ON THE VORONOĬ FORMULA FOR $GL(n)$

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Abstract. We prove a general Voronoĭ formula for cuspidal automorphic representations of $GL(n)$ over number fields. This generalizes recent work by Miller-Schmid and Goldfeld-Li on Maass forms. Our method follows closely the adelic framework of integral representations of L-functions. The proof is flexible enough to allow ramification and we propose possible variants. For example the assumption that the additive twist is trivial at places where the representation is ramified is sufficient to obtain an explicit final statement with hyper-Kloosterman sums.

1. Introduction.

A Voronoĭ summation formula is an equality between a weighted sum of Fourier coefficients of an automorphic form twisted by an additive character and a dual weighted sum of Fourier coefficients of the dual form twisted by Kloosterman sums. The weights are related by a Bessel transform.

Our purpose in this article is to formulate a $GL(n)$ Voronoĭ formula in general and derive it from the classical theory of integral representation and the formalism of local/global functional equations of L-functions on $GL(n)$, as may be found in the work of Jacquet, Piatetskii-Shapiro and Shalika [17, 18, 20–24]. We hope the present article might help to open the road for new developments.

Such a generalization was probably expected, although not yet available especially when ramifications come in. Several distinct approaches have been developed before and it has not been clear so far which techniques are the more appropriate for the Voronoĭ formula, see below for a brief summary. We feel that the present article gives the proper framework to understand this problem. In particular the exact relationship with the functional equation may now be clearly seen.

The Voronoĭ formula for $GL(2)$ is a basic tool for the study of analytic properties of automorphic forms, see the series of papers by Duke-Friedlander-Iwaniec [5] or Kowalski-Michel-VanderKam [25] and Harcos-Michel [14], and the references herein. Recent applications of the formula for $GL(3)$ may be found in Miller-Schmid [29, 31], Sarnak-Watson [32] and Li [26, 27].

1.1. **Main result.** Let F be a number field; denote by $\mathbb{A} = \mathbb{A}_F$ the ring of adeles. Let $n \geq 2$ and let $\mathcal{A}_{\mathrm{cusp}}(GL_n)$ be the space of automorphic cusp forms on $GL_n(\mathbb{A})$. Let $\pi = \otimes_v \pi_v \subset \mathcal{A}_{\mathrm{cusp}}(GL_n)$ be an irreducible cuspidal automorphic representation of $GL_n(\mathbb{A})$. Let $\psi = \otimes_v \psi_v$ be a non-trivial additive character on $\mathbb{A}/F$. More notations will be recalled in § 1.7, most of them being standard.

For all places v of F, to a smooth compactly supported function $w_v \in \mathcal{C}_c^\infty(F_v^\times)$ is associated a dual function $\tilde{w}_v$ so that:

$$\int_{F_v^\times} \tilde{w}_v(y)\chi(y)^{-1} |y|_v^{s-\frac{n-1}{2}} dy = \chi(-1)^{n-1}\gamma(1-s, \pi_v \times \chi, \psi_v) \int_{F_v^\times} w_v(y)\chi(y) |y|_v^{1-s-\frac{n-1}{2}} dy \tag{1.1}$$

for all $s \in \mathbb{C}$ of real part sufficiently large and all unitary characters $\chi : F_v^\times \to S^1$. The equality (1.1) is independent of the chosen Haar measure dy on $F_v^\times$ and defines $\tilde{w}_v$ uniquely in terms of π_v, ψ_v and w_v. Such an explicit relation makes the Voronoĭ formula very useful in practice, especially if one is willing to determine the asymptotic behavior of $\tilde{w}_v$. Details on this generalized Bessel transform are given in § 5.3. The function $\tilde{w}_v$ is smooth of rapid decay at infinity but not necessarily compactly supported.

Let S be a finite set of places of F including the places where π ramifies, where ψ ramifies and all archimedean places. Denote by $\mathbb{A}^S$ the subring of adeles with trivial component above S. Denote by $W_\circ^S = \prod_{v \notin S} W_{\circ v}$ the unramified Whittaker function of $\pi^S = \otimes_{v\notin S}\pi_v$ above the complement of S. Let

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$\tilde{W}_\circ^S = \prod_{v\notin S} \tilde{W}_{\circ v}$ be the unramified Whittaker function of $\tilde{\pi}^S = \otimes_{v\notin S}\tilde{\pi}_v$. We recall that $\tilde{W}_\circ^S(g) = W_\circ^S(w\,{}^t g^{-1})$ for all $g \in GL_n(\mathbb{A}^S)$ where w is the longest Weyl element of GL_n.

Our main result is the following.

Theorem 1. *Let $\zeta \in \mathbb{A}^S$, let R be the set of places v such that $|\zeta|_v > 1$ (then R and S are disjoint), and for all $v \in S$ let $w_v \in \mathcal{C}_c^\infty(F_v^\times)$. Then:*

$$\sum_{\gamma\in F^\times} \psi(\gamma\zeta) W_\circ^S\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right)\prod_{v\in S} w_v(\gamma) = \sum_{\gamma\in F^\times} K_R(\gamma,\zeta,\tilde{W}_{\circ R})\tilde{W}_\circ^{R\cup S}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right)\prod_{v\in S}\tilde{w}_v(\gamma) \tag{1.2}$$

where $K_R(\gamma,\zeta,\tilde{W}_{\circ R})$ is a certain Kloosterman integral as in Definition 2.2.

Let $T \subset GL_n$ be the maximal torus of diagonal matrices. The Kloosterman integral may be further decomposed into a finite sum, see § 6.2:

$$K_R(\gamma,\zeta,\tilde{W}_{\circ R}) = \sum_{t\in T(\mathbb{A}_R)/T(\mathfrak{o}_R)} \tilde{W}_{\circ R}(t)\mathcal{K}\ell(\gamma\zeta^{-1},t) \tag{1.3}$$

where $\mathcal{K}\ell$ is the hyper-Kloosterman sum of dimension $n-1$, and $t = \mathrm{diag}(t_1,\ldots,t_n)$ is a diagonal matrix. The Whittaker function $\tilde{W}_{\circ R}(t)$ is zero unless for all $v \in R$:

$$|t_1|_v \le |t_2|_v \le \cdots \le |t_n|_v\,. \tag{1.4}$$

Then the hyper-Kloosterman sum $\mathcal{K}\ell(\gamma\zeta^{-1},t)$ is zero unless for all $v \in R$:

$$1 \le |t_2|_v\,, \quad |\det(t)|_v = |t_1\cdots t_n|_v = |\gamma|_v\,, \quad |t_n|_v = |\zeta|_v\,. \tag{1.5}$$

When the conditions (1.4) and (1.5) hold for all $v \in R$, we have (Corollary 6.7):

$$\mathcal{K}\ell(\gamma\zeta^{-1},t) = \sum_{v_{n-1}\in t_{n-1}\mathfrak{o}_R^\times/\mathfrak{o}_R}\cdots\sum_{v_2\in t_2\mathfrak{o}_R^\times/\mathfrak{o}_R} \psi(v_{n-1}+\cdots+v_2)\psi((-1)^n\gamma\zeta^{-1}v_2^{-1}\ldots v_{n-1}^{-1}). \tag{1.6}$$

Note that this sum degenerates into a product of the Ramanujan sums at places $v \in R$ such that $|t_2|_v = 1$.

Denote by T_ζ the set of elements $(t_2,\cdots,t_{n-1})$ in $\mathbb{A}_R^{n-2}$ such that for all $v \in R$:

$$1 \le |t_2|_v \le \cdots \le |t_{n-1}|_v \le |\zeta|_v\,.$$

It is clearly invariant by multiplication by $T_\circ = (\mathfrak{o}_R^\times)^{n-2}$ and there are finitely many $T_\circ$-orbits. For $t \in T_\zeta/T_\circ$, we denote by $a(t)$ the diagonal matrix $\mathrm{diag}(t_1,\ldots,t_n) \in T(\mathbb{A}_R)/T(\mathfrak{o}_R)$ uniquely completed so that $|t_n|_v = |\zeta|_v$ and $|t_1\cdots t_n|_v = 1$ for all $v \in R$.

Replacing $K_R(\gamma,\zeta,\tilde{W}_{\circ R})$ in (1.2) in the main theorem by the decomposition (1.3), the right-hand side of (1.2) may be written as:

$$\sum_{t\in T_\zeta/T_\circ}\sum_{\gamma\in F^\times} \mathcal{K}\ell(\gamma\zeta^{-1},t)\tilde{W}_\circ^S\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}a(t)\right)\prod_{v\in S}\tilde{w}_v(\gamma).$$

The range of the first sum over t is finite and independent of γ. This traditional fact makes the Voronoĭ formula of practical interest. As far as one is concerned with upper bounds it is possible to focus on the inner sum only. The inner sum is really dual to the left-hand side of (1.2), with the additive twist replaced by the hyper-Kloosterman sum.

Remark 1. We briefly address the convergence issues in (1.2) which are classical. The left-hand side is finitely supported for $\gamma \in F^\times$. For the right-hand side, the conditions (1.4) and (1.5) imply that $|\gamma|_v$ is bounded for $v \in R$; for $v \notin R\cup S$, the Whittaker functions being unramified, we have $\gamma \in \mathfrak{o}_{R\cup S}$ (Dedekind ring of R- and S-integers); for $v \in S$, the function $x \mapsto \tilde{w}_v(x)$ is of rapid decay as $|x|_v \to \infty$. Together this implies that the sum $\sum_{\gamma\in F^\times}$ is rapidly convergent.

Remark 2. In § 1.5 we explicate the formula in the most studied case where $F = \mathbb{Q}$ and $S = \{\infty\}$. Over number fields our formula is more general than one could get from the functional equation which in some sense is one-dimensional, see the next § 1.2 for a review of previous methods. For instance above the infinite places our truncation function has the form $\gamma \mapsto \prod_{v|\infty} w_v(\gamma)$ whereas making use of the functional equation one could achieve only $\gamma \mapsto w(\mathbf{N}_{F/\mathbb{Q}}\gamma)$.

1.2. **Brief summary of the literature.** For $GL(2)$, the formula is well established. The treatment of the divisor function τ goes back to Voronoĭ himself. His purpose was to establish a good remainder term for the asymptotic of $\sum_{n\leq X}\tau(n)$ as $X\to\infty$.

The version for holomorphic and Maass cusp forms has been worked out at several places. For full level, Duke-Iwaniec [6–9] have derived the formula from the functional equations of the L-functions twisted by a character on $GL(1)$. This is also implicit at several other places. A general version of the Voronoĭ formula for $GL(2)$ may be found in [25, Appendix].

Progress in higher rank are more recent. A proof of the Voronoĭ formula for $GL(3)$-cusp forms of full level has first been given by Miller-Schmid [28, 29] introducing the heavy machinery of automorphic distributions. It should be noted that Sarnak (oral communication) had also developed a version of the Voronoĭ formula for prime denominators, based on the functional equation and contour integration. This was motivated by his work with Watson on L^4-norms of Maass cusp forms (unpublished). Miller-Schmid [28] were then able to establish the formula for non-prime denominators. The machinery of automorphic distribution is natural from the point of view of Fourier duality and interesting because many independent problems have been raised in the process. When revising the article we learned that in [30] Miller-Schmid have extended their results to $GL(n)$-cusp forms of full level.

Voronoĭ type formulas for $GL(n)$ and full level have also been established by Goldfeld-Li [11] using the functional equations of the twisted L-functions, as in the above Duke-Iwaniec approach. The proof has been written for additive characters of prime conductors, although the method should extend to square-free conductors. These assumptions are removed in a second paper [12] by another method. Note that [30] precedes [12] (this is acknowledged in the addendum to [12]).

The method of [12] relies on an explicit description of Maass forms on $GL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})$ which is developed in the book [13]. The authors provide a nonadelic proof of the functional equation for the L-functions of these Maass forms, a variant of which yields the desired Voronoĭ formula. We point out that the proof of [12, Proposition 5.3] is rather difficult, see below.

As noted by several authors, implicitly and explicitly, the knowledge of the functional equation for the L-functions twisted by characters is not sufficient for a general formula. This is partly because an explicit assessment of the twisted local ε- and L-factors would be fairly complicated in general.

When applying Fourier transforms back and forth, both on the $GL(1)$-characters and on $GL(n)$, one quickly feels the need of a more organic setting. This is partially achieved: in [25, Appendix] for $GL(2)$ where the identities are derived from the action of the modular group, *à la Hecke*; in [29, 30] for $GL(n)$ where automorphic distributions are introduced; and in the approach of [12] where a nonadelic approach of the functional equation is involved.

In this paper we go further by making full use of the adelic formalism of integral representations of L-functions. Our method works over number fields and is well-suited to address all kinds of ramification.

1.3. **Our approach.** Implicit in those previous works is the belief that a new ingredient or a new setting has to be developed to be able to establish the Voronoĭ formula. On the contrary, in this paper we rely on the classical analytic theory of L-functions on $GL(n)$ and derive the Voronoĭ formula without extra input and in complete generality.

This is done in several independent steps, each owning its own motivation; thereby we gain flexibility. We begin with a global identity, which is related to the global functional equation. Then we are reduced to local identities. Above the twisted places, the Kloosterman integral K_R arises. Above the ramified places, the local functional equations are used to characterize explicitly the dual weight functions with a Mellin transform (1.1). The reader is referred to § 1.6 for a detailed outline.

Formally we are close to [12]; the attentive reader will recognize related calculations in the detail of the proofs. The Proposition 5.3 of [12] is really delicate because the proof includes at the same time a global identity (which corresponds to our Proposition 1.1), a manipulation of archimedean Whittaker functions (which corresponds to the local functional equation, see § 5) and a computation of Kloosterman integrals (that we perform independently in § 6).

Also the Proposition 5.6 of [30] is similar to our Proposition 1.1 below. Independently, both [30] and the present article isolate that proposition as a cornerstone to the Voronoĭ formula. Our approach differs mainly in that we then use Kirillov models to construct the duality (1.1) from the local functional equations (which we recall in § 5.1). Another difference with [30] is that we do not make explicit use of principal series representations. We recall that [30] proceeds to construct the automorphic distribution by applying Casselman's theorem to embed the archimedean component into a principal series.

1.4. **Pseudo-Whittaker functionals.** Consider the following Weyl element:

$$w' = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \end{pmatrix} \in GL_n.$$

We introduce the unipotent subgroup Y which consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & u_{1,3} & \cdots & u_{1,n-1} & u_{1,n} \\ 0 & 1 & u_{2,3} & \cdots & u_{2,n-1} & u_{2,n} \\ 0 & 0 & 1 & \cdots & u_{3,n-1} & u_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in GL_n,$$

and the character ψ on $Y(\mathbb{A})$, trivial on $Y(F)$, given by $\psi(u_{2,3} + \cdots + u_{n-1,n})$. We denote by du the Tamagawa measure on $Y(\mathbb{A})$, see § 1.7 for details.

We define a linear functional $\mathcal{P} : \mathcal{A}_{\text{cusp}}(GL_n) \to \mathbb{C}$ by

$$\mathcal{P}\varphi = \int_{Y(F)\backslash Y(\mathbb{A})} \varphi(u)\overline{\psi(u)}\, du, \quad \varphi \in \mathcal{A}_{\text{cusp}}(GL_n).$$

We refer to $\mathcal{P}$ as a *pseudo-Whittaker functional.* Let X be the unipotent subgroup which consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ x_2 & 1 & 0 & \cdots & 0 & 0 \\ x_3 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in GL_n.$$

We define a dual pseudo-Whittaker functional $\tilde{\mathcal{P}} : \mathcal{A}_{\text{cusp}}(GL_n) \to \mathbb{C}$ by:

$$\tilde{\mathcal{P}}\varphi = \int_{X(\mathbb{A})} \left[\int_{Y(F)\backslash Y(\mathbb{A})} \varphi(uxw')\psi(u)\, du \right] dx, \quad \varphi \in \mathcal{A}_{\text{cusp}}(GL_n).$$

The absolute convergence of the outer integral follows from the Whittaker expansion of the pseudo-Whittaker functional (see Lemma 2.1) and an estimate of a Whittaker function by a gauge (see [20, Lemma 2.6] and [23, Lemma 5.1]). Both functionals are constructed from integrations over unipotent subgroups, like Whittaker functions. This explains our terminology. We note that the dual pseudo-Whittaker functional is defined by a more sophisticated formula.

We have an involution ι on GL_n and then on $\mathcal{A}_{\text{cusp}}(GL_n)$ defined by $\iota\varphi(g) = \varphi(g^\iota) = \varphi({}^t g^{-1})$. Important in our proof is the following identity between the two pseudo-Whittaker functionals. This identity or possibly a variant thereof is implicitly used by Jacquet, Piatetskii-Shapiro and Shalika to establish the global functional equation of the L-functions for $GL(n) \times GL(1)$ from the Rankin-Selberg method (see § 4.1).

Proposition 1.1. *The following holds:*

$$\mathcal{P} = \tilde{\mathcal{P}} \circ \iota. \tag{1.7}$$

The above identity is the basic ingredient of both the Voronoĭ formula and the global functional equation. We will include a proof of this proposition in section 4 for the sake of completeness. We proceed by a recursive application of Fourier expansion. We have learned recently that a related proof will appear in [30, Appendix A].

1.5. **Classical formulation.** We assume in this section that $F = \mathbb{Q}$ and $S = \{\infty\}$ so that π is unramified. The ring of finite adeles $\mathbb{A}^S = \mathbb{A}_f$ is denoted with a subscript f; this convention is valid for the Whittaker functions as well. For the sake of clarity we state our main Theorem 1 in that case.

Theorem 2. *Let π be an irreducible unramified cuspidal automorphic representation of $GL_n(\mathbb{A})$. Assume also that ψ is unramified above all finite places. Let $\zeta \in \mathbb{A}_f$ and $w \in \mathcal{C}_c^\infty(\mathbb{R}^\times)$. Then:*

$$\sum_{\gamma\in\mathbb{Q}^\times} \psi(\gamma\zeta) W_{\circ f}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right) w(\gamma) = \sum_{t\in T_\zeta/T_\circ}\sum_{\gamma\in\mathbb{Q}^\times} \mathcal{K}\ell(\gamma\zeta^{-1}, t)\tilde{W}_{\circ f}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix} a(t)\right)\tilde{w}(\gamma).$$

With different notation, the above is exactly Theorem 1.10 of Miller-Schmid [30]. The Theorem 1.1 of Goldfeld-Li [12] is a special case, see the remark below. A minor improvement compared to [12] is on the assumption that the Maass form be even. In our formulation this aspect (even or odd) is encoded in the duality (1.1). More precisely, that identity takes into account the two possible gamma factors $\gamma(s, \pi_\infty \times \chi, \psi_\infty)$ built from the two unitary characters χ of $\mathbb{R}^\times$. As we have seen this rephrasing is convenient for a generalization to all places.

As in [30] there is no assumption on π_∞, the formula is not restricted to spherical Maass forms. Unitary representations of $GL_n(\mathbb{R})$ besides the principal series may occur in $\mathcal{A}_{\text{cusp}}(GL_n)$, see Sarnak [33] for an account on the generalized Ramanujan conjectures and Vogan [36] for a complete classification of the unitary dual of $GL_n(\mathbb{R})$.

Remark 3. For the sake of clarity, we make explicit the link between the notation in [12, Theorem 1.1] and ours. We take our additive character ψ to be the standard one. Namely $\psi(x) = e^{-2\pi i x}$ for all $x \in \mathbb{R} \subset \mathbb{A}$. For $x \in \mathbb{Q} \subset \mathbb{A}_f$ we then get the usual character with a plus sign, traditionally denoted by $e(x) = e^{2\pi i x}$. Our ζ corresponds to their $\frac{\overline{h}}{q}$. The set of places R contains the prime divisors of q. Their function ϕ is our function w (with an evenness restriction) and their function Φ is our $\tilde{w}$. Their $m \in \mathbb{Z} - \{0\}$ in the left-hand side is our $\gamma \in \mathbb{Q}^\times$. Our γ in the right-hand side corresponds to their $\frac{m\prod_{i=1}^{n-2} d_i^{n-i}}{q^n}$. Up to normalization their Fourier coefficients A correspond to our $W_{\circ f}$ and $\tilde{W}_{\circ f}$. Their sequence $d = (d_1, d_2, \ldots, d_{n-2})$ corresponds to our $t \in T_\zeta/T_\circ$. More precisely, d_1 is qt_{n-1} up to a unit, $d_1 d_2$ is qt_{n-2} and so on until $d_1 \cdots d_{n-2}$ is qt_2, up to units as well. Their hyper-Kloosterman sum $KL(h, m; d, q)$ equals our $\mathcal{K}\ell(\gamma\zeta^{-1}, t)$ in (1.6). To see this, one first moves their $(-1)^n h$ to the last exponential in their expression. Then $t_1/(q/d_1)$ corresponds to our v_{n-1}, $\overline{t_1}t_2/(q/d_1 d_2)$ corresponds to our v_{n-2} and so on until $\overline{t_{n-3}}t_{n-2}/(q/d_1 \cdots d_{n-2})$ corresponds to our v_2. Then their $(m/q) \cdot \overline{t_{n-2}}/(q/d_1 \cdots d_{n-2})$ corresponds to our $\gamma v_2^{-1} \cdots v_{n-1}^{-1}$.

1.6. **Outline.** Although our proof is short, several points were not obvious. In this section we provide a detailed outline.

In § 2.1 we choose a suitable factorizable vector $\varphi = \otimes_v \varphi_v \in \pi$ depending on the initial data ζ and w_v. This is made possible because the Kirillov model of a cuspidal representation is large enough: it contains all smooth compactly supported functions, see § 5.2.

We denote $\iota\varphi$ by $\tilde{\varphi}$. The starting point is the identity from Proposition 1.1 (whose proof is given in § 4):

$$\mathcal{P}\varphi = \tilde{\mathcal{P}}\tilde{\varphi}.$$

In § 2.2 we give the Fourier expansions of the two sides in terms of the global Whittaker functions W_φ of φ and $W_{\tilde{\varphi}}$ of $\tilde{\varphi}$. This is not difficult because the functional $\tilde{\mathcal{P}}$ is already chosen so as to factorize in an Euler product. In words, the $X(\mathbb{A})$-integral breaks into a product of local integrals and the $Y(F)\backslash Y(\mathbb{A})$-integral is made of *upper-triangular* matrices so that the link with Whittaker functions is straightforward.

In § 2.3 we verify that the Fourier expansion of $\mathcal{P}\varphi$ is equal to the left-hand side of the Voronoï formula. The identification of the right-hand side is less obvious. Preliminary simplifications are made in § 2.4. We are reduced to computing for each place v the following integral

$$(1.8) \qquad \int_{F_v^{n-2}} \tilde{W}_v\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\begin{pmatrix}1 & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix} w'\begin{pmatrix}1 & & \\ -\zeta & 1 & \\ & & 1_{n-2}\end{pmatrix}\right) dx.$$

In § 2.5 we deal with unramified places $v \notin R \cup S$, which are much easier.

The § 2.6 is concerned with the evaluation of local Kloosterman integrals. Their properties may be found in § 6 where we extend work of Stevens [34] and Friedberg [10].

The § 2.7 is concerned with the ramified places $v \in S$ (for those $\zeta_v = 1$). We emphasize that the integral (1.8) cannot be evaluated explicitly. Rather it is identified with the Bessel transform $\gamma \mapsto \tilde{w}_v(\gamma)$. This is done thanks to the local functional equation.

In § 2.8 we gather the above computations to complete the proof of Theorem 1.

1.7. **Notation.** All local and global fields are of characteristic zero. If v is a place of a number field F, we denote by F_v the associated local field and consider F as embedded in F_v without further mention. When F_v is non-archimedean, we denote by $\mathfrak{o}_v$ the ring of integers and by $\mathfrak{m}_v$ the maximal ideal of $\mathfrak{o}_v$.

Let $G = GL_n$. We recall that $\mathcal{A}_{\text{cusp}}(GL_n)$ is the space of automorphic cusp forms on $G(\mathbb{A})$. We denote by ρ the action of $G(\mathbb{A})$ by right translation. Let $\pi \subset \mathcal{A}_{\text{cusp}}(GL_n)$ be an irreducible cuspidal automorphic representation.

Let $\iota(g) = g^\iota = {}^tg^{-1}$ for $g \in G$. For $\varphi \in \pi \subset \mathcal{A}_{\text{cusp}}(GL_n)$, we define $\iota\varphi \in \tilde{\pi} \subset \mathcal{A}_{\text{cusp}}(GL_n)$ by $\iota\varphi(g) = \varphi(g^\iota)$, where $\tilde{\pi}$ is the contragredient representation of π. Often we denote $\iota\varphi$ by $\tilde{\varphi}$.

Let T be the maximal torus of G consisting of diagonal matrices. We let $U = U_n$ be the maximal unipotent subgroup of G of upper-triangular matrices, and U^- the opposite maximal unipotent subgroup of lower-triangular matrices. As in the introduction, ψ is a non-trivial unitary character on $F\backslash\mathbb{A}$. We denote by the same letter ψ the character on $U(\mathbb{A})$ trivial on $U(F)$ given by

$$\psi(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$$

for $u = (u_{i,j}) \in U(\mathbb{A})$.

For $\varphi \in \mathcal{A}_{\text{cusp}}(GL_n)$, let W_φ be the ψ-Whittaker function of φ given by

$$W_\varphi(g) = \int_{U(F)\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\, du$$

for $g \in G(\mathbb{A})$, where du is the Tamagawa measure on $U(\mathbb{A})$.

For $n \geq 0$ we denote by 1_n the identity $n \times n$ matrix and $J_n = \begin{pmatrix} 0 & & 1 \\ & \reflectbox{$\ddots$} & \\ 1 & & 0 \end{pmatrix}$ the matrix filled with 1's on the opposite diagonal. In particular $J_n^2 = 1_n$. We often let $w = J_n$.

The $\bar{\psi}$-Whittaker function $\tilde{W}_\varphi$ of $\tilde{\varphi}$ is given by $\tilde{W}_\varphi(g) = W_\varphi(wg^\iota)$ for $g \in G(\mathbb{A})$. Note that $\tilde{W}_\varphi(ug) = \overline{\psi(u)}\tilde{W}_\varphi(g)$ for $u \in U(\mathbb{A})$. We have $\rho(h)\tilde{W}_\varphi(g) = \tilde{W}_{\rho(h^\iota)\varphi}(g)$, where ρ is the right translation.

We recall briefly the construction of Tamagawa measures that shall be used in the rest of the text on the unipotent groups $U(\mathbb{A})$, $Y(\mathbb{A})$ and $\mathbb{A}^{n-2}$. First one constructs the Tamagawa measure $dx = \prod_v dx_v$ on $\mathbb{A}$ by requiring that dx_v is the self-dual measure on F_v with respect to the additive character ψ_v. Then we fix a non-zero

$$\omega \in \text{Hom}_F(\wedge^{\text{top}}\text{Lie } U, F),$$

and the same for Y. The form ω_v together with the measure dx_v defines a measure on Lie $U(F_v)$, which then corresponds to an invariant measure on $U(F_v)$. The product of these measures is the global Tamagawa measure. The volume of $Y(F)\backslash Y(\mathbb{A})$, $U(F)\backslash U(\mathbb{A})$ and $F\backslash\mathbb{A}$ is one.

We consider the further Weyl elements:

$$w' := \begin{pmatrix} 1 & \\ & J_{n-1} \end{pmatrix}, \quad \sigma := \begin{pmatrix} & & 1 \\ 1_{n-2} & & \\ & & & 1 \end{pmatrix}.$$

1.8. **Structure of the article.** In § 2 we give a proof of the main result, taking for granted the Proposition 1.1, some properties of Kloosterman sums and Whittaker integrals, which are established in subsequent sections. In § 3 we give two variants of the main result and explain how to modify the proof under these new assumptions.

The integral representations and analytic properties of L-functions on $GL(n)$ are now well-understood ; the reader may refer to the excellent survey by Cogdell [3] for an introduction to the work of Jacquet, Piatetskii-Shapiro and Shalika. Because the Voronoĭ formula is intimately related to the functional equation of L-functions, we have chosen to provide a detailed proof of the key Proposition 1.1 in § 4 so that the paper is essentially self-contained. We hope the relationship with the functional equation will now appear transparent.

Background on representations of $GL(n)$ over local fields and on Kloosterman sums may be found in § 5 and § 6 respectively.

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2. Proof of Theorem 1.

2.1. **Local vectors.** In this section we construct a suitable vector $\varphi \in \pi$ whose image under $\mathcal{P}$ and $\tilde{\mathcal{P}}$ will produce the two sides of the identity (1.2) in Theorem 1. We have chosen to explain our choice at the outset for the sake of clarity and concreteness. Several arguments below are valid for a general $\varphi \in \pi$.

Recall that $\zeta \in \mathbb{A}^S$; consider the following matrix $A \in G(\mathbb{A}^S) \subset G(\mathbb{A})$:

$$A := \begin{pmatrix} 1 & \zeta & \\ & 1 & \\ & & 1_{n-2} \end{pmatrix}. \tag{2.1}$$

We take φ as a pure tensor: $\varphi = \otimes_v \varphi_v$. Below we shall often omit the $\otimes$ symbol when no confusion is possible.

The vectors $\varphi^S = \otimes_{v \notin S} \varphi_v$ outside S are chosen to be a translation of the unramified vector $\varphi_\circ^S$ by A.

The vectors $\varphi_S = \otimes_{v \in S} \varphi_v$ above S are chosen in the following way. According to Lemma 5.1, there exist Whittaker functions $W_v \in \mathcal{W}(\pi_v, \psi_v)$ so that:

$$w_v(y) = W_v\left(\begin{pmatrix} y & \\ & 1_{n-1} \end{pmatrix}\right), \quad \text{for all } y \in F_v^\times.$$

We choose such a Whittaker function W_v and obtain a corresponding vector $\varphi_v \in \pi_v$.

In summary, the vector $\varphi \in \pi$ is defined to be:

$$\varphi = \rho(A)\varphi_\circ^S \varphi_S. \tag{2.2}$$

Before continuing we introduce some more conventions. We shall abbreviate W_S for W_{φ_S}; W_v for W_{φ_v}; $W_{\circ v}$ for $W_{\varphi_{\circ v}}$; W^S for W_{φ^S} and $W_\circ^S$ for $W_{\varphi_\circ^S}$. We adopt similar conventions for the dual Whittaker functions: we write $\tilde{W}_v$ for $\tilde{W}_{\varphi_v}$ and so on.

2.2. **Whittaker expansions.** Let $\tilde{W}_\varphi \in \mathcal{W}(\tilde{\pi}, \psi^{-1})$ be the Whittaker function of $\tilde{\varphi}$.

Lemma 2.1. *The following holds:*

$$\mathcal{P}\varphi = \sum_{\gamma \in F^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix}\right)$$

$$\tilde{\mathcal{P}}\tilde{\varphi} = \sum_{\gamma \in F^\times} \int_{\mathbb{A}^{n-2}} \tilde{W}_\varphi\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & & \\ x & 1_{n-2} & \\ & & 1 \end{pmatrix} w'\right) dx.$$

Proof. The first identity is stated in [21, § 3.1] (see also [3, Proposition 5.2]). Consider

$$\mathsf{F} : x \longmapsto \int_{Y(F)\backslash Y(\mathbb{A})} \varphi\left(u\begin{pmatrix}1 & x & \\ & 1 & \\ & & 1_{n-2}\end{pmatrix}\right)\overline{\psi(u)}\,du.$$

A possible proof consists in expanding in Fourier series. Indeed we infer that:

$$\int_{F\backslash\mathbb{A}} \mathsf{F}(x)\overline{\psi(\gamma x)}\,dx = \begin{cases} W_\varphi\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right), & \text{if } \gamma\in F^\times \\ 0, & \text{if } \gamma = 0.\end{cases}$$

The second identity follows from the first. □

2.3. **The left-hand side.** From the expression (2.2) and the previous Lemma 2.1 we infer that:

$$\mathcal{P}\varphi = \sum_{\gamma\in F^\times} W_S\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right) W_\circ^S\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\cdot A\right).$$

It is easily seen that for all $\gamma\in F^\times$:

$$\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\cdot A = \begin{pmatrix}1 & \gamma\zeta & \\ & 1 & \\ & & 1_{n-2}\end{pmatrix}\cdot\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}.$$

This may be combined with the behavior of $W_\circ^S$ under left multiplication by $U(\mathbb{A})$, which pulls out the multiplication by $\psi(\gamma\zeta)$.

From our choices in § 2.1, $W_S\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right)$ is equal to $\prod_{v\in S} w_v(\gamma)$.

We thus arrive at the expression of the left-hand side of (1.2). The remaining part of the proof consists in showing that $\tilde{\mathcal{P}}\tilde{\varphi}$ is equal to the right-hand side of (1.2).

2.4. **The dual functional.** We apply Lemma 2.1 and incorporate the translation by the matrix:

$$A^\iota = \begin{pmatrix}1 & & \\ -\zeta & 1 & \\ & & 1_{n-2}\end{pmatrix}.$$

The dual functional $\tilde{\mathcal{P}}\tilde{\varphi}$ is equal to the sum over $\gamma\in F^\times$ of the integral:

$$\int_{\mathbb{A}^{n-2}} \tilde{W}_S\tilde{W}_{\circ R}\tilde{W}_\circ^{R\cup S}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\begin{pmatrix}1 & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & \\ & J_{n-1}\end{pmatrix}\begin{pmatrix}1 & & \\ -\zeta & 1 & \\ & & 1_{n-2}\end{pmatrix}\right)dx. \tag{2.3}$$

This integral breaks into a product of corresponding local integrals I_v over F_v^{n-2}. In § 2.5 (resp. § 2.6, resp. § 2.7) we shall study the integral I_v when v is an unramified place (resp. twisted place, resp. ramified place). By abuse of notation, we shall often denote by A, γ, ζ their local components A_v, γ_v, ζ_v respectively.

2.5. **The unramified places.** Let $v\notin R\cup S$, a non-archimedean place of F. Recall that ψ_v is an unramified character on F_v, π_v is an irreducible unramified representation of $G(F_v)$ and $W_{\circ v}\in\mathcal{W}(\pi_v,\psi_v)$ is a non-zero unramified vector. The dual Whittaker function $\tilde{W}_{\circ v}\in\mathcal{W}(\tilde{\pi}_v,\psi_v^{-1})$ satisfies $\tilde{W}_{\circ v}(g) = W_{\circ v}(wg^\iota)$ for $g\in G(F_v)$.

Since $\begin{pmatrix}1 & \\ & J_{n-1}\end{pmatrix}$ and A^ι are in $G(\mathfrak{o}_v)$, these leave $\tilde{W}_{\circ v}$ invariant. We claim that:

$$I_v = \int_{F_v^{n-2}} \tilde{W}_{\circ v}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\begin{pmatrix}1 & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix}\right)dx = \tilde{W}_{\circ v}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right). \tag{2.4}$$

This would follow from the computations at the twisted places from the next section and § 6.2. For the sake of completeness we provide a more direct argument.

We have that $\tilde{W}_{\circ v}$ is supported on $N(F_v)T(F_v)^+G(\mathfrak{o}_v)$, see (6.2). For all $x \in F_v^{n-2}$ it is possible to express the Iwasawa decomposition of $\begin{pmatrix} 1 & & \\ x & 1_{n-2} & \\ & & 1 \end{pmatrix}$. Then we infer that the integrand is non-zero if and only if $x \in \mathfrak{o}_v^{n-2}$ and $|\gamma|_v \leq 1$. The integrand is constant under these conditions and this establishes the claim.

2.6. **The twisted places.** Let $v \in R$, a non-archimedean place of F. We have that π_v and ψ_v are unramified and $|\zeta|_v \geq 1$ (the latter inequality is actually strict but the computations below are also valid when $|\zeta|_v = 1$). We will compute

$$I_v = \int_{F_v^{n-2}} \tilde{W}_{\circ v}\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix}\begin{pmatrix} 1 & & \\ x & 1_{n-2} & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & J_{n-1} \end{pmatrix}\begin{pmatrix} 1 & & \\ -\zeta & 1 & \\ & & 1_{n-2} \end{pmatrix}\right) dx.$$

Recall that $w' = \begin{pmatrix} 1 & \\ & J_{n-1} \end{pmatrix}$ and $\sigma = \begin{pmatrix} & 1 & \\ 1_{n-2} & & \\ & & 1 \end{pmatrix}$. Let $U_\sigma^- = U \cap \sigma^{-1}U^-\sigma$. We have

$$U_\sigma^- = \left\{\begin{pmatrix} 1_{n-2} & * & \\ & 1 & \\ & & 1 \end{pmatrix}\right\}, \qquad \sigma U_\sigma^- \sigma^{-1} = \left\{\begin{pmatrix} 1 & & \\ * & 1_{n-2} & \\ & & 1 \end{pmatrix}\right\},$$

so that

$$\begin{aligned} I_v &= |\gamma|_v^{n-2} \int_{F_v^{n-2}} \tilde{W}_{\circ v}\left(\begin{pmatrix} 1 & & \\ x & 1_{n-2} & \\ & & 1 \end{pmatrix}\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix} w' A^\iota\right) dx \\ &= |\gamma|_v^{n-2} \int_{U_\sigma^-(F_v)} \tilde{W}_{\circ v}\left(\sigma u \sigma^{-1}\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix} w' A^\iota\right) du. \end{aligned}$$

The measure on $U_\sigma^-(F_v)$ is transported from F_v^{n-2}. Since $w'\sigma \in G(\mathfrak{o}_v)$ and

$$\begin{aligned} \sigma^{-1} w' A^\iota w' \sigma = \sigma^{-1}\begin{pmatrix} 1 & & \\ & 1_{n-2} & \\ -\zeta & & 1 \end{pmatrix}\sigma &= \begin{pmatrix} 1_{n-2} & & \\ & 1 & \\ & -\zeta & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1_{n-2} & & \\ & 1 & -\zeta^{-1} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1_{n-2} & & \\ & -\zeta^{-1} & \\ & & -\zeta \end{pmatrix}\begin{pmatrix} 1_{n-2} & & \\ & & -1 \\ & 1 & -\zeta^{-1} \end{pmatrix}, \end{aligned}$$

we have (the last matrix above is also in $G(\mathfrak{o}_v)$):

$$\begin{aligned} I_v &= |\gamma|_v^{n-2} \int_{U_\sigma^-(F_v)} \tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix} 1_{n-2} & & \\ & \gamma & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1_{n-2} & & \\ & 1 & -\zeta^{-1} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1_{n-2} & & \\ & -\zeta^{-1} & \\ & & -\zeta \end{pmatrix}\right) du \\ &= |\gamma|_v^{n-2} \int_{U_\sigma^-(F_v)} \tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix} 1_{n-2} & & \\ & 1 & -\gamma\zeta^{-1} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1_{n-2} & & \\ & -\gamma\zeta^{-1} & \\ & & -\zeta \end{pmatrix}\right) du. \end{aligned}$$

For $u = \begin{pmatrix} 1_{n-2} & x & \\ & 1 & \\ & & 1 \end{pmatrix} \in U_\sigma^-$, we have

$$\sigma u \begin{pmatrix} 1_{n-2} & & \\ & 1 & -\gamma\zeta^{-1} \\ & & 1 \end{pmatrix} = \sigma \begin{pmatrix} 1_{n-2} & & -\gamma\zeta^{-1}x \\ & 1 & -\gamma\zeta^{-1} \\ & & 1 \end{pmatrix} u = \begin{pmatrix} 1 & & -\gamma\zeta^{-1} \\ & 1_{n-2} & -\gamma\zeta^{-1}x \\ & & 1 \end{pmatrix}\sigma u.$$

If $n = 2$, then we have

$$I_v = |\gamma|_v^{n-2}\, \psi_v(\gamma\zeta^{-1})\tilde{W}_{\circ v}\left(\begin{pmatrix} -\gamma\zeta^{-1} & \\ & -\zeta \end{pmatrix}\right).$$

If $n \geq 3$, then we have

$$I_v = |\gamma|_v^{n-2} \int_{U_\sigma^-(F_v)} \psi_v(\gamma\zeta^{-1}u_{n-2,n-1})\tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix} 1_{n-2} & & \\ & -\gamma\zeta^{-1} & \\ & & -\zeta \end{pmatrix}\right) du.$$

With a few more transformations we have $I_v = K_v(\gamma, \zeta, \tilde{W}_{\circ v})$, where K_v is as follows.

Definition 2.2 (Hyper-Kloosterman integral)**.** For $\gamma,\zeta\in F_v^\times$, let:

$$K_v(\gamma,\zeta,\tilde W_{\circ v}) = |\zeta|_v^{n-2}\int_{U_\sigma^-(F_v)} \overline{\psi_v(u_{n-2,n-1})}\tilde W_{\circ v}(\tau u)\,du, \tag{2.5}$$

where

$$\tau = \begin{pmatrix} & & 1\\ 1_{n-2} & & \\ & 1 & \end{pmatrix}\begin{pmatrix} 1_{n-2} & & \\ & -\gamma\zeta^{-1} & \\ & & -\zeta\end{pmatrix}. \tag{2.6}$$

We let $K_R(\gamma,\zeta,\tilde W_{\circ R})=\prod_{v\in R}K_v(\gamma_v,\zeta_v,\tilde W_{\circ v})$ for $\gamma,\zeta\in\mathbb{A}_R^\times$.

The properties of this integral will be studied in detail in section 6. In § 6.2 we evaluate it explicitly in terms of hyper-Kloosterman sums.

2.7. **The ramified places.** Let $v\in S$ be an archimedean or non-archimedean place of F. In this section we make use of the local functional equation, recalled in § 5.1.

Because $\zeta_v=0$, the last matrix in (2.3) is the identity 1_n. We claim that the corresponding integral is equal to

$$I_v = \tilde w_v(\gamma), \tag{2.7}$$

where $\tilde w_v$ is defined via the duality (1.1), see also Lemma 5.2. This follows at once from the following lemma, whose proof is postponed to § 5.3.

Lemma 2.3. *Let $W\in\mathcal{W}(\pi_v,\psi_v)$ be a ψ_v-Whittaker function so that*

$$w_v(y) = W\left(\begin{pmatrix} y & \\ & 1_{n-1}\end{pmatrix}\right),\quad \textit{for all } y\in F_v^\times.$$

Let $\tilde W(g)=W(wg^\iota)$ be the dual Whittaker function (it belongs to the space $\mathcal{W}(\tilde\pi_v,\psi_v^{-1})$). Then for all $y\in F_v^\times$, one has (dx is the self-dual measure with respect to ψ_v):

$$\tilde w_v(y) = \int_{F_v^{n-2}} \tilde W\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix} w'\right)dx. \tag{2.8}$$

2.8. **Conclusion.** Let $\varphi\in\pi$ be as in § 2.1. From Proposition 1.1, we have $\mathcal{P}\varphi=\tilde{\mathcal{P}}\tilde\varphi$. We have seen in § 2.3 that $\mathcal{P}\varphi$ is equal to the left-hand side of (1.2) in Theorem 1. The dual functional $\tilde{\mathcal{P}}\tilde\varphi$ is equal to the sum over $\gamma\in F^\times$ of (2.3). The latter is the product over all places v of F of the respective expressions (2.4), (2.5) and (2.7), which yields the right-hand side of (1.2). This concludes the proof of Theorem 1. □

3. Variants of the main identity.

In this section, we give two variants of Theorem 1. In the Theorem 3 below we allow a shift of the unramified vector. In the Theorem 4 below we consider a summation of Fourier coefficients of a new-vector, which is not necessarily unramified.

Recall that S is a finite set of places of F including the places where π ramifies, where ψ ramifies and all archimedean places. Let $v\notin S$. For $\gamma,\zeta\in F_v^\times$ and $\xi=\mathrm{diag}(\xi_1,\dots,\xi_n)\in T(F_v)$, let:

$$K_v^\sharp(\gamma,\zeta,\xi,\tilde W_{\circ v}) = \begin{cases} \left|\xi_1^{n-2}\xi_3^{-1}\cdots\xi_n^{-1}\right|_v \displaystyle\int_{U_\sigma^-(F_v)} \tilde W_{\circ v}(\tau^\sharp u)\,du & \text{if } \left|\zeta\xi_1^{-1}\xi_2\right|_v\le 1,\\ \left|\zeta^{n-2}\xi_2^{n-2}\xi_3^{-1}\cdots\xi_n^{-1}\right|_v \displaystyle\int_{U_\sigma^-(F_v)} \overline{\psi_v(\xi_2\xi_3^{-1}u_{n-2,n-1})}\tilde W_{\circ v}(\tau^\sharp u)\,du & \text{if } \left|\zeta\xi_1^{-1}\xi_2\right|_v\ge 1,\end{cases}$$

where

$$\tau^\sharp = \sigma\times\begin{cases}\mathrm{diag}(\xi_n^{-1},\xi_{n-1}^{-1},\dots,\xi_3^{-1},\gamma\xi_1^{-1},\xi_2^{-1}) & \text{if } \left|\zeta\xi_1^{-1}\xi_2\right|_v\le 1,\\ \mathrm{diag}(\xi_n^{-1},\xi_{n-1}^{-1},\dots,\xi_3^{-1},-\gamma\zeta^{-1}\xi_2^{-1},-\zeta\xi_1^{-1}) & \text{if } \left|\zeta\xi_1^{-1}\xi_2\right|_v\ge 1.\end{cases}$$

At least when $\xi_2\xi_3^{-1}$ is an integer it is not difficult to express $K_v^\sharp$ in terms of hyper-Kloosterman sums as in § 6.2, details are left to the reader.

Theorem 3. *Let $\zeta \in \mathbb{A}^S$ and $\xi = \mathrm{diag}(\xi_1, \ldots, \xi_n) \in T(\mathbb{A}^S)$. For $v \in S$, let $w_v \in \mathcal{C}_c^\infty(F_v^\times)$ and let $\tilde{w}_v$ be the dual function associated to w_v given by (1.1). Let R be the set of places v such that $|\zeta|_v > 1$ or $\xi_v \notin T(\mathfrak{o}_v)$. Then:*

$$\sum_{\gamma\in F^\times} \psi(\gamma\zeta) W_\circ^S\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\xi\right)\prod_{v\in S} w_v(\gamma) = \sum_{\gamma\in F^\times} K_R^\sharp(\gamma,\zeta,\xi,\tilde{W}_{\circ R})\tilde{W}_\circ^{R\cup S}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\right)\prod_{v\in S}\tilde{w}_v(\gamma)$$

where $K_R^\sharp(\gamma,\zeta,\xi,\tilde{W}_{\circ R}) = \prod_{v\in R} K_v^\sharp(\gamma_v,\zeta_v,\xi_v,\tilde{W}_{\circ v})$.

Proof. We explain how to modify the arguments of § 2. Let $A \in G(\mathbb{A}^S)$ be as in (2.1). We take $\varphi \in \pi$ given by

$$\varphi = \rho(A\xi)\varphi_\circ^S\varphi_S,$$

where $\varphi_\circ^S$ and φ_S are as in § 2.1.

For $v \in R$, $\gamma, \zeta \in F_v^\times$, and $\xi = \mathrm{diag}(\xi_1, \ldots, \xi_n) \in T(F_v)$, let

$$I_v^\sharp = \int_{F_v^{n-2}} \tilde{W}_{\circ v}\left(\begin{pmatrix}\gamma & \\ & 1_{n-1}\end{pmatrix}\begin{pmatrix}1 & & \\ x & 1_{n-2} & \\ & & 1\end{pmatrix} w' A^\iota \xi^{-1}\right) dx.$$

As in § 2.6, we have

$$I_v^\sharp = |\gamma|_v^{n-2}\int_{U_\sigma^-(F_v)} \tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix}1_{n-2} & & \\ & \gamma & \\ & & 1\end{pmatrix}\sigma^{-1} w' A^\iota \xi^{-1}\right) du.$$

If $\left|\zeta\xi_1^{-1}\xi_2\right|_v \le 1$, then we have $\xi A^\iota \xi^{-1} \in G(\mathfrak{o}_v)$, so that

$$I_v^\sharp = |\gamma|_v^{n-2}\int_{U_\sigma^-(F_v)} \tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix}1_{n-2} & & \\ & \gamma & \\ & & 1\end{pmatrix}\sigma^{-1} w' \xi^{-1}\right) du.$$

Since $\sigma^{-1}w'\xi^{-1}w'\sigma = \mathrm{diag}(\xi_n^{-1}, \xi_{n-1}^{-1}, \ldots, \xi_3^{-1}, \xi_1^{-1}, \xi_2^{-1})$ and $w'\sigma \in G(\mathfrak{o}_v)$, we have

$$I_v^\sharp = |\gamma|_v^{n-2}\int_{U_\sigma^-(F_v)} \tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix}\xi_n^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1} & & \\ & & & \gamma\xi_1^{-1} & \\ & & & & \xi_2^{-1}\end{pmatrix}\right) du = K_v^\sharp(\gamma,\zeta,\xi,\tilde{W}_{\circ v}).$$

Assume that $\left|\zeta\xi_1^{-1}\xi_2\right|_v \ge 1$. Since $\sigma^{-1}w'\xi^{-1}w'\sigma \cdot \sigma^{-1}w'\xi A^\iota \xi^{-1} w'\sigma$ is equal to

$$\begin{pmatrix}\xi_n^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1} & & \\ & & & \xi_1^{-1} & \\ & & & & \xi_2^{-1}\end{pmatrix}\begin{pmatrix}1_{n-2} & & \\ & 1 & -\zeta^{-1}\xi_1\xi_2^{-1} \\ & & 1\end{pmatrix}\begin{pmatrix}1_{n-2} & & \\ & -\zeta^{-1}\xi_1\xi_2^{-1} & \\ & & -\zeta\xi_1^{-1}\xi_2\end{pmatrix}\begin{pmatrix}1_{n-2} & & \\ & & -1 \\ & 1 & -\zeta^{-1}\xi_1\xi_2^{-1}\end{pmatrix},$$

we have

$$\begin{aligned} I_v^\sharp &= |\gamma|_v^{n-2}\int_{U_\sigma^-(F_v)} \tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix}1_{n-2} & & \\ & 1 & -\gamma\zeta^{-1} \\ & & 1\end{pmatrix}\begin{pmatrix}\xi_n^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1} & & \\ & & & -\gamma\zeta^{-1}\xi_2^{-1} & \\ & & & & -\zeta\xi_1^{-1}\end{pmatrix}\right) du \\ &= |\gamma|_v^{n-2}\int_{U_\sigma^-(F_v)} \overline{\psi_v(-\gamma\zeta^{-1}u_{n-2,n-1})}\tilde{W}_{\circ v}\left(\sigma u \begin{pmatrix}\xi_n^{-1} & & & & \\ & \ddots & & & \\ & & \xi_3^{-1} & & \\ & & & -\gamma\zeta^{-1}\xi_2^{-1} & \\ & & & & -\zeta\xi_1^{-1}\end{pmatrix}\right) du \\ &= K_v^\sharp(\gamma,\zeta,\xi,\tilde{W}_{\circ v}). \end{aligned}$$

Now the theorem follows from the arguments of § 2. □

We give another variant of Theorem 1. We fix a subset S' of S which does not contain any archimedean places. For $v \in S - S'$, let $w_v \in \mathcal{C}_c^\infty(F_v^\times)$ and let $\tilde{w}_v$ be the dual function associated to w_v given by (1.1). For $v \in S'$, let $W_{\circ v} \in \mathcal{W}(\pi_v, \psi_v)$ be a non-zero new-vector (see [19, § 5]). Namely, $W_{\circ v}$ is right $G(\mathfrak{o}_v)$-invariant when π_v is unramified, and $W_{\circ v}$ is right invariant by the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(\mathfrak{o}_v) \,\middle|\, a \in GL_{n-1}(\mathfrak{o}_v),\, d \in \mathfrak{o}_v^\times,\, d \equiv 1 \bmod \mathfrak{m}_v^{f_{\pi_v}},\, c \equiv 0 \bmod \mathfrak{m}_v^{f_{\pi_v}} \right\}$$

when π_v is ramified, where $f_{\pi_v} \geq 1$ is the conductor of π_v. Note that $W_{\circ v}$ is unique up to scalar. Set

$$w_v(y) = W_{\circ v}\left(\begin{pmatrix} y & \\ & 1_{n-1} \end{pmatrix}\right).$$

We define a smooth function $\tilde{w}_v$ on $F_v^\times$ by

$$\tilde{w}_v(y) = \int_{F_v^{n-2}} \tilde{W}_{\circ v}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ x & 1_{n-2} & \\ & & 1 \end{pmatrix} w'\right) dx,$$

where $\tilde{W}_{\circ v} \in \mathcal{W}(\tilde{\pi}_v, \psi_v^{-1})$ satisfies $\tilde{W}_{\circ v}(g) = W_{\circ v}(wg^\iota)$ for $g \in G(F_v)$. Note that this integral is absolutely convergent by [20, Lemma 2.6].

Theorem 4. *Let $\zeta \in \prod_{v \in S'} \mathfrak{o}_v \times \mathbb{A}^S$. Assume that $\zeta_v \in \mathfrak{m}_v^{f_{\pi_v}}$ for all $v \in S'$ if $n = 2$. For $v \in S$, let w_v and $\tilde{w}_v$ be as above. Let R be the set of places v such that $|\zeta|_v > 1$. Then:*

$$\sum_{\gamma \in F^\times} \psi(\gamma\zeta) W_\circ^S\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix}\right) \prod_{v \in S} w_v(\gamma) = \sum_{\gamma \in F^\times} K_R(\gamma, \zeta, \tilde{W}_{\circ R}) \tilde{W}_\circ^{R \cup S}\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix}\right) \prod_{v \in S} \tilde{w}_v(\gamma)$$

where $K_R(\gamma, \zeta, \tilde{W}_{\circ R})$ is the Kloosterman integral as in Definition 2.2.

Proof. We explain how to modify the arguments of § 2. Let $A \in \prod_{v \in S'} G(\mathfrak{o}_v) \times G(\mathbb{A}^S)$ be as in (2.1). We take $\varphi \in \pi$ given by

$$\varphi = \rho(A)\varphi_\circ^S \varphi_{\circ S'} \varphi_{S-S'},$$

with $\varphi_{\circ S'} = \otimes_{v \in S'} \varphi_{\circ v}$ and $\varphi_{S-S'} = \otimes_{v \in S-S'} \varphi_v$. Here $\varphi_{\circ v} \in \pi_v$ is a non-zero new-vector for $v \in S'$ and $\varphi_v \in \pi_v$ is a vector associated to w_v as in § 2.1 for $v \in S - S'$. For $v \in S'$ and $\gamma \in F^\times$, we have

$$\int_{F_v^{n-2}} \tilde{W}_{\circ v}\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix}\begin{pmatrix} 1 & & \\ x & 1_{n-2} & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & J_{n-1} \end{pmatrix}\begin{pmatrix} 1 & & \\ -\zeta_v & 1 & \\ & & 1_{n-2} \end{pmatrix}\right) dx = \tilde{w}_v(\gamma)$$

since $W_{\circ v}$ is a new-vector and $\zeta_v \in \mathfrak{o}_v$ ($\zeta_v \in \mathfrak{m}_v^{f_{\pi_v}}$ if $n = 2$). Now the theorem follows from the arguments of § 2. □

4. Proof of Proposition 1.1.

Proposition 1.1 is implicitly used by Jacquet, Piatetskii-Shapiro and Shalika to establish the global functional equation of the L-functions for $GL(n) \times GL(1)$, but does not seem to have been written down. In this section, we give details of the proof for the sake of completeness. Note that a similar proof is also given in the appendix of Miller-Schmid [30].

All the measures of this section are normalized to be Tamagawa. To prove Proposition 1.1, it suffices to show that

$$(4.1)\qquad \int_{(F\backslash\mathbb{A})^{(n-1)n/2-1}} \varphi\left(w'\iota\begin{pmatrix} 1 & 0 & u_{1,3} & \cdots & * & * \\ 0 & 1 & u_{2,3} & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} w' \right) \psi(-u_{2,3} - \cdots - u_{n-1,n})\, du$$

$$= \int_{\mathbb{A}^{n-2}} \left[\int_{(F\backslash\mathbb{A})^{(n-1)n/2-1}} \varphi\left(\begin{pmatrix} 1 & 0 & u_{1,3} & \cdots & * & * \\ 0 & 1 & u_{2,3} & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ x_2 & 1 & 0 & \cdots & 0 & 0 \\ x_3 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1} & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right) \right.$$

$$\left. \times \psi(u_{2,3} + \cdots + u_{n-1,n})\, du \right] dx$$

for $\varphi \in \mathcal{A}_{\mathrm{cusp}}(GL_n)$. The left-hand side of (4.1) is equal to

$$\int_{(F\backslash\mathbb{A})^{(n-1)n/2-1}} \varphi\left(w'\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ * & -u_{2,3} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & 1 & 0 \\ * & * & * & \cdots & -u_{n-1,n} & 1 \end{pmatrix} w' \right) \psi(-u_{2,3} - \cdots - u_{n-1,n})\, du$$

$$= \int_{(F\backslash\mathbb{A})^{(n-1)n/2-1}} \varphi\left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & -u_{n-1,n} & \cdots & * & * \\ * & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & 0 & \cdots & 1 & -u_{2,3} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right) \psi(-u_{2,3} - \cdots - u_{n-1,n})\, du$$

$$= \int_{(F\backslash\mathbb{A})^{(n-1)n/2-1}} \varphi\left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ * & 1 & u_{n-1,n} & \cdots & * & * \\ * & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & 0 & \cdots & 1 & u_{2,3} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right) \psi(u_{2,3} + \cdots + u_{n-1,n})\, du.$$

Let

$$U' = \{u' = (u_{i,j}) \in U \mid u_{1,2} = \cdots = u_{1,n} = 0\} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & u_{2,3} & \cdots & * & * \\ 0 & 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & u_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

For $2 \le k \le n$, let

$$\mathbf{U}_k^+ = \left\{ \mathsf{e}_{1,n}(x_n) \cdots \mathsf{e}_{1,k}(x_k) = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_k & \cdots & x_n \\ 0 & 1 & & & \cdots & & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & & & \cdots & & 1 \end{pmatrix} \right\} \subset U,$$

$$\mathbf{U}_k^- = \left\{ \mathsf{e}_{2,1}(y_2) \cdots \mathsf{e}_{k,1}(y_k) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ y_2 & 1 & \cdots & 0 \\ \vdots & & & \\ y_k & \vdots & \ddots & \vdots \\ 0 & & & \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix} \right\} \subset U^-,$$

where $\mathsf{e}_{i,j}(x) = 1_n + x\mathsf{E}_{i,j}$ and $\mathsf{E}_{i,j}$ is the matrix with 1 at the (i,j)-th entry and 0 elsewhere. Let $\mathbf{U}_{n+1}^+ = \mathbf{U}_1^- = \{1_n\}$. Then U' normalizes $\mathbf{U}_{k-1}^-$, and $\mathbf{U}_l^+$ normalizes $\mathbf{U}_{k-1}^- U' \mathbf{U}_{l+1}^+$ for $k+1 \le l \le n$. In particular $\mathfrak{U}_k := \mathbf{U}_{k-1}^- U' \mathbf{U}_{k+1}^+$ is a subgroup of GL_n. Note that the natural map $\mathbf{U}_{k-1}^- \times U' \times \mathbf{U}_{k+1}^+ \to \mathfrak{U}_k$ is bijective. Put

$$\psi(u^- u' u^+) = \psi(u_{2,3} + \cdots + u_{n-1,n})$$

for $u^- \in \mathbf{U}_{k-1}^-(\mathbb{A})$, $u' = (u_{i,j}) \in U'(\mathbb{A})$, $u^+ \in \mathbf{U}_{k+1}^+(\mathbb{A})$. Then ψ defines a character on $\mathfrak{U}_k(\mathbb{A})$ trivial on $\mathfrak{U}_k(F)$.

For $\varphi \in \mathcal{A}_{\mathrm{cusp}}(GL_n)$, let

$$
\begin{aligned}
&\mathsf{F}_k(y_k, \dots, y_{n-1}) \\
&= \int_{\mathfrak{U}_k(F)\backslash\mathfrak{U}_k(\mathbb{A})} \varphi(u\mathsf{e}_{k,1}(y_k)\cdots\mathsf{e}_{n-1,1}(y_{n-1}))\psi(u)\,du \\
&= \int_{\mathfrak{U}_k(F)\backslash\mathfrak{U}_k(\mathbb{A})} \varphi\left(\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & u_{1,k+1} & \cdots & u_{1,n-1} & u_{1,n} \\ u_{2,1} & 1 & u_{2,3} & & & \cdots & & * & * \\ u_{3,1} & 0 & 1 & & & \cdots & & * & * \\ \vdots & & & & & & & & \\ u_{k-1,1} & \vdots & \vdots & & & \ddots & & \vdots & \vdots \\ 0 & & & & & & & & \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & & & \cdots & & 1 & u_{n-1,n} \\ 0 & 0 & 0 & & & \cdots & & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & & & \\ y_k & \vdots & \ddots & \vdots \\ \vdots & & & \\ y_{n-1} & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}\right) \psi(u_{2,3} + \cdots + u_{n-1,n})\,du
\end{aligned}
$$

and

$$\mathcal{I}_k = \int_{\mathbb{A}^{n-k}} \mathsf{F}_k(y_k, \dots, y_{n-1})\,dy_k \cdots dy_{n-1}.$$

The absolute convergence of $\mathcal{I}_k$ will be given later. Note that $\mathcal{I}_n$ is the left-hand side of (4.1) and $\mathcal{I}_2$ is the right-hand side of (4.1).

For each $2 \le k \le n-1$, we first show that

$$\int_{\mathbb{A}} \mathsf{F}_k(y_k, y_{k+1}, \dots, y_{n-1})\,dy_k = \mathsf{F}_{k+1}(y_{k+1}, \dots, y_{n-1}), \tag{4.2}$$

at least formally. We may assume that $y_{k+1} = \cdots = y_{n-1} = 0$. Write $u' = (u_{i,j}) \in U'(\mathbb{A})$ and $u_+ = \mathsf{e}_{1,n}(x_n)\cdots\mathsf{e}_{1,k+1}(x_{k+1}) \in \mathbf{U}_{k+1}^+(\mathbb{A})$. Since

$$\mathsf{e}_{k,1}(y_k)^{-1}u'u_+\mathsf{e}_{k,1}(y_k) = u'_-u'u''u_+ \tag{4.3}$$

with $u'_- = \mathsf{e}_{2,1}(u_{2,k}y_k)\cdots\mathsf{e}_{k-1,1}(u_{k-1,k}y_k)$ and $u'' = 1_n - y_kx_{k+1}\mathsf{E}_{k,k+1} - \cdots - y_kx_n\mathsf{E}_{k,n}$, we have

$$
\begin{aligned}
\mathsf{F}_k(\xi + y_k, 0, \dots, 0) &= \int_{F\backslash\mathbb{A}} \cdots \int_{F\backslash\mathbb{A}} \int_{U'(F)\backslash U'(\mathbb{A})} \int_{\mathbf{U}_{k-1}^-(F)\backslash\mathbf{U}_{k-1}^-(\mathbb{A})} \\
&\quad \times \varphi(u_-u'\mathsf{e}_{1,n}(x_n)\cdots\mathsf{e}_{1,k+1}(x_{k+1})\mathsf{e}_{k,1}(\xi + y_k))\psi(u')\,du_-\,du'\,dx_n\cdots dx_{k+1} \\
&= \int_{F\backslash\mathbb{A}} \cdots \int_{F\backslash\mathbb{A}} \int_{U'(F)\backslash U'(\mathbb{A})} \int_{\mathbf{U}_{k-1}^-(F)\backslash\mathbf{U}_{k-1}^-(\mathbb{A})} \\
&\quad \times \varphi(u_-u'\mathsf{e}_{1,n}(x_n)\cdots\mathsf{e}_{1,k+1}(x_{k+1})\mathsf{e}_{k,1}(y_k))\psi(u')\psi(\xi x_{k+1})\,du_-\,du'\,dx_n\cdots dx_{k+1}
\end{aligned}
$$

for $\xi \in F$ and $y_k \in \mathbb{A}$. By applying the Fourier expansion to the function of x_{k+1}, we have

$$
\begin{aligned}
\sum_{\xi\in F} \mathsf{F}_k(\xi + y_k, 0, \dots, 0) &= \int_{F\backslash\mathbb{A}} \cdots \int_{F\backslash\mathbb{A}} \int_{U'(F)\backslash U'(\mathbb{A})} \int_{\mathbf{U}_{k-1}^-(F)\backslash\mathbf{U}_{k-1}^-(\mathbb{A})} \\
&\quad \times \varphi(u_-u'\mathsf{e}_{1,n}(x_n)\cdots\mathsf{e}_{1,k+2}(x_{k+2})\mathsf{e}_{k,1}(y_k))\psi(u')\,du_-\,du'\,dx_n\cdots dx_{k+2}.
\end{aligned}
$$

Using (4.3) with $x_{k+1} = 0$ and changing variables, we obtain

$$
\begin{aligned}
\sum_{\xi\in F} \mathsf{F}_k(\xi + y_k, 0, \dots, 0) &= \int_{F\backslash\mathbb{A}} \cdots \int_{F\backslash\mathbb{A}} \int_{U'(F)\backslash U'(\mathbb{A})} \int_{\mathbf{U}_{k-1}^-(F)\backslash\mathbf{U}_{k-1}^-(\mathbb{A})} \\
&\quad \times \varphi(\mathsf{e}_{k,1}(y_k)u_-u'\mathsf{e}_{1,n}(x_n)\cdots\mathsf{e}_{1,k+2}(x_{k+2}))\psi(u')\,du_-\,du'\,dx_n\cdots dx_{k+2}.
\end{aligned}
$$

This implies (4.2), at least formally.

Now we have

$$\mathcal{I}_k = \int_{\mathbb{A}^{n-k}} \mathsf{F}_k(y_k, \cdots, y_{n-1})\,dy_k\cdots dy_{n-1} = \int_{\mathbb{A}^{n-k-1}} \mathsf{F}_{k+1}(y_{k+1}, \cdots, y_{n-1})\,dy_{k+1}\cdots dy_{n-1} = \mathcal{I}_{k+1},$$

at least formally. Moreover, the absolute convergence of $\mathcal{I}_k$ implies that of $\mathcal{I}_{k+1}$. Since $\mathcal{I}_2$ is absolutely convergent, this justifies the calculation and we have

$$\mathcal{I}_2 = \mathcal{I}_n.$$

This completes the proof of Proposition 1.1.

4.1. **Integral representations of L-functions.** In this section we briefly recall the constructions of the Eulerian global integrals of the $GL(n) \times GL(1)$ convolution. This enables us to place Proposition 1.1 in that context.

Recall that $Y = \{u = (u_{i,j}) \in U \mid u_{1,2} = 0\}$ and let

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2 \hookrightarrow GL_2 \times GL_1 \times \cdots \times GL_1 \subset G.$$

Then P is the stabilizer in GL_2 of the character $\psi|_Y$.

For $\varphi \in \mathcal{A}_{\mathrm{cusp}}(GL_n)$, we define a cusp form $\mathbb{P}\varphi$ on $P(\mathbb{A})$ by

$$\mathbb{P}\varphi(p) = |\det(p)|^{-(n-2)/2} \int_{Y(F)\backslash Y(\mathbb{A})} \varphi\left(u \begin{pmatrix} p & \\ & 1_{n-2} \end{pmatrix}\right) \overline{\psi(u)}\, du$$

for $p \in P(\mathbb{A})$. Then we have the Fourier expansion

$$\mathbb{P}\varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) = |y|^{-(n-2)/2} \sum_{\gamma \in F^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1_{n-1} \end{pmatrix}\begin{pmatrix} y & \\ & 1_{n-1} \end{pmatrix}\right)$$

for $y \in \mathbb{A}^\times$. It is clear that $\mathbb{P}\varphi(1) = \mathcal{P}\varphi$ for all $\varphi \in \mathcal{A}_{\mathrm{cusp}}(GL_n)$ so that the above is equivalent to Lemma 2.3.

Let χ be a character on $\mathbb{A}^\times/F^\times$. We consider the integral

$$I(s, \varphi, \chi) = \int_{F^\times\backslash\mathbb{A}^\times} \mathbb{P}\varphi\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \chi(y)\, |y|^{s-1/2}\, dy.$$

For $\varphi = \otimes_v \varphi_v$, we choose a decomposition $W_\varphi = \prod_v W_{\varphi_v}$. Then we have

$$I(s, \varphi, \chi) = \prod_v \Psi_v(s, W_{\varphi_v}, \chi_v),$$

where

$$\Psi_v(s, W_{\varphi_v}, \chi_v) = \int_{F_v^\times} W_{\varphi_v}\left(\begin{pmatrix} y_v & \\ & 1_{n-1} \end{pmatrix}\right) \chi_v(y_v)|y_v|_v^{s-(n-1)/2}\, dy_v. \tag{4.4}$$

Recall that $\iota(g) = g^\iota = {}^t g^{-1}$ for $g \in G$. Let $\tilde{\mathbb{P}} = \iota \circ \mathbb{P} \circ \iota$. Then we have

$$\tilde{\mathbb{P}}\tilde{\varphi}(p) = |\det(p)|^{(n-2)/2} \int_{Y(F)\backslash Y(\mathbb{A})} \tilde{\varphi}\left(u^\iota \begin{pmatrix} p & \\ & 1_{n-2} \end{pmatrix}\right) \overline{\psi(u)}\, du$$

for $p \in P(\mathbb{A})$. It is clear that $\tilde{\mathbb{P}}\varphi(1) = \mathcal{P}\tilde{\varphi}$. We consider the integral

$$\tilde{I}(s, \tilde{\varphi}, \chi^{-1}) = \int_{F^\times\backslash\mathbb{A}^\times} \tilde{\mathbb{P}}\tilde{\varphi}\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \chi(y)^{-1}\, |y|^{s-1/2}\, dy. \tag{4.5}$$

Then we have the global functional equation

$$\tilde{I}(1-s, \tilde{\varphi}, \chi^{-1}) = I(s, \varphi, \chi).$$

We have

$$\tilde{I}(s, \tilde{\varphi}, \chi^{-1}) = \prod_v \tilde{\Psi}_v(s, \rho(w')\tilde{W}_{\varphi_v}, \chi_v^{-1}), \tag{4.6}$$

where ρ is the right translation, see §1.7 and where (the measure dy_v has to be the same as in the definition of Ψ_v and dx_v is the Tamagawa measure with respect to ψ_v):

$$\tilde{\Psi}_v(s, \tilde{W}_{\varphi_v}, \chi_v^{-1}) := \int_{F_v^\times} \int_{F_v^{n-2}} \tilde{W}_{\varphi_v} \left(\begin{pmatrix} y_v & 0 & 0 \\ x_v & 1_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \chi_v(y_v)^{-1} |y_v|_v^{s-(n-1)/2} \, dx_v \, dy_v. \tag{4.7}$$

The identity (4.6) is not obvious and is fundamental to the global functional equation. It follows by inserting (1.7) from Proposition 1.1, which reads $\tilde{\mathbb{P}}\varphi(1) = \tilde{\mathcal{P}}\varphi$, into the integral (4.5). Surprisingly details on a proof of (4.6) do not seem to have been published, see [2, section 2] and [30, Appendix A].

5. Background on representations of $GL(n)$ over local fields.

5.1. **Local functional equation.** The reader is referred to the survey by Cogdell [3] for a comprehensive introduction.

Let F be a local field. Let π be an irreducible admissible unitary generic representation of $G(F)$ and χ a character on $F^\times$. Let $W \in \mathcal{W}(\pi, \psi)$ be a ψ-Whittaker function. Jacquet and Shalika [20, 23] proved that $\Psi(s, W, \chi)$ and $\tilde{\Psi}(s, W, \chi)$ given by (4.4) and (4.7) extend to meromorphic functions of s, that the ratios $\dfrac{\Psi(s, W, \chi)}{L(s, \pi \times \chi)}$ and $\dfrac{\tilde{\Psi}(s, W, \chi)}{L(s, \pi \times \chi)}$ are entire (in our notations, Ψ and $\tilde{\Psi}$ correspond to the integrals Ψ_0 and Ψ_{n-2}) respectively. The following functional equation holds (see also Jacquet [24] for another proof when F is archimedean):

$$\tilde{\Psi}(1-s, \rho(w')\tilde{W}, \chi^{-1}) = \chi(-1)^{n-1} \gamma(s, \pi \times \chi, \psi) \Psi(s, W, \chi). \tag{5.1}$$

Recall that ρ is the right translation and $\tilde{W}(g) = W(wg^\iota)$ for $g \in G(F)$ and $\gamma(s, \pi \times \chi, \psi)$ is defined as the ratio $\varepsilon(s, \pi \times \chi, \psi) \cdot \dfrac{L(1-s, \tilde{\pi} \times \chi^{-1})}{L(s, \pi \times \chi)}$.

When F is a number field, π is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, and χ is a character on $\mathbb{A}^\times / F^\times$, we recall that we have the global functional equation

$$L^S(s, \pi \times \chi) = \prod_{v \in S} \gamma(s, \pi_v \times \chi_v, \psi_v) \cdot L^S(1-s, \tilde{\pi} \times \chi^{-1})$$

where S is a sufficiently large finite set of places of F, although we do not use it in the proofs.

Example 1. When $F = \mathbb{R}$, let $\tau(\pi)$ be the semi-simple n-dimensional representation of the Weil group $W_\mathbb{R}$ associated to π by the local Langlands correspondence. The L- and ε-factors may be defined as the Artin-Weil L- and ε-factors of $\tau(\pi)$.

Tempered Maass forms correspond to principal series representations π of $G(\mathbb{R})$. We recall the expression of the L-factors as a product of Γ-functions. These representations are fully induced: $\pi \simeq \mathrm{Ind}_B^G(|.|^{r_1} \otimes \cdots \otimes |.|^{r_n})$, where B is the Borel subgroup of G of upper-triangular matrices. Then $L(s, \pi) = \prod_{i=1}^n \Gamma_\mathbb{R}(s + r_i)$, where

$$\Gamma_\mathbb{R}(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right).$$

Recall that the corresponding local factors over the complex field is $\Gamma_\mathbb{C}(s) := 2(2\pi)^{-s}\Gamma(s)$.

5.2. **Kirillov models.** In this section we show that the space of Whittaker functions is sufficiently large. Let F be a local field and π an irreducible admissible unitary generic representation of $G(F)$.

Lemma 5.1. *Let $w \in \mathcal{C}_c^\infty(F^\times)$ be a smooth compactly supported function. Then there exists a smooth Whittaker function $W \in \mathcal{W}(\pi, \psi)$ so that*

$$w(y) = W\left(\begin{pmatrix} y & \\ & 1_{n-1} \end{pmatrix} \right), \quad \textit{for all } y \in F^\times. \tag{5.2}$$

Proof. Let M be the miraboloic subgroup of G. When F is non-archimedean, according to Bernstein-Zelevinsky [1, p.50] (see also [22, § 1.3]), the Kirillov model $\mathcal{K}(\pi, \psi)$ contains the space c-$\mathrm{ind}_U^M(\psi)$ of smooth functions $f : M(F) \to \mathbb{C}$ so that $f(um) = \psi(u)f(m)$ for all $u \in U(F)$ and $m \in M(F)$ which are compactly supported modulo $U(F)$. When F is archimedean, the corresponding statement is proved in Jacquet-Shalika [22, § 3.8].

Let B^- be the Borel subgroup of G of lower-triangular matrices. According to the Bruhat decomposition, $U(F)B^-(F)$ is an open cell in $G(F)$. If g is a smooth compactly supported function on $B^-(F)$, the function f on $U(F)B^-(F)$ so that $f(ub) = \psi(u)g(b)$ is smooth and well-defined, and extends to a smooth function on $G(F)$ by letting f to be zero on the remaining cells. Restricting f to $M(F)$ we get a function in $\text{c-ind}_U^M(\psi)$. Choosing g suitably we can satisfy (5.2). This concludes the proof of the lemma. □

5.3. **A transform.** In this section we define a generalized Bessel transform $w \rightsquigarrow \tilde{w}$ which is the duality (1.1) and establish some general properties. Both w and $\tilde{w}$ are smooth functions on $F^\times$ where F is a local field (of class $\mathcal{C}^\infty$ when F is archimedean and locally constant when F is non-archimedean). When $F = \mathbb{R}$, the reader may refer to [12, § 6-7] for an alternative approach.

Lemma 5.2. *Let $w \in \mathcal{C}_c^\infty(F^\times)$. There is a unique smooth function $\tilde{w}$ on $F^\times$, of rapid decay at infinity and with at most polynomial growth at zero so that*

$$\int_{F^\times} \tilde{w}(y)\chi(y)^{-1} |y|^{s-\frac{n-1}{2}}\, dy = \chi(-1)^{n-1}\gamma(1-s, \pi\times\chi, \psi) \int_{F^\times} w(y)\chi(y) |y|^{1-s-\frac{n-1}{2}}\, dy \tag{5.3}$$

for all $s \in \mathbb{C}$ of real part sufficiently large and all unitary character $\chi : F^\times \to S^1$. The measures on the left- and right-hand side are the same.

As in the main theorem we assume that w is compactly supported. We note however that the lemma is valid even if we only assume that w is a Schwartz function: smooth and rapidly decaying (i.e. faster than any polynomial) at zero and infinity. In practice, when one applies the Voronoĭ formula to analytic problems, the assumption on the compactness of the support is not an obstacle. One usually applies a dyadic decomposition and estimates trivially the boundary terms.

A familiar example is when $F = \mathbb{R}$, the presence of the characters χ in the above lemma is crucial in order to prescribe the behavior of $\tilde{w}$ on the two components of $\mathbb{R}^\times$.

Proof. First observe that the integral on the right-hand side of (5.3) is defined for all $s \in \mathbb{C}$ and that the γ-factor has no pole when $\Re\mathrm{e}\, s$ is large enough. The existence and unicity of $\tilde{w}$ then easily follow from the properties of the Mellin transform over local fields (see the chapter 2 of [35] for example). More precisely, if $A(s, \chi)$ denotes the function on the right-hand side of (5.3), we have for σ large enough:

$$\tilde{w}(y) = c\,|y|^{\frac{n-1}{2}-\sigma} \int A(\sigma, \chi)\chi(y)\, d\chi$$

where c is a certain constant which depends on the normalization of the Haar measures, and the integration is over the group of unitary characters on $F^\times$. □

It is noteworthy to observe that the same transform appears in the approximate functional equation method for the special values of L-functions (this is essentially the case $\zeta = 0$ in the Voronoĭ formula). The reader may refer to [15] or [16, section 5.2] for more details.

Next we shall give a proof of Lemma 2.3 which explains the relevance of the transform in the context of Whittaker integrals. This lemma would give an alternative definition of the function $\tilde{w}$, since by Lemma 5.1, the Whittaker function W always exists. However it is more difficult to extract from the integral (2.8) the analytic properties of $\tilde{w}$ (e.g. rapid decay at infinity).

Proof of Lemma 2.3. This is achieved by a Mellin transform and an application of the local functional equation.

First the integral in the right-hand side of (2.8) is absolutely convergent, it defines a smooth function $y \mapsto \mathfrak{H}(y)$ of rapid decay at infinity and moderate growth at zero. This is not obvious and may be achieved by majorizing the Whittaker function by a gauge (see [23, p.134] and [17, § 2.3]). Uniform bounds may be found in [20, Lemma 2.6] for non-archimedean fields and in Jacquet-Shalika [23, Lemma 5.1] for archimedean fields (see also [24, § 11]).

All the Mellin integrals will be well-defined for $\Re e\, s$ sufficiently large. By the definition (4.7) of $\tilde{\Psi}$ we have:

$$\int_{F^\times} \mathfrak{H}(y)\chi(y)^{-1}\,|y|^{s-\frac{n-1}{2}}\,dy = \tilde{\Psi}(s,\rho(w')\tilde{W},\chi^{-1}).$$

On the other hand by the definition (4.4):

$$\int_{F^\times} w(y)\chi(y)\,|y|^{s-\frac{n-1}{2}}\,dy = \Psi(s,W,\chi).$$

The function $y \mapsto \mathfrak{H}(y)$ satisfies (5.3), as follows from (5.1) with $1-s$ in place of s. Indeed the corresponding integral of $\mathfrak{H}$ in the left-hand side of (5.3) is equal to $\tilde{\Psi}(s,\rho(w')\tilde{W},\chi^{-1})$ and the corresponding integral of $\mathfrak{H}$ in the right-hand side of (5.3) is equal to $\Psi(1-s,W,\chi)$. Thus we have $\mathfrak{H} = \tilde{w}$ because of the unicity of such a function. This completes the proof of the Lemma 2.3. □

6. Background on Kloosterman sums.

A good reference is Stevens [34]. Let N be the normalizer of the standard torus T. Let $U^+ = U$ be the subgroup of upper-triangular unipotent matrices and U^- the opposite unipotent subgroup. For $\tau \in N(F)$, we have the decomposition:

$$U = U_\tau^+ U_\tau^- = U_\tau^- U_\tau^+,$$

where $U_\tau^\pm := U \cap \tau^{-1}U^\pm\tau$.

From now on the ground field is a non-archimedean local field F with ring of integers $\mathfrak{o}$ and residue field $\mathbb{F}_q$.

Definition 6.1 (Kloosterman integral)**.** Let $\tau \in N(F)$. Let ψ and ψ' be unitary characters on $U(F)$ and $U_\tau^-(F)$ respectively. Let W be a ψ-Whittaker function. The Kloosterman integral associated to these data is:

$$K(W,\psi',g) := \int_{U_\tau^-(F)} W(\tau u g)\overline{\psi'(u)}\,du, \quad g \in G(F).$$

Below are some remarks to draw comparison with conventions in previous works. These observations will not be used in our computation of Kloosterman sums and the reader might go directly to the next section.

Remark 4. Our definition differs from the one given in Stevens [34]. There the character ψ' is defined on $U(F)$ and it is required that ψ' coincides with $\psi(\tau.\tau^{-1})$ on $U_\tau^+(F)$. In the context of the Coxeter element in the hyper-Kloosterman integral from Definition 2.2, the character ψ' could always be extended from $U_\tau^-(F)$ to $U(F)$ so that there is no real difference. For the sake of clarity we have preferred the simpler definition above.

Remark 5. Assume that the character ψ' on $U_\tau^-(F)$ may be extended on $U(F)$ so as to coincide with $\psi(\tau.\tau^{-1})$ on $U_\tau^+(F)$. Then the function $u \mapsto W(\tau u g)\overline{\psi'(u)}$ on $U(F)$ is left $U_\tau^+(F)$-invariant. Recall that the Haar measure on $U(F)$ coincides with the product of the Haar measures on $U_\tau^+(F)$ and $U_\tau^-(F)$. As a consequence the integration $\int_{U_\tau^-(F)} du$ could be seen over $U_\tau^+(F)\backslash U(F)$ as well. In particular the Kloosterman integral $g \mapsto K(W,\psi',g)$ is left $U(F)$-invariant under that assumption.

Remark 6. The effect of a left translation from $T(F)$ may be inferred by a change of variable: the character ψ' will be replaced by a conjugate. When W is unramified and the assumption in the previous remark is satisfied, the Kloosterman integral is both left $U(F)$-invariant and right $G(\mathfrak{o})$-invariant. Therefore it is possible to apply the Iwasawa decomposition to reduce to the case $g = 1$.

6.1. **Kloosterman sums.**

Now we want to obtain more traditional formulae in the unramified case. We recall in this paragraph a standard definition of Kloosterman sums. For $\tau \in N(F)$, consider the following set:

$$C(\tau) := U(F)\tau U_\tau^-(F) \cap G(\mathfrak{o}).$$

The intersection will be non-empty in the cases we are interested in. Form the following double quotient:

$$X(\tau) := U(\mathfrak{o})\backslash C(\tau)/U_\tau^-(\mathfrak{o}).$$

It follows from the Bruhat decomposition that we have well-defined maps $u : X(\tau) \to U(\mathfrak{o})\backslash U(F)$ and $u' : X(\tau) \to U_\tau^-(F)/U_\tau^-(\mathfrak{o})$ so that:

$$x = u(x) \cdot \tau \cdot u'(x), \quad \text{for } x \in X(\tau).$$

Indeed we recall that $U\tau U = U\tau U_\tau^-$ (Bruhat cell). Observe that both u and u' are injective.

Definition 6.2 (Kloosterman sums)**.** Let $\tau \in N(F)$. Let ψ be a character on $U(F)$ trivial on $U(\mathfrak{o})$ and ψ' a character on $U_\tau^-(F)$ trivial on $U_\tau^-(\mathfrak{o})$. The Kloosterman sum is defined as:

$$\mathcal{K}\ell(\psi, \psi', \tau) := \sum_{x \in X(\tau)} \psi(u(x))\psi'(u'(x)).$$

When W is invariant under $G(\mathfrak{o})$, the Kloosterman integral from Definition 6.1 may be expressed as a finite sum of Kloosterman sums:

$$K(W, \psi', 1) = \sum_{t \in T(F)/T(\mathfrak{o})} W(t)\mathcal{K}\ell(t^{-1}\tau, \bar{\psi}(t.t^{-1}), \bar{\psi}'). \tag{6.1}$$

The idea is to apply the Iwasawa decomposition to τu with $u \in U_\tau^-(F)$ which yields $\tau u = tv^{-1}x$ with $v \in U(F)$, $t \in T(F)$ and $x \in G(\mathfrak{o})$; so that $W(\tau u) = \psi(tvt^{-1})^{-1}W(t)$. Furthermore, since:

$$vt^{-1}\tau u = x$$

is a Bruhat cell decomposition we infer that $x \in C(t^{-1}\tau)$. When u is in the quotient $U_\tau^-(F)/U_\tau^-(\mathfrak{o})$, t is in $T(F)/T(\mathfrak{o})$, v is in $U(\mathfrak{o})\backslash U(F)$ and x then runs though the quotient $X(t^{-1}\tau)$; moreover $u = u'(x)$ and $v = u(x)$. Observe that $C(t^{-1}\tau)$ is empty for all but finitely many $t \in T(F)/T(\mathfrak{o})$.

The previous proof is an adaptation of the proof in [34, Theorem 2.12]. We do not require the assumption that ψ' extends on $U(F)$ and coincides with $\psi(\tau.\tau^{-1})$ for assertion (6.1) to be valid, but one can check that it is not used in the proof. We recall that $\mathrm{vol}(U_\tau^-(\mathfrak{o})) = 1$ which is also the normalization used in [34].

6.2. **Hyper-Kloosterman integral.** In this section we shall compute the following Kloosterman integral which appeared in § 2.6, Definition 2.2. Assume that $n \geq 3$ and that π, ψ, and $\tilde{W} \in \mathcal{W}(\tilde{\pi}, \psi^{-1})$ are unramified. Let $\gamma, \zeta \in F^\times$. The integral was:

$$K = K(\gamma, \zeta, \tilde{W}) = |\zeta|^{n-2} \int_{U_\tau^-(F)} \overline{\psi(u_{n-2,n-1})}\tilde{W}(\tau u)\, du,$$

where

$$\tau = \begin{pmatrix} & & 1 \\ 1_{n-2} & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1_{n-2} & & \\ & -\gamma\zeta^{-1} & \\ & & -\zeta \end{pmatrix}.$$

By (6.1), we have

$$K = |\zeta|^{n-2} \sum_{t \in T(F)/T(\mathfrak{o})} \tilde{W}(t)\mathcal{K}\ell(t^{-1}\tau, \psi_t, \bar{\psi}),$$

where $\psi_t(u) = \psi(tut^{-1})$ for $u \in U(F)$. Note that $\tilde{W}$ is a $\bar{\psi}$-Whittaker function unlike in Definition 6.1. For $t \in T(F)$, we have $\tilde{W}(t) = 0$ unless $t \in T(F)^+$, where

$$T(F)^+ = \{t = \mathrm{diag}(t_1, \ldots, t_n) \in T(F) \mid |t_1| \leq |t_2| \leq \cdots \leq |t_n|\}. \tag{6.2}$$

We have chosen to work with Kloosterman sums because this enables to quote a few lemmas from Stevens [34], but we feel that the framework of Kloosterman integrals is more natural in several aspects.

We have

$$t^{-1}\tau = \begin{pmatrix} 0 & \cdots & 0 & -\gamma\zeta^{-1}t_1^{-1} & 0 \\ t_2^{-1} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & t_{n-1}^{-1} & 0 & 0 \\ 0 & \cdots & 0 & 0 & -\zeta t_n^{-1} \end{pmatrix}.$$

By [34, Corollary 3.11], we have:

$$K = |\zeta|^{n-2} \sum_{\substack{t=\operatorname{diag}(t_1,\ldots,t_n)\in T(F)^+/T(\mathfrak{o}) \\ |t_n|=|\zeta|}} \tilde{W}(t)\mathcal{K}\ell(s,\psi_t,\bar{\psi}),$$

where

$$s = s(t,\gamma,\zeta) = \begin{pmatrix} 0 & \cdots & 0 & -\gamma\zeta^{-1}t_1^{-1} \\ t_2^{-1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & t_{n-1}^{-1} & 0 \end{pmatrix}$$

($\mathcal{K}\ell$ is the $(n-1)$-dimensional Kloosterman sum). Note that ψ_t is $U(\mathfrak{o})$-invariant for $t \in T(F)^+$, and the 1-dimensional Kloosterman sum:

$$\mathcal{K}\ell(-\zeta t_n^{-1}, \psi_t, \bar{\psi}) = \begin{cases} 1 & \text{if } |t_n| = |\zeta|, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.3. *Let $t = \operatorname{diag}(t_1,\ldots,t_n) \in T(F)^+$ with $|t_n| = |\zeta|$. The Kloosterman sum $\mathcal{K}\ell(s,\psi_t,\bar{\psi})$ is zero unless:*

$$|t_2| \geq 1, \ldots, |t_{n-1}| \geq 1,$$

and

$$|t_1 t_2 \cdots t_{n-1}| = \left|\gamma\zeta^{-1}\right|.$$

Proof. In general, according to Stevens [34, th. 3.12], $C(\tau)$ is non-empty if and if only $\det(\tau) \in \mathfrak{o}^\times$ and every exposed subdeterminant of τ is integral. We do not recall the definition of exposed subdeterminant, but it is not difficult to see that this criterion is equivalent to the statement above in our case. ☐

Thus we have

$$K = |\zeta|^{n-2} \sum_{\substack{t=\operatorname{diag}(t_1,\ldots,t_n)\in T(F)^+/T(\mathfrak{o}) \\ 1\leq|t_2|\leq\cdots\leq|t_{n-1}|\leq|t_n|=|\zeta| \\ |t_1t_2\cdots t_{n-1}|=\left|\gamma\zeta^{-1}\right|}} \tilde{W}(t)\mathcal{K}\ell(s,\psi_t,\bar{\psi}). \tag{6.3}$$

6.3. **Hyper-Kloosterman sums.** In this section we shall compute the hyper-Kloosterman sum $\mathcal{K}\ell(s,\psi_t,\bar{\psi})$ in (6.3). Let $t = \operatorname{diag}(t_1,\ldots,t_n) \in T(F)^+$ so that $1 \leq |t_2| \leq \cdots \leq |t_{n-1}| \leq |t_n| = |\zeta|$ and $|t_1 t_2 \cdots t_{n-1}| = \left|\gamma\zeta^{-1}\right|$. To ease the following computations we change notations slightly. Let $r = n-1 \geq 2$; we work in GL_r. Let $\rho = \begin{pmatrix} & 1 \\ 1_{r-1} & \end{pmatrix}$. Then $U_\rho^- = U \cap \rho^{-1}U^-\rho$ consists of matrices of the form:

$$U_\rho^- = \left\{\begin{pmatrix} 1_{r-1} & * \\ & 1 \end{pmatrix}\right\}, \qquad \rho U_\rho^- \rho^{-1} = \left\{\begin{pmatrix} 1 & \\ * & 1_{r-1} \end{pmatrix}\right\}.$$

Write $s_1 = -\gamma\zeta^{-1}t_1^{-1}$, and $s_i = t_i^{-1}$ for $2 \leq i \leq r$. Let $a = \operatorname{diag}(s_1,\ldots,s_r)$ be the r times r diagonal matrix with entries $s_1,\ldots,s_r$. We now have:

$$s = a\rho = \begin{pmatrix} 0 & \cdots & 0 & s_1 \\ s_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & s_r & 0 \end{pmatrix}.$$

Furthermore $|\det(s)| = |s_1 s_2 \cdots s_r| = 1$ and $s_2, \dots, s_r \in \mathfrak{o}$.

From now on we differ from Stevens [34] because we need to consider the general case where some of the s_i are allowed to be units. The following proposition provides a recursive construction of $X(s)$ which is of great practical use when evaluating the Kloosterman sum. It is close in its principle to Dabrowski-Reeder [4] who provide an explicit description of the sets $X(s)$ in terms of various root data. Their description is complicated even in the present case of GL_r, although it is sufficient for the purpose of obtaining the cardinality of $X(s)$. With some work it should be possible to compute the hyper-Kloosterman sum $\mathcal{K}\ell(s, \psi_t, \bar{\psi})$ from [4]. Anyway we follow a third approach for which we provide a self-contained treatment. A fourth approach would be to use Plücker coordinates as in Friedberg [10], a brief sketch is outlined in Remark 7.

Proposition 6.4. *Let notations be as above.*

(i) When $s_i \in \mathfrak{o}^\times$ for all $1 \le i \le r$, $X(s)$ is a singleton.

(ii) Assume that $|s_r| < 1$. For $v \in s_r^{-1}\mathfrak{o}^\times$, let $s_v = \begin{pmatrix} 0 & \cdots & 0 & -s_1 v^{-1} \\ s_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & s_{r-1} & 0 \end{pmatrix}$ *be an $r-1$ times $r-1$ matrix. There is a canonical surjective map*

$$X(s) \longrightarrow \bigsqcup_{v \in s_r^{-1}\mathfrak{o}^\times/\mathfrak{o}} X(s_v), \quad x \longmapsto y$$

whose fibers have the same cardinality $|s_r|^{2-r} = \left|s_r^{-1}\mathfrak{o}^\times/\mathfrak{o}\right|^{r-2}$. It satisfies the following recursive property: the $(r-1,r)$-entry of $u'(x)$ is equal to v; $u(y)$ is the restriction of $u(x)$ to the $r-1$ times $r-1$ upper left corner; and the $(r-1,r)$-entry of $u(x)$ is equal to $s_r^{-1}s_{r-1}$ times the $(r-2,r-1)$-entry of $u'(y)$.

In the quotient $s_r^{-1}\mathfrak{o}^\times/\mathfrak{o}$ above, the group $\mathfrak{o}$ acts additively on $s_r^{-1}\mathfrak{o}^\times$. One could extend the formula to $|s_r| = 1$ so as to include case (i) with the convention that $s_r^{-1}\mathfrak{o}^\times/\mathfrak{o} = \{1\}$, i.e. imposing that v is equal to 1. We prefer distinguish the two cases (i) and (ii) for the sake of clarity.

Proof. (i) When $s \in GL_r(\mathfrak{o})$, it is not difficult to check that the condition $usu' \in GL_r(\mathfrak{o})$ implies that $u, u' \in GL_r(\mathfrak{o})$. Therefore $X(s)$ is a singleton.

(ii) For $1 \le i \le r-1$, let v_i be the (i,r)-entry of $u'(x)$. Put $v = v_{r-1}$. We recall the Bruhat decomposition in $C(s)$ given by $x = u(x)su'(x)$, which we look upon in the following way:

$$u(x)^{-1}x = su'(x) = \begin{pmatrix} 0 & \cdots & 0 & s_1 \\ s_2 & \cdots & 0 & s_2 v_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & s_r & s_r v_{r-1} \end{pmatrix}. \tag{6.4}$$

Since $|s_r| < 1$ one necessarily has $s_r v = s_r v_{r-1} \in \mathfrak{o}^\times$ because $x \in GL_r(\mathfrak{o})$. Therefore there is a unique unipotent matrix $u_1 = \begin{pmatrix} 1_{r-1} & * \\ 0 & 1 \end{pmatrix} \in U_\rho^-$ with the following property. The last column of $u_1 x$ consists of zeros except the (r,r)-entry. Namely:

$$u_1 x = \begin{pmatrix} \begin{matrix} & y & \\ 0 & & s_r \end{matrix} & \begin{matrix} 0 \\ s_r v \end{matrix} \end{pmatrix} \tag{6.5}$$

for some $r-1$ times $r-1$ matrix y. We now proceed to verify that the map $x \mapsto y$ has the required properties.

Since $x \in GL_r(\mathfrak{o})$ one sees by inspection on the last column of (6.5) that $u_1 \in U_\rho^-(\mathfrak{o})$. Therefore $u_1 x \in GL_r(\mathfrak{o})$ and thus $y \in GL_{r-1}(\mathfrak{o})$. Also the $(r-1,r)$-entry of $u(x)$ is equal to $-s_{r-1}v_{r-2}s_r^{-1}v^{-1}$ modulo $\mathfrak{o}$.

Let $\rho' = \begin{pmatrix} & 1 \\ 1_{r-2} & \end{pmatrix}$. We deduce from (6.4) the following equality

$$su'(x) = u_5 \begin{pmatrix} s_v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & s_r & s_r v \end{pmatrix}, \tag{6.6}$$

where $u_5 \in U_\rho^-$ and $u_3 \in U_{\rho'}^-$. Indeed we shall give an explicitly construction for u_5 and u_3 which will be uniquely determined by $u'(x)$.

Put

$$u_5 = \begin{pmatrix} 1 & 0 & \cdots & 0 & s_1 s_r^{-1} v^{-1} \\ 0 & 1 & \cdots & 0 & s_2 v_1 s_r^{-1} v^{-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s_{r-1} v_{r-2} s_r^{-1} v^{-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

In U_ρ^-, we write $u'(x) = u_v u_2$ where $u_v = \begin{pmatrix} 1_{r-2} & & \\ & 1 & v \\ & & 1 \end{pmatrix}$. We construct u_3 as follows:

$$\begin{pmatrix} u_3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1_{r-2} & & \\ & & -v \\ & 1 & \end{pmatrix} u_2 \begin{pmatrix} 1_{r-2} & & \\ & & 1 \\ & -v^{-1} & \end{pmatrix}.$$

It is not difficult to see that $u_3 \in U_{\rho'}^-$. Inserting the above in the right-hand side of (6.6) and comparing with the right-hand side of (6.4) we infer the equality (6.6).

It follows from $u_1 x = u_1 u(x) s u'(x)$, (6.5) and (6.6) that

$$u_1 x = \begin{pmatrix} u_4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & 1 & 0 \\ 0 & \cdots & s_r & s_r v \end{pmatrix}, \tag{6.7}$$

for a certain unipotent upper-triangular matrix u_4. We deduce that $y = u_4 s_v u_3$. Therefore $y \in C(s_v)$, $u_4 = u(y)$ and $u_3 = u'(y)$. Note that $u_1 u(x) u_5 = \begin{pmatrix} u_4 & 0 \\ 0 & 1 \end{pmatrix}$ and the $(r-2, r-1)$-entry of u_3 is $-v_{r-2} v^{-1}$.

The map $x \mapsto y$ descends to a well-defined map from $X(s)$ to $X(s_v)$. Indeed $u_3 \in U_{\rho'}^-(\mathfrak{o})$ as soon as $u_2 \in U_\rho^-(\mathfrak{o})$ (recall that $|v| > 1$). Similarly, if $u(x) \in U(\mathfrak{o})$ then $u_4 \in U_{r-1}(\mathfrak{o})$.

Conversely, given $y = u_4 s_v u_3 \in C(s_v)$, it is not difficult to see that the right-hand side in (6.7) belongs to $C(s)$, and that its image in $C(s_v)$ is equal to y. Furthermore when the image of x is y, the $(r-1, r)$-entry of $u'(x)$ is equal to v and the remaining entries are uniquely determined modulo $v^{-1}\mathfrak{o}$ by u_3 and v. Therefore there are $|v|^{r-2}$ possibilities for $u'(x)$. Then because of (6.5), $u(x)$ is uniquely determined up to multiplication by u_1. Since $u_1 \in U_\rho^-(\mathfrak{o})$ we have that $x \in X(s)$ is uniquely determined by y and $u'(x)$ as well. This concludes the proof of (ii). □

Corollary 6.5. *Let notations be as above. Let ψ' be a character on F, trivial on $\mathfrak{o}$.*

(i) If $t_{n-1} \in \mathfrak{o}^\times$ then $t_i \in \mathfrak{o}^\times$ for all $2 \leq i \leq n-1$ and we have

$$\mathcal{K}\ell(s, \psi_t, \psi') = 1.$$

(ii) Assume that $|t_{n-1}| > 1$. Then:

$$\mathcal{K}\ell(s, \psi_t, \psi') = \sum_{v \in t_{n-1}\mathfrak{o}^\times/\mathfrak{o}} \psi'(v) \mathcal{K}\ell(s_v, \psi_t^\flat, \psi),$$

where $\psi_t^\flat$ is the restriction of ψ_t to $U_{n-2}(F)$.

Proof. We apply Proposition 6.4. The claim (i) is easy. For (ii), we use the definition of Kloosterman sums and the recursive description of $X(s)$:

$$\begin{aligned}\mathcal{K}\ell(s,\psi_t,\psi') &= \sum_{x\in X(s)} \psi_t(u(x))\psi'(u'(x)) \\ &= \sum_{v\in s_r^{-1}\mathfrak{o}^\times/\mathfrak{o}} \sum_{y\in X(s_v)} \psi_t^\flat(u(y))\psi^\diamond(u'(y))\psi'(v)\end{aligned}$$

where $\psi^\diamond$ is given by the following rule: form the upper-triangular unipotent matrix with only non-trivial entry $(r-1,r)$ which is given by $s_r^{-1}s_{r-1}$ times the $(r-2,r-1)$ entry of $u'(y)$, then apply the character ψ_t. Since $\psi_t(u)=\psi(tut^{-1})$, $t=\mathrm{diag}(t_1,\ldots,t_n)$ and $s_i=t_i^{-1}$, we see that $\psi^\diamond(u'(y))$ is equal to $\psi(u'(y))$. This concludes the proof of the corollary. □

The following is usual, we provide a proof in our notation for consistency.

Lemma 6.6. *Let $s=\begin{pmatrix}0 & s_1\\ s_2 & 0\end{pmatrix}$ with $s_1\in F$ and $s_2\in\mathfrak{o}$ so that $|s_1s_2|=1$. Let ψ_1 and ψ_2 be two characters on F, trivial on $\mathfrak{o}$. Then the Kloosterman sum is given by:*

$$\mathcal{K}\ell(s,\psi_1,\psi_2)=\begin{cases}1 & \text{if } |s_2|=1,\\ \displaystyle\sum_{v\in s_2^{-1}\mathfrak{o}^\times/\mathfrak{o}}\psi_1(-s_2^{-1}s_1v^{-1})\psi_2(v) & \text{if } |s_2|<1.\end{cases}$$

Proof. Consider $x=u(x)su'(x)\in X(s)$, and let $u'(x)=\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}$ and $u(x)=\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$, then:

$$x=\begin{pmatrix}s_2u & s_1+s_2uv\\ s_2 & s_2v\end{pmatrix}.$$

The claim easily follows from the fact that $x\in GL_2(\mathfrak{o})$. □

Corollary 6.7. *Let notations be as above. Then the hyper-Kloosterman sum is given by:*

$$\mathcal{K}\ell(s,\psi_t,\bar{\psi})=\sum_{v_{n-1}\in t_{n-1}\mathfrak{o}^\times/\mathfrak{o}}\cdots\sum_{v_2\in t_2\mathfrak{o}^\times/\mathfrak{o}}\psi(v_{n-1}+\cdots+v_2)\psi((-1)^n\gamma\zeta^{-1}v_2^{-1}\ldots v_{n-1}^{-1}).$$

Proof. Recall that $1\le|t_2|\le\cdots\le|t_{n-1}|$. If $|t_2|>1$, then we have

$$\mathcal{K}\ell(s,\psi_t,\bar{\psi})=\sum_{v_{n-1}\in t_{n-1}\mathfrak{o}^\times/\mathfrak{o}}\cdots\sum_{v_2\in t_2\mathfrak{o}^\times/\mathfrak{o}}\psi(-v_{n-1}+v_{n-2}+\cdots+v_2)\psi((-1)^{n-2}t_1t_2^{-1}s_1s_2^{-1}v_2^{-1}\ldots v_{n-1}^{-1})$$

by Corollary 6.5 and Lemma 6.6. Thus the claim follows from the assumption that $s_1t_1=-\gamma\zeta^{-1}$ and $s_2t_2=1$. If $|t_2|=1$, we let $k=\max\{2\le i\le n-1\mid |t_i|=1\}$. Similarly, we have

$$\sum_{v_{n-1}\in t_{n-1}\mathfrak{o}^\times/\mathfrak{o}}\cdots\sum_{v_{k+1}\in t_{k+1}\mathfrak{o}^\times/\mathfrak{o}}\psi(-v_{n-1}+v_{n-2}+\cdots+v_{k+1})=\begin{cases}(-1)^{n-k-1} & \text{if } |t_{n-1}|\le q,\\ 0 & \text{if } |t_{n-1}|>q.\end{cases}$$

Thus the claim follows from the assumption that $|t_1|\le|t_2|=1$, $|t_1t_2\cdots t_{n-1}|=\left|\gamma\zeta^{-1}\right|$, and ψ is trivial on $\mathfrak{o}$. □

Remark 7. In this remark we briefly outline the relationship with the computations by Friedberg [10]. The computations in [10] are performed over the base ring $\mathbb{Z}$, but one may check that the proofs are valid over a local ring as well. We consider $\mathcal{K}\ell(s,\psi_t,\bar{\psi})$ as given in (6.3). In [10] it is required that $\det(s)=1$, so without loss of generality we assume that $t_1\cdots t_r=-\gamma\zeta^{-1}$. The dimension n in [10] corresponds to our integer $r=n-1$. The character θ_1 in [10] corresponds to ψ_t and $\alpha_i=t_it_{i+1}^{-1}$ for $1\le i\le r-1$ in the notations of [10]. The character θ_2 corresponds to $\bar{\psi}$, and $\beta_{r-1}=-1$; the other values of β are unimportant and uniquely determined, see our remarks following Definition 6.1. The diagonal matrix c in [10] corresponds to our a, and $c_1=t_r^{-1}$, $c_2/c_1=t_{r-1}^{-1}$, $\ldots$, $c_{r-1}/c_{r-2}=t_2^{-1}$ (the value of c_{r-1} is unimportant). The sum $S(\theta_1,\theta_2,c,w)$ in [10] is then equal to our $\mathcal{K}\ell(s,\psi_t,\bar{\psi})$. The Theorem 4.3 from [10] corresponds to our Corollary 6.7 under the following modifications for the right-hand sides: $x_1 = -v_r t_r^{-1}$, $x_2\overline{x_1} = v_{r-1}t_{r-1}^{-1}$, ..., $x_{r-1}\overline{x_{r-2}} = v_2 t_2^{-1}$. The first entry in the sum is equal to $t_1\overline{x_{r-1}}$ which equals our $\gamma\zeta^{-1}v_k^{-1}\dots v_{n-1}^{-1}$ by a straightforward verification. The other entries correspond to our v_i for $2 \le i \le r$.

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