WEAK DIAMOND AND OPEN COLORINGS

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Abstract. The purpose of this article is to prove the relative consistency of certain statements about open colorings with $2^{\aleph_0} < 2^{\aleph_1}$. In particular both OCA and the statement that every 1-1 function of size $\aleph_1$ is σ-monotonic are consistent with $2^{\aleph_0} < 2^{\aleph_1}$. As a corollary we have that $2^{\aleph_0} < 2^{\aleph_1}$ does not admit a $\mathbb{P}_{\max}$ variation (in the presence of an inaccessible cardinal).

The open coloring axiom, OCA, is a Ramsey theoretic statement relating to the real numbers. Since its introduction in [17] it has found a wealth of applications to problems closely related to the real line (see [5], [7], [13], [17], [19], and [20]). Frequently, applications of OCA require an application of $\mathrm{MA}_{\aleph_1}$ to complete the argument (see, e.g., [7], [20]). Farah gave some explanation of this phenomenon through the following theorem.

Theorem 0.1. [6] *OCA is relatively consistent with*

(1) $\mathfrak{t} = \aleph_1$.
(2) There are no Q-sets.
(3) There are two $\aleph_1$ dense sets of reals which are not order isomorphic.

Since items 1-3 are all consequences of weak diamond (an equivalent of $2^{\aleph_0} < 2^{\aleph_1}$ — see [4]) this suggests the question of whether OCA is relatively consistent with weak diamond. In this paper I will show that this is indeed the case. I will also show the relative consistency of weak diamond with "Every partial 1-1 function $f \subseteq \mathbb{R}^2$ of size $\aleph_1$ is the union of countably many monotonic functions." An application of the later result is that there is no $\mathbb{P}_{\max}$ variation for weak diamond (see [21] for a discussion of $\mathbb{P}_{\max}$ models and their variants). In particular there are two consequences of the bounded form of PFA which jointly imply $2^{\aleph_0} = 2^{\aleph_1}$ but which are each relatively consistent with $2^{\aleph_0} < 2^{\aleph_1}$ (one of them modulo an inaccessible cardinal).

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1. Weak diamond and the open coloring axiom

In this section I will prove that the open coloring axiom of [17] is relatively consistent with $2^{\aleph_0} < 2^{\aleph_1}$. Recall that the open coloring axiom is the following assertion:

OCA: If X is a separable metric space and $G \subseteq [X]^2$ is open then either
(1) there is a decomposition of X into countably many pieces X_i such that $[X_i]^2 \cap G$ is empty for all i or
(2) there is an uncountable $H \subseteq X$ such that $[H]^2 \subseteq G$.

Here $[X]^2$ is the set of unordered pairs of distinct elements of X and open subsets of $[X]^2$ are those induced by symmetric open subsets of X^2.

Definition 1.1. If $G \subseteq [X]^2$ is an open graph then let $\mathcal{H}(X, G)$ be the partial order of all finite complete subgraphs of (X, G) ordered by reverse inclusion. Let $\mathcal{H}^\omega(X, G)$ be the finite support product of ω many copies of $\mathcal{H}(X, G)$.

We will need the following lemma which is a corollary of the proof of Theorem 4.4 in [17].

Lemma 1.2. *Let $G \subseteq [\omega^\omega]^2$ be an open graph. There is a countably closed forcing $\mathcal{P}_G$ which introduces a set $\dot{X} \subseteq \omega^\omega$ such that*
(1) if $Y \subseteq \omega^\omega$ is in the ground model and induces an uncountably chromatic subgraph of G then $\check{Y} \cap \dot{X}$ is forced to be uncountable and
(2) $\mathcal{H}^\omega(\dot{X}, G)$ is forced to satisfy the countable chain condition.

Proof. If $U \subseteq (\omega^\omega)^n$ is open and $p \in (\omega^\omega)^n$ set

$$U_p = \{q \in U : \forall i < n\{q(i), p(i)\} \in G\}.$$

If $f : (\omega^\omega)^n \to \omega^\omega$ is a partial function with domain A then let

$$\omega^*_f(p) = \bigcap\{\overline{f''(U_p \cap A)} : p \in U \text{ and } U \text{ is open}\}$$

Let $\mathcal{P}_G$ be all triples of countable sequences $\langle T_\xi : \xi < \alpha\rangle$, $\langle g_\xi : \xi < \alpha\rangle$, and $\langle x_\xi : \xi < \alpha\rangle$ of closed subgraphs of G, countable partial functions from $(\omega^\omega)^n$ to ω^ω, and elements of ω^ω respectively which satisfy
(1) $[T_\xi]^2 \cap G$ is empty,
(2) if $\xi < \eta < \alpha$ then x_η is not in T_ξ, and
(3) if $\xi < \eta < \alpha$, the domain of g_ξ is contained in $(\omega^\omega)^n$, p is an n-tuple of distinct elements of $\{x_\gamma : \gamma < \eta\}$, and $\omega^*_{g_\xi}(p)$ is an empty subgraph of G then x_η is not in $\omega^*_{g_\xi}(p)$.

Set $\dot{X}$ to be the elements of the generic sequence $\langle x_\xi : \xi < \omega_1 \rangle$. The first conclusion of our theorem follows from genericity. Also note that the generic sequences $\langle T_\xi : \xi < \omega_1 \rangle$ and $\langle g_\xi : \xi < \omega_1 \rangle$ enumerate the closed subgraphs of G satisfying (1) above and the countable partial functions in $\mathbf{V}^{\mathcal{P}_G}$. The proof of Theorem 4.4 now yields that $\mathcal{H}^\omega(\dot{X}, G)$ satisfies the c.c.c.. [1] □

Theorem 1.3. *OCA is relatively consistent with weak diamond.*

Proof. Start with a ground model $\mathbf{V}$ satisfying CH together with $\aleph_2 < 2^{\aleph_1}$. Construct a countable support iteration of length ω_2 of partial orders of size $\aleph_1$ as follows. At stage $\alpha < \omega_2$ we have a book-keeping device which gives us a $\mathcal{Q}_\alpha$ name $\dot{G}_\alpha$ for an open graph. Define

$$\mathcal{Q}_{\alpha+1} = \mathcal{Q}_\alpha * \mathcal{P}_{\dot{G}_\alpha} * \mathcal{H}^\omega(\dot{X}_\alpha, \dot{G}_\alpha).$$

The book-keeping is done in such a way that every $\mathcal{Q}_{\omega_2}$ name for an open graph $\dot{G}$ appears as $\dot{G}_\alpha$ for stationarity many $\alpha < \omega_2$ (see [15]).

Since properness is preserved by countable support iterations, $\mathcal{Q}_{\omega_2}$ is proper. Moreover, since $\mathcal{Q}_\alpha$ has size $\aleph_1$ for all α, $\mathcal{Q}_{\omega_2}$ satisfies the $\aleph_2$-c.c.. It follows immediately that $\mathbf{V}^{\mathcal{Q}_{\omega_2}}$ satisfies $2^{\aleph_0} = \aleph_2 < 2^{\aleph_1}$.

To see that $\mathcal{Q}_{\omega_2}$ forces OCA, let $(\dot{Y}, \dot{G})$ be a $\mathcal{Q}_{\omega_2}$ name for an open graph which is not countably chromatic. Since $\mathcal{Q}_{\omega_2}$ has the $\aleph_2$-c.c. there is a closed unbounded set $C \subseteq \omega_2$ such that for every δ in C, $\dot{G}$ is a $\mathcal{Q}_\delta$ name and $(\dot{Y} \restriction \mathcal{Q}_\delta, \dot{G})$ is a $\mathcal{Q}_\delta$ name for an open graph which is not countably chromatic. Thus for some δ in C, $\dot{G} = \dot{G}_\delta$. Now $\dot{X}_\delta$ is forced to have an uncountable intersection with $\dot{Y}$ and since $\mathcal{H}^\omega(\dot{X}_\delta, \dot{G}_\delta)$ decomposes $(\dot{X}_\delta, \dot{G})$ into countably many complete subgraphs, it also must introduce an uncountable $\dot{H} \subseteq \dot{Y}$ such that $[\dot{H}]^2 \subseteq \dot{G}$. □

2. Weak diamond and σ-monotonic functions

The purpose of this section is to demonstrate that it is relatively consistent with weak diamond that all 1-1 functions $f \subseteq \mathbb{R}^2$ of size $\aleph_1$ can be decomposed into countably many monotonic functions.

Lemma 2.1. *Suppose that there is a cofinal subset of $(\omega_1^{\omega_1}, <)$ of size $\aleph_2$ and that the continuum is $\aleph_2$. There is a sequence $\vec{g} = \langle g_\xi : \xi < \omega_2 \rangle$ of functions from $\mathbb{R}$ to $\mathbb{R}$ such that if f is a partial function from $\mathbb{R}$ to $\mathbb{R}$ of size $\aleph_1$ then f can be covered by countably many functions from the sequence $\vec{g}$.*

[1] Theorem 4.4 is cast in the language of closed set mappings. To see the translation to open colorings, see Theorems 8.0 and 8.1 of [17].

Proof. It will be easiest to index $\vec{g}$ by the set $\omega_2 \times \omega_2 \times \omega$. Let $\mathbb{R} = \{x_\xi : \xi < \omega_2\}$ be a fixed enumeration and for $\delta < \omega_2$ set $X_\delta = \{x_\xi : \xi < \delta\}$. Fix $\delta < \omega_2$. I will first show that there is a sequence $\Gamma_{\delta,\xi}$ $(\xi < \omega_2)$ of subsets of X_δ^2 such that

(1) for all $\xi < \omega_2$ and all x in X_δ there are only countably many y such that (x, y) is in $\Gamma_{\delta,\xi}$ and
(2) if $f : X_\delta \to X_\delta$ then there is a $\xi < \omega_2$ such that $f \subseteq \Gamma_{\delta,\xi}$.

Fix a cofinal sequence h_γ $(\gamma < \omega_2)$ in $(\omega_1^{\omega_1}, <)$ and set

$$\Gamma_\gamma = \{(x_\xi, x_\eta) : \eta < h_\gamma(\xi)\}.$$

If $f \subseteq X^2$ is a function, let $h(\xi)$ be the index of $f(x_\xi)$ ($h(\xi) = 0$ if $f(x_\xi)$ is not defined). Clearly if h_γ dominates h, then $f \subseteq \Gamma_\gamma$.

Notice that condition (1) implies that each of the sets $\Gamma_{\delta,\xi}$ can be covered by countably many functions $g_{\delta,\xi,n}$ $(n < \omega)$ from $\mathbb{R}$ to $\mathbb{R}$. By property (2),

$$\vec{g} = \langle g_{\delta,\xi,n} : (\delta, \xi, n) \in \omega_2 \times \omega_2 \times \omega\rangle$$

has the desired covering property. □

Theorem 2.2. *It is relatively consistent with weak diamond that every partial 1-1 function f from $\mathbb{R}$ to $\mathbb{R}$ of size $\aleph_1$ can be decomposed into countably many monotonic functions.*

Remark 2.3. A similar result is proven in Section 7 of [1] and similar techniques are employed there (this was brought to my attention during the final revision of this paper). We will need Theorem 2.2, however, for the application in the next section.

Proof. First notice that the sequence $\vec{g}$ of Lemma 2.1 can be chosen in such a way that if f is 1-1 then it can be covered by countably many 1-1 elements of $\vec{g}$. To see this take the original sequence together with all functions of the form

$$\{(x, y) : (x, y) \in g_\xi \text{ and } (y, x) \in g_\eta\}.$$

It is easily verified that countable unions of these functions cover all partial 1-1 functions of size $\aleph_1$.

Start with a ground model which satisfies CH, $\aleph_2 < 2^{\aleph_1}$, and "There is a cofinal subset of $(\omega_1^{\omega_1}, <)$ of size $\aleph_2$." We will now build a c.c.c. partial order $\mathcal{Q}$ of size $\aleph_2$ such that in $\mathbf{V}^{\mathcal{Q}}$ all 1-1 functions $f \subseteq \mathbb{R}^2$ of size $\aleph_1$ can be decomposed into countably many monotonic functions. In $\mathbf{V}^{\mathcal{Q}}$ the continuum will be $\aleph_2$ and there will be a cofinal subset of $(\omega_1^{\omega_1}, <)$ of size $\aleph_2$ and hence Lemma 2.1 will apply. Using the same book-keeping technique as in [15], build $\mathcal{Q}$ as an iteration of partial orders of size $\aleph_1$ in such a way that the functions in the sequence $\vec{g}$ given by Lemma 2.1 can all be decomposed into countably many monotonic functions in $\mathbf{V}^{\mathcal{Q}}$. The following lemma takes care of individual stages of the iteration.

Lemma 2.4. [2] *(CH) If f is a 1-1 function from $\mathbb{R}$ to $\mathbb{R}$ then there is a c.c.c. forcing $\mathcal{P}$ of size $\aleph_1$ such that forcing with $\mathcal{P}$ introduces a decomposition of f into countably many monotonic functions.*

□

3. An application to $\mathbb{P}_{\max}$ theory.

In this section I will show that there is no $\mathbb{P}_{\max}$ variation conditioned to the sentence "$2^{\aleph_0} < 2^{\aleph_1}$." This will be done by showing that there are two Π_2 sentences for the structure $(H(\omega_2), \in)$ each of which can individually be forced hold together with $2^{\aleph_0} < 2^{\aleph_1}$ over any ground model[2] but which jointly imply $2^{\aleph_0} = 2^{\aleph_1}$.

The first sentence is the following assertion due to Baumgartner [3]:

ψ_A: For every tree T of size $\aleph_1$ with countable levels there is a map $f : T \to \mathbb{N}$ such that for all s, t, u with $f(s) = f(t) = f(u)$, $s < t, u$ implies t and u are comparable.

This statement can readily be seen to be equivalent to the assertion that all Aronszajn trees are special and that there are no Kurepa trees (see [3]). It is known that this statement can be forced together with CH (and hence $2^{\aleph_0} < 2^{\aleph_1}$) over any model in which there is an inaccessible cardinal (see chapter V section 8 in [14]).

The second sentence is

ψ_B: If $f \subseteq \mathbb{R}^2$ is a 1-1 function of size $\aleph_1$ then f can be covered by countably many monotonic functions.

The fact that $\psi_A \wedge \psi_B$ implies $2^{\aleph_0} = 2^{\aleph_1}$ is a consequence of the next theorem.

Theorem 3.1. *(ψ_B) There is a sequence T_ξ ($\xi < 2^{\aleph_0}$) of subtrees of $2^{<\omega_1}$, each with countable levels, such that every element of 2^{ω_1} is a branch through some T_ξ.*

Proof. Fix a sequence x_α ($\alpha < \omega_1$) of distinct elements of $\mathbb{R}$ and a sequence of bijections $H_\alpha : 2^\alpha \leftrightarrow \mathbb{R}$ for each $\alpha < \omega_1$. Also fix an enumeration $\vec{f}_\xi$ ($\xi < 2^{\aleph_0}$) of all objects $\vec{f}$ such that

(1) $\vec{f}$ is a countable sequence of monotonic functions,
(2) the domain of each element of $\vec{f}$ is a Borel subset of $\mathbb{R}$, and
(3) the domains of the elements of $\vec{f}$ cover $X = \{x_\alpha : \alpha < \omega_1\}$.

[2] In one case an inaccessible cardinal is needed in the ground model.

For each $\xi < 2^{\aleph_0}$, let T_ξ be the collection of all t in $2^{<\omega_1}$ such that if $\alpha \leq |t|$, there is a member of the sequence $\vec{f_\xi}$ which sends x_α to $H_\alpha(t \restriction \alpha)$. Clearly T_ξ is a downwards closed subtree of $2^{<\omega_1}$ and, since $\vec{f_\xi}$ is countable, T_ξ has countable levels. It suffices to show that every element of 2^{ω_1} is a branch though some T_ξ. To this end, fix z in 2^{ω_1} and define $f : X \to 2^\omega$ by setting $f(x_\alpha) = H_\alpha(z \restriction \alpha)$. Now, by ψ_B, it is possible to find a $\xi < 2^{\aleph_0}$ such that the members of the sequence $\vec{f_\xi}$ cover f. It is now easy to see that z is a branch through T_ξ. □

Remark 3.2. It is perhaps worth mentioning the following result due to Baumgartner (see Theorem 3.5.11 of [9]): Assuming PFA, after forcing with any measure algebra ψ_A holds. The analysis of random graphs under $\mathrm{MA}_{\aleph_1}$ has been one of the main techniques used to produce models of $2^{\aleph_0} < 2^{\aleph_1}$ in which certain consequences of $\mathrm{MA}_{\aleph_1}$ hold. Thus the method of analyzing random graphs (see, e.g., [12], [18]) and techniques presented in Section 2 take a fundamentally different approaches to maximizing consequences of forcing axioms in the context of $2^{\aleph_0} < 2^{\aleph_1}$.

4. Closing remarks and questions

The real motivation behind the results of this paper is the program of studying which Π_2 sentences for $(H(\omega_2), \in)$ are compatible with weak diamond. Set-theoretic topologists are often interested in questions of this sort, particularly in the context of studying the structure of normal and hereditarily normal spaces (see [8], [16]). For example

Question 4.1. Is weak diamond compatible with the statement that every c.c.c. partial order satisfies Knaster's condition?

Here Knaster's condition is the statement that ever uncountable family contains an uncountable linked subcollection. The papers [6], [10], and [11] provide some answers to problems of this nature but with weak diamond replaced by some of its useful consequences.

Todorčević has shown (unpublished) that if $\mathfrak{p} > \omega_1$ then the open coloring axiom of [1] (which I will denote $\mathrm{OCA}_{[\mathrm{ARS}]}$ — see [1] for a definition) is equivalent to the statement that every 1-1 function $f \subseteq \mathbb{R}^2$ is the union of countably many monotonic functions. The following question, however, remains open.

Question 4.2. Is $\mathrm{OCA}_{[\mathrm{ARS}]}$ relatively consistent with $2^{\aleph_0} < 2^{\aleph_1}$?

Finally, the fact that ψ_A is not only consistent with weak diamond but with CH as well suggests the following question (for undefined terms, see chapter 10 of [21]; compare to question 2 on page 906 of [21]).

Question 4.3. (LC)[3] Are there two Π_2 sentences for the structure $(H(\omega_2), \in)$ which can individually be forced to hold together with CH over any ground model but which jointly imply $2^{\aleph_0} = 2^{\aleph_1}$?

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