the collection of all $R(x, \Pi)$ as x and Π vary over elements of 2^ω and partitions consisting of finite non-empty intervals, respectively. Elements of $\mathscr{R}$ will be referred to as *test spaces*. The ideal $\mathscr{I}_0$ of interest to us is the collection of all compact $K \subseteq 2^\omega$ such that $K \cap R$ is nowhere dense in R for every test space R.

While $\mathscr{I}_0$ is clearly co-analytic, it is not immediately apparent that $\mathscr{I}_0$ is G_δ. If s is in FF, let $\mathscr{R}_s$ denote the set of all non-empty intersections of the form $R \cap [s]$ such that R is in $\mathscr{R}$. For brevity, $R \cap [s]$ will be denoted by $R[s]$. Notice that membership to $\mathscr{I}_0$ is equivalent to not containing an element of some $\mathscr{R}_s$.

If Π is a partition of ω into finitely many intervals, then we say that Π is *degenerate*. In this case we can still define $R(x, \Pi)$ provided some extra care is taken. Let l be the number of elements of Π. If l is even, then $R(x, \Pi)$ is defined as in the non degenerate case except that $i < \omega$ is replaced by $i < l/2$. If $l = 2k + 1$, then $R(x, \Pi)$ is the set of all y such that if $i < k$, then there is a $j < 2$ such that

$$x \upharpoonright \Pi(2i + j) = y \upharpoonright \Pi(2i + j).$$

Observe that if Π consists of an odd number of intervals, then $R(x, \Pi)$ is clopen and if Π consists of an even number of intervals, then $R(x, \Pi)$ is the union of a clopen set and a finite set of points which are each eventually equal to x.

The reason for considering degenerate partitions is that the set of all partitions of ω into intervals is compact when equipped with its natural topology. Moreover, the map sending a pair (x, Π) to $R(x, \Pi)$ is easily seen to be continuous. Since degenerate partitions give rise to sets $R(x, \Pi)$ with interior, the complement of $\mathscr{I}_0$ is the union of the closures of the sets $\mathscr{R}_s$ as s ranges over FF. In particular, $\mathscr{I}_0$ is a G_δ set.

The following propositions will be needed to establish that NWD is below $\mathscr{I}_0$ in the Tukey order.

Proposition 2.1. *If $R(x_0, \Pi_0) \subseteq R(x_1, \Pi_1)$ are test spaces, then $x_0 = x_1$ and $\Pi_0 = \Pi_1$.*

Proof. We will first show that if $R(x_0, \Pi_0) \subseteq R(x_1, \Pi_1)$, then $\Pi_0 = \Pi_1$. First note that for each i there is k such that $\max \Pi_1(2i) = \max \Pi_0(2k)$; otherwise it is easy to find a point in $R(x_0, \Pi_0)$ which differs from x_1 at $\max \Pi_1(2i)$ and $\min \Pi_1(2i+1)$ (and hence is not in $R(x_1, \Pi_1)$). Second, for each i, $\Pi_1(2i) \cup \Pi_1(2i + 1)$ is intersected by at most two intervals of Π_0; otherwise there would be an element of $R(x_0, \Pi_0)$ which differs from x_1 at $\min(\Pi_1(2i))$ and $\max(\Pi_1(2i + 1))$. These two conditions together imply that $\Pi_0 = \Pi_1$.

Now we are left to show that if $R(x_0, \Pi) \subseteq R(x_1, \Pi)$, then $x_0 = x_1$. Suppose that $x_0(n) \neq x_1(n)$ for some n and let $i < \omega$ and $j < 2$ be such that n is in $\Pi(2i + j)$. Define y in 2^ω so that $y(k) = x_0(k)$ if k is not in $\Pi(2i + 1 - j)$ and $y(k) = 1 - x_1(k)$ if k is in $\Pi(2i + 1 - j)$. Notice that

$$y \upharpoonright \Pi(2i + j) = x_0 \upharpoonright \Pi(2i + j) \neq x_1 \upharpoonright \Pi(2i + j)$$

$$y \upharpoonright \Pi(2i + 1 - j) \neq x_1 \upharpoonright \Pi(2i + 1 - j).$$

Hence y is in $R(x_0, \Pi) \setminus R(x_1, \Pi)$.   □

Proposition 2.2. *If an element of $\mathscr{R}$ contains an element of $\mathscr{R}_s$, then the latter has non-empty interior in the former.*

Proof. Suppose that $R(x_1, \Pi_1)$ is an element of $\mathscr{R}$ and $R(x_0, \Pi_0)[s]$ is an element of $\mathscr{R}_s$ such that $R(x_0, \Pi_0)[s] \subseteq R(x_1, \Pi_1)$. Fix i_0 and i_1 such that

$$\max(\mathrm{dom}(s)) < \min(\Pi_0(2i_0)) \leq \min(\Pi_1(2i_1)).$$

Set $m = \min(\Pi_1(2i_1))$.

We will first show that m is in an element of Π_0 of even index. Suppose that this is not the case and let $i < \omega$ be such that m is in $\Pi_0(2i + 1)$. If $n = \min(\Pi_1(2i_1 + 1))$ is in $\Pi_0(2i + 1)$, then since $\mathrm{dom}(s)$ does not intersect $\Pi_0(2i)$ (by choice of i_0 and i_1), there is a y in $R(x_0, \Pi_0) \cap [s]$ which differs from x_1 at m and n. The point here is that we are free to put $y \upharpoonright \Pi_0(2i) = x_0 \upharpoonright \Pi_0(2i)$, leaving us uncommitted to y's restriction to $\Pi_0(2i + 1)$ and in particular to its values at m and

n. If n is not in $\Pi_0(2i+1)$, then pick $l > i$ and j such that n is in $\Pi_0(2l+j)$. Then find a y in $R(x_0, \Pi_0) \cap [s]$ by first arranging that y agrees with x_0 on $\Pi_0(2i)$ and $\Pi_0(2l+j)$. We are then free to set the values of y at m and n which are in $\Pi_0(2i+1)$ and $\Pi_0(2l+1-j)$ to something other than $x_1(m)$ and $x_1(n)$.

Hence m in an element of Π_0 of even index. Extend s to $\bar{s}$ such that $R(x_0, \Pi_0) \cap [\bar{s}]$ is non-empty and $\mathrm{dom}(\bar{s}) = m$. Observe that, for $j < 2$,

$$R_j = \{y \in 2^\omega : \bar{s}^\frown y \in R(x_j, \Pi_j)\}$$

is a test space and that $R_0 \subseteq R_1$. It follows from Proposition 2.1 that Π_0 and Π_1 define the same partition on $\omega \setminus m$ and that x_0 and x_1 agree on $\omega \setminus m$. Consequently,

$$R(x_0, \Pi_0) \cap [\bar{s}] = R(x_1, \Pi_1) \cap [\bar{s}]$$

and therefore $R(x_0, \Pi_0)[s]$ has interior in $R(x_0, \Pi_0)$.

$\square$

3. $\mathscr{I}_0$ IS STRICTLY ABOVE NWD

Theorem 3.1. *The ideal $\mathscr{I}_0$ is strictly above* NWD *in the Tukey ordering.*

First we will show that NWD $\leq_T \mathscr{I}_0$. Let R be any test space. Since R is homeomorphic to 2^ω, it is sufficient to show that $\mathrm{NWD}(R) \leq_T \mathscr{I}_0$. It follows immediately from Proposition 2.2, however, that $\mathrm{NWD}(R) \subseteq \mathscr{I}_0$ and that, moreover, the inclusion map is a Tukey reduction.

We will now prove the following lemma which will be used to show that $\mathscr{I}_0$ is strictly above NWD in the Tukey order.

Lemma 3.2. *Suppose $f : 2^\omega \to$ NWD is Baire measurable and V_i $(i < \omega)$ enumerates a clopen basis for 2^ω. There are x in 2^ω and sequences $\langle m_i : i < \omega \rangle$ and $\langle n_i : i < \omega \rangle$ in ω such that:*

    (1) *$n_0 = 0$ and $n_i < n_{i+1}$;*

    (2) *V_{m_i} is a subset of V_i and if $i < j$ and $V_i = V_j$, then $V_{m_j} \subseteq V_{m_i}$;*

    (3) *if y is in 2^ω, then either the set of $i < \omega$ such that y extends $x \upharpoonright [n_i, n_{i+1})$ is finite or else whenever y extends $x \upharpoonright [n_i, n_{i+1})$, then $f(y)$ is disjoint from V_{m_i}.*

Proof. Let f and V_i $(i < \omega)$ be given as in the statement of the lemma. Suppose that, for some k, we have constructed $\langle m_i : i < k \rangle$, $\langle n_i : i < k \rangle$, and $x \upharpoonright n_{k-1}$. Suppose further that we have arranged that, for each

$i < k - 1$, $f^{-1}(\{K \in \mathrm{NWD} : K \cap V_{m_i} = \emptyset\})$ is comeager in $[\sigma_i]$ where $\sigma_i = x \restriction [n_i, n_{i+1})$ and that we have fixed a decreasing sequence of open sets $U_i^j \subseteq [\sigma_i]$ $(j < \omega)$ with U_i^j dense in σ_i for all j and with

$$\bigcap_{j=0}^{\infty} U_i^j \subseteq f^{-1}(\{K \in \mathrm{NWD} : K \cap V_{m_i} = \emptyset\}).$$

The following claim will be useful.

Claim. Suppose that for some $k < \omega$, $\{\sigma_i\}_{i<k}$ is a sequence of elements of FF and $\{U_i\}_{i<k}$ is a sequence of open sets such that $U_i \subseteq [\sigma_i]$ is dense in $[\sigma_i]$ for each $i < k$. If $n < \omega$ is such that $\mathrm{dom}(\sigma_i) \subseteq n$ for all i, then there is a τ in FF with $n = \min(\mathrm{dom}(\tau))$ such that $[\sigma_i \cup \tau] \subseteq U_i$ for each $i < k$.

Proof of Claim. Let $\{\sigma_i\}_{i<k}$, $\{U_i\}_{i<k}$, and n be given as in the statement of the lemma. Construct τ_ξ $(\xi \in 2^n)$ such that the domain of τ_ξ is interval, τ_ξ is an initial part of τ_η whenever $\xi <_{\mathrm{lex}} \eta$, $\min(\mathrm{dom}(\tau_\xi)) = n$ for all ξ in 2^n, and if ξ extends σ_i for some $i < k$, then $[\xi \cup \tau_\xi]$ is a subset of $U_i \cap [\xi]$. The proof is finished by setting τ equal to $\tau_{\bar{1}}$. $\square$

Returning to the proof of Lemma 3.2, the claim allows us to find a τ in FF such that $\min(\mathrm{dom}(\tau)) = n_{k-1}$ and for all $i < k$, $[\sigma_i \cup \tau] \subseteq U_i^k$. By the Baire category theorem, there is an $m_k < \omega$ such that $V_{m_k} \subseteq V_k$ and

$$[\sigma \cup \tau] \cap f^{-1}(\{K \in \mathrm{NWD} : K \cap V_{m_k} = \emptyset\})$$

is non-meager where $\sigma = \bigcup_{i<k} \sigma_i$. Since f is Baire measurable, we can fix n_k and $\sigma_k : [n_{k-1}, n_k) \to 2$ such that σ_k extends τ and $f^{-1}(\{K \in \mathrm{NWD} : K \cap V_{m_k} = \emptyset\})$ is comeager in $[\sigma \cup \sigma_k]$. The recursion is finished by fixing a decreasing sequence U_k^j $(j < \omega)$ of open dense subsets of $[\sigma \cup \sigma_k]$ satisfying

$$\bigcap_{j=0}^{\infty} U_k^j \subseteq f^{-1}(\{K \in \mathrm{NWD} : K \cap V_{m_k} = \emptyset\}).$$

Now suppose that y is in 2^ω and that there are infinitely many $i < \omega$ such that $\sigma_i \subseteq y$. If $\sigma_i \subseteq y$, then we claim that y is in U_i^j for all $j < \omega$ and consequently $f(y)$ is disjoint from V_{m_i}. In order to see this, let j be given. Pick a $k > j$ such that $\sigma_k \subseteq y$. Then y is in $U_k \subseteq U_j$, as desired. $\square$

We are now ready to finish the proof of Theorem 3.1. Suppose for contradiction that $F : \mathscr{I}_0 \to$ NWD is a Tukey reduction. By [8, Theorem 5.3(i)], we may assume without loss of generality that F is measurable with respect to the σ-algebra generated by analytic sets. Define $f : 2^\omega \to$ NWD by setting $f(x) = F(\{x\})$. Now, f is also measurable with respect to the σ-algebra generated by analytic sets and, therefore, it is Baire measurable. Let V_i $(i < \omega)$ be an enumeration of the non-empty clopen subsets of 2^ω such that for all i, $V_{2i+1} = V_{2i}$. Let x, $\langle m_i : i < \omega \rangle$, and $\langle n_i : i < \omega \rangle$ be given as in the conclusion of Lemma 3.2. Define

$$E = 2^\omega \setminus \bigcup_{i=0}^\infty V_{m_{2i+1}}$$

and let R be the set of all y in 2^ω such that for every $i < \omega$, there is a $j < 2$ such that

$$y \upharpoonright [n_{2i+j}, n_{2i+j+1}) = x \upharpoonright [n_{2i+j}, n_{2i+j+1}).$$

The set R is a test space and hence $\{\{y\} : y \in R\}$ is unbounded in $\mathscr{I}_0$. On the other hand, E is nowhere dense since if $U \subseteq 2^\omega$ is open, there is an $i < \omega$ such that $V_i \subseteq U$ and then $V_{m_{2i+1}} \subseteq V_{m_{2i}} \subseteq V_i$ is disjoint from E. Also, if y is in R, then $f(y)$ is disjoint from $V_{m_{2i+1}}$ for any $i < \omega$ and consequently $F(\{y\}) \subseteq E$ for every y in R. It follows that $F^{-1}(\{K \in \text{NWD} : K \subseteq E\})$ is unbounded, contradicting our assumption that F was a Tukey map. The theorem is therefore proved.

Two remarks about the proof above are in order. First, Proposition 2.2 shows that whenever L is a compact set which is positive with respect to $\mathscr{I}_0$, there is a positive compact set $K \subseteq L$ such that both $\mathscr{I}_0 \upharpoonright K \leq_T$ NWD and NWD $\leq_T \mathscr{I}_0 \upharpoonright K$. This suggests the following question: If $\mathscr{J}$ is a G_δ σ-ideal of compact sets, must every $\mathscr{J}$-positive compact set contain a $\mathscr{J}$-positive compact set K such that $\mathscr{J} \upharpoonright K \leq_T$ NWD? Second, in order to deny the existence of a Tukey function from $\mathscr{I}_0$ to NWD, we needed to analyze its restriction to the set of all singletons of the underlying space. This occurs frequently in proofs of non-existence of Tukey functions defined on ideals of compact sets. Its first appearance we found in [1, 3M Proposition (a)].

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