Multicorns are not path connected

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Milnor, that intrepid explorer,
Traveled cubics in hopes to discover
Some exotic new beast:
Northwest and southeast
He found tricorns lurking there under cover.

ABSTRACT. The tricorn is the connectedness locus in the space of antiholomorphic quadratic polynomials $z \mapsto \bar{z}^2 + c$. We prove that the tricorn is not locally connected and not even pathwise connected, confirming an observation of John Milnor from 1992. We extend this discussion more generally for antiholomorphic unicritical polynomials of degrees $d \geq 2$ and their connectedness loci, known as multicorns.

1 INTRODUCTION

The **multicorn** $\mathcal{M}_d^*$ is the connectedness locus in the space of antiholomorphic unicritical polynomials $p_c(z) = \bar{z}^d + c$ of degree d, i.e., the set of parameters for which the Julia set is connected. The special case $d = 2$ is the **tricorn**, which is the formal antiholomorphic analog to the Mandelbrot set.
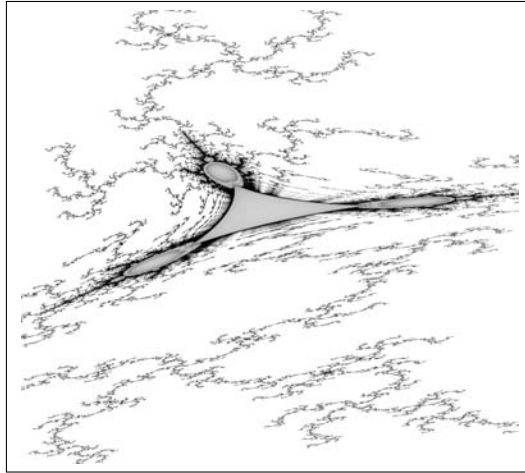

FIGURE 1.1: A "little tricorn" within the tricorn $\mathcal{M}_2^*$ illustrating that the "umbilical cord" converges to the little tricorn without landing at it. (See also Plate 2.)

The second iterate is

$$p_c^{\circ 2}(z) = \overline{\left(\overline{z^d} + c\right)^d} + c = (z^d + \overline{c})^d + c$$

and thus is holomorphic in the dynamical variable z but no longer complex analytic in the parameter c. Much of the dynamical theory of antiholomorphic polynomials (in short, antipolynomials) is thus in analogy to the theory of holomorphic polynomials, except for certain features near periodic points of odd periods. For instance, a periodic point of odd period k may be the simultaneous landing point of dynamic rays of periods k and $2k$ (which is invisible from the holomorphic second iterate of the first return map); see [NS, Lemma 3.1].

However, the theory of parameter space of multicorns is quite different from that of the Mandelbrot set and its higher-degree cousins, the multibrot sets of degree d, because the parameter dependence is only real analytic. Already the open mapping principle of the multiplier map fails, so it is not a priori clear that every indifferent orbit is on the boundary of a hyperbolic component and that bifurcations multiplying periods occur densely on boundaries of hyperbolic components. However, it turns out that many properties of parameter space are quite similar to that of the Mandelbrot set, except near hyperbolic components of odd period. For instance, there is a simple recursive relation for the number of hyperbolic components of period n for the multibrot set, given by $s_{d,n} = d^{n-1} - \sum_{k|n, k<n} s_{d,k}$: for multicorns, the same result holds, except if n is twice an odd number; in that case, the number of hyperbolic components equals $s_{d,n} + 2s_{d,n/2}$ [MNS]. Similarly, the multiplier map is an open map on the closure of any hyperbolic component of even period, except where it intersects the boundary of an odd-period-hyperbolic component.

However, boundaries of odd-period hyperbolic components have some quite interesting properties. The multiplier map is constant along their boundaries (all boundary points have parabolic orbits of multiplier $+1$); bifurcations only double the period (no higher factors), and these period doublings occur along arcs rather than at isolated points (see Corollary 3.9). Adjacent to these parabolic arcs, there are comb-like structures where the multicorn fails to be locally connected, and $(\sin 1/x)$-like structures accumulate on the centers of many parabolic arcs: even pathwise connectivity fails there. Nonetheless, some boundary arcs of odd-period hyperbolic components also feature "open beaches" with sub-arcs of positive length that form part of boundary of a hyperbolic component without any further decorations, so the hyperbolic component and the escape locus meet along a smooth arc. (We do not know whether the number of such arcs is finite or not.)

Overview of Paper and Results. In this paper we study the boundaries of hyperbolic components of $\mathcal{M}_d^*$ of odd period, focusing on local connectivity and pathwise connectivity. In Section 2, we investigate parabolic dynamics, especially of odd period, and review Ecalle cylinders and their special features in antiholomorphic dynamics: the existence of an invariant curve called the *equator*. We then discuss parabolic arcs on the boundary of hyperbolic components of odd period. In Section 3, we investigate these arcs from the point of view of the holomorphic fixed-point index, and we show that period-doubling bifurcations occur near both ends of all parabolic arcs. We then discuss, in Section 4, perturbations of parabolic periodic points and introduce continuous coordinates for the perturbed dynamics. In Section 5 we introduce an invariant tree in parabolic dynamics, similar to the Hub-

bard tree for postcritically finite polynomials, and discuss the dynamical properties of parabolic maps that we will later transfer into parameter space. This transfer is then done in Section 6, using perturbed Fatou coordinates: these are somewhat simplified in the antiholomorphic setting because of the existence of the invariant equator. Some concluding remarks and further results are discussed in Section 7.

Relations to Holomorphic Parameter Spaces. Much of the relevance of the tricorn (and the higher-dimensional multicorns) comes from the fact that it is related to natural holomorphic parameter spaces. Clearly, the tricorn space is the (real two-dimensional but not complex-analytic) slice $c = \bar{a} = b$ in the complex two-dimensional space of maps $z \mapsto (z^2 + a)^2 + b$, one of the natural complex two-dimensional spaces of polynomials. Perhaps more interestingly, the tricorn is naturally related to the space of *real* cubic polynomials: this space can be parametrized as $z \mapsto \pm z^3 - 3a^2z + b$ with $a, b \in \mathbb{R}$. It was in the context of this space that Milnor discovered and explored the tricorn [M2, M3] as one of the prototypical local dynamical features in the presence of two active critical points; compare Figure 1.2. To see how antiholomorphic dynamics occurs naturally in the dynamics of a real cubic polynomial p, suppose there is an open bounded topological disk $U \subset \mathbb{C}$ containing one critical point so that $p(U)$ contains the closure of the complex conjugate of U. Denoting complex conjugation of p by p^* and the closure of U by $\overline{U}$, we have $p^*(U) \supset \overline{U}$. If, possibly by suitable restriction, the map $p\colon U \to p(U)$ is proper holomorphic, then $p^*\colon U \to p^*(U)$ is the antiholomorphic analogue of a polynomial-like map in the sense of Douady and Hubbard. Since p commutes with complex conjugation, the dynamics of p and of p^* are the same (the even iterates coincide), so the dynamics of p near one critical point is naturally described by the antiholomorphic polynomial p^* (and the other critical point is related by conjugation). The advantage of the antiholomorphic point of view is that, while U and $p(U)$ may be disjoint domains in $\mathbb{C}$ without obvious dynamical relation, there is a well-defined antiholomorphic dynamical system $p^*\colon U \to p^*(U)$. This is even more useful when $\overline{U}$ is a subset not of $p^*(U)$, but of a higher iterate: in this case, like for ordinary polynomial-like maps, the interesting dynamics of a high-degree polynomial is captured by a low-degree polynomial or, in this case, antipolynomial.
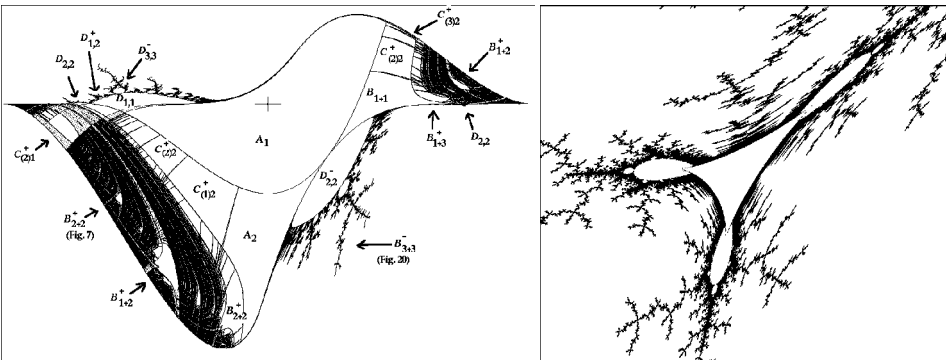

FIGURE 1.2: The connectedness locus of real cubic polynomials and a detail from the southeast quadrant, showing a tricorn-like structure. (See also Plate 4.) Pictures from Milnor [M2], reprinted by permission of Taylor & Francis.

Are There Embedded Tricorns? It was numerically "observed" by several people that the tricorn contains, around each hyperbolic component of even period, a small copy of the Mandelbrot set, and around each odd period component a small copy of the tricorn itself; and similar statements hold for certain regions of the real cubic connectedness locus — much as the well-known fact that the Mandelbrot set contains a small copy of itself around each hyperbolic component. A small tricorn within the big one is shown in Figure 1.3. However, we believe that most, if not all, "little tricorns" are not homeomorphic to the actual tricorn (both within the tricorn space and within the real cubic locus); quite possibly most little tricorns might not even be homeomorphic to each other. Indeed, a subset of the real axis connects the main hyperbolic component (of period 1) to the period 3 "airplane" component (along the real axis, the tricorn and the Mandelbrot set coincide obviously): we say that the "umbilical cord" of the period 3 tricorn lands. However, we prove for many little tricorns that their umbilical cords do not land but rather form some kind of $(\sin 1/x)$-structure. Our methods apply only to "prime" little tricorns: these are the ones not contained in larger "little tricorns," so we do not disprove continuity of the empirically observed embedding map given by the straightening theorem (even though this seems very likely). Two little tricorns could be homeomorphic to each other only if they have matching sizes of the wiggles of the umbilical cords of all the infinitely many little tricorns they contain, where the size of such a wiggle is measured in terms of Ecalle heights, as introduced later in this paper; see Definition 2.4.
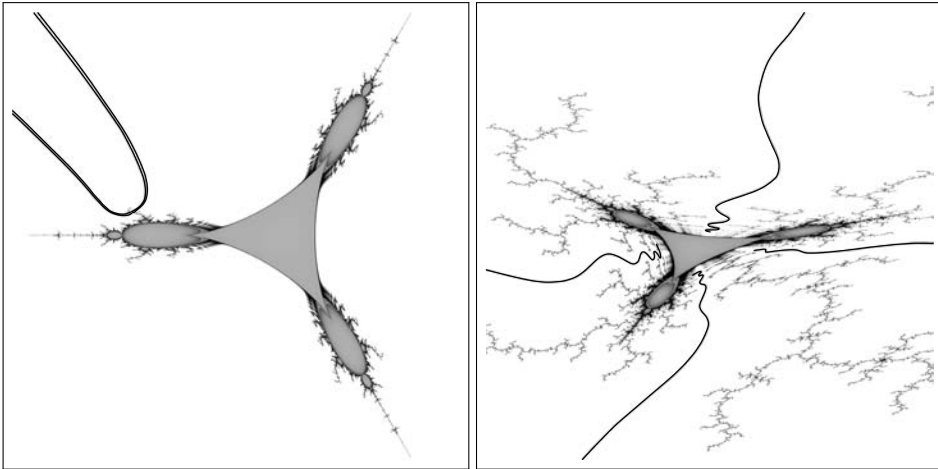

FIGURE 1.3: The tricorn and a blow-up showing a "small tricorn" of period 5. Shown in both pictures are the four parameter rays accumulating at the boundary of the period 5 hyperbolic component (at angles 371/1023, 12/33, 13/33, and 1004/1023). The wiggly features of these non-landing rays are clearly visible in the blow-up. (See also Plate 5.)

Failure of continuity of the straightening map was shown in other contexts, for instance by Epstein and by Inou. Failure of local connectivity and of pathwise connectivity was numerically observed by Milnor [M2] for the tricorn. For complex parameter spaces, failure of local connectivity was observed by Lavaurs for the cubic connectedness locus (a brief remark in his thesis) and by Epstein and

Yampolsky [EY] for real slices of cubic polynomials. Nakane and Komori [NK] showed that certain "stretching rays" in the space of real cubic polynomials do not land.

REMARK 1.4. This work was inspired by John Milnor in many ways: he was the first to have observed the tricorn and its relevance in the space of iterated (real) cubic maps, he made systematic studies about the local behavior of parameter spaces and under which conditions little tricorns appear there, he observed the loss of local connectivity and even of path connectivity of the tricorn, he introduced the term "tricorn"—and his home page shows non-landing rays of the kind that he observed and that we discuss here.

REMARK 1.5. In this paper, we need certain background results from the (still unpublished) earlier manuscript [MNS], which has more detailed results of bifurcations especially at hyperbolic components of odd period. In setting up notation and background, it seems more convenient to complete these proofs here rather than to strictly avoid overlap.

2 ANTIHOLOMORPHIC AND PARABOLIC DYNAMICS

In many ways, antiholomorphic maps have similar dynamical properties as holomorphic ones because the second iterate is holomorphic. There are a number of interesting features specific to antiholomorphic dynamics though, especially near periodic points of odd period k. The multiplier of a periodic point of odd period k is not a conformal invariant; only its absolute value is, and the multiplier of the $(2k)$th iterate (the second return map) is always non-negative real. This has interesting consequences on boundaries of hyperbolic components of odd period: all boundary parameters are parabolic with multiplier $+1$ (for the holomorphic second return map).

Another unusual feature is that dynamic rays landing at the same point of odd period k need not all have the same period. These rays can have period k or $2k$ (not higher), and both periods of rays can coexist: see [NS, Lemma 3.1].

We will also show that the number of periodic points of odd period k can change: but of course the number of periodic points of periods k and $2k$, which are both periodic points of period k for the holomorphic second iterate, must remain constant; the only thing that can happen is that two orbits of odd period k turn into one orbit of period $2k$, and this always occurs on boundaries of hyperbolic components of odd period k: see Lemma 3.5.

The straightening theorem [DH2] for polynomials has an antiholomorphic ana-
logue. We state it here for easier reference; the proof is the same as in the holomor-
phic case.

THEOREM 2.1 (The Antiholomorphic Straightening Theorem). *Suppose that U and
V are two bounded topological disks so that the closure of U is contained in V. Suppose
also that $f\colon U \to V$ is an antiholomorphic proper map of degree d. Then $f|_U$ is hybrid
equivalent to an antiholomorphic polynomial p of the same degree d. If the filled-in Julia
set of $f\colon U \to V$ (the set of points that can be iterated infinitely often) is connected, then
p is unique up to conformal conjugation.*

As usual, two maps are hybrid equivalent if they are quasiconformally conjug-
ation so that the complex dilatation vanishes on the filled-in Julia set. In the rest
of this section, we discuss the local dynamics of parabolic periodic points of odd
period k specifically for antipolynomials $p_c(z) = \overline{z}^d + c$.

LEMMA 2.2 (Simple and Double Parabolics). *Every parabolic periodic point of p_c of
odd period, when viewed as a fixed point of an even period iterate of p_c, has parabolic
multiplicity 1 or 2.*

PROOF. The first return map of any parabolic periodic point of odd period is an-
tiholomorphic, but the second iterate of the first return map is holomorphic and
has multiplier $+1$. This second iterate can thus be written in local coordinates as
$z \mapsto z + z^{q+1} + \dots$, where $q \geq 1$ is the multiplicity of the parabolic orbit. There
are then q attracting Fatou petals, and each must absorb an infinite critical orbit
of $p_c^{\circ 2}$. But $p_c^{\circ 2}$ has two critical orbits (the single critical orbit of p_c splits up into
two orbits of $p_c^{\circ 2}$, for even and odd iterates), hence $q \leq 2$. (Viewing this periodic
point as a fixed point of a higher iterate of $p_c^{\circ 2}$ does not change q: in the same local
coordinates as before, the higher iterate takes the form $z \mapsto z + az^{q+1} + \dots$, where
$a \in \mathbb{N}$ measures which higher iterate we are considering.)   □

A parabolic periodic point with multiplicity 1 (resp. 2) is called a *simple (resp.
double) parabolic point*. A parameter c so that p_c has a double parabolic periodic
point is called a *parabolic cusp*.

LEMMA 2.3 (Ecalle Cylinders). *Let z_0 be a simple parabolic periodic point of odd pe-
riod k of an antiholomorphic map f and let V be the attracting basin of z_0. Then there is a
neighborhood U of z_0 and an analytic map $\varphi\colon U \cap V \to \mathbb{C}$ that is an isomorphism to the
half-plane $\mathrm{Re}\, w > 0$ such that*

$$\varphi \circ f^{\circ k} \circ \varphi^{-1}(w) = \overline{w} + \frac{1}{2}.$$

The map φ is unique up to an additive real constant.

It follows that the quotient of $V \cap U$ by $f^{\circ 2k}$ is isomorphic to $\mathbb{C}/\mathbb{Z}$, and on this
quotient cylinder f induces the map $x + iy \mapsto x + 1/2 - iy$ with $x \in \mathbb{R}/\mathbb{Z}$, $y \in \mathbb{R}$.

PROOF. The second iterate $f^{\circ 2}$ is holomorphic, and for this map z_0 is parabolic
with period k. Since the parabolic point is simple, we have the usual conformal Fa-
tou coordinates $\varphi\colon V \cap U \to \mathbb{C}$ with $\varphi \circ f^{\circ 2k} \circ \varphi^{-1}(w) = w + 1$ for a certain neigh-
borhood U of z_0, where $\varphi(V \cap U)$ covers some right half-plane and φ is unique up
to addition of a complex constant. Adjusting this constant and restricting U
(which will no longer be a neighborhood of z_0), we may assume that $\varphi(V \cap U)$ is

the right half-plane $\operatorname{Re} w > 0$. It follows that $(V \cap U)/(f^{\circ 2k})$ is conformally iso-morphic to the bi-infinite $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$, so that $f^{\circ k}$ has to send this cylinder to itself in an antiholomorphic way. The only antiholomorphic automorphisms of $\mathbb{C}/\mathbb{Z}$ are $w \mapsto \pm\overline{w} + \alpha'$ with $\alpha' \in \mathbb{C}/\mathbb{Z}$ (depending on the sign, the two ends of $\mathbb{C}/\mathbb{Z}$ are either fixed or interchanged), and lifting this to the right half-plane we get $\varphi \circ f^{\circ k} \circ \varphi^{-1}(w) = \pm\overline{w} + \alpha$ with $\alpha \in \mathbb{C}$, hence

$$\varphi \circ f^{\circ 2k} \circ \varphi^{-1}(w) = w + \pm\overline{\alpha} + \alpha \overset{!}{=} w + 1$$

so either $2\operatorname{Re}\alpha = 1$ or $2i\operatorname{Im}\alpha = 1$. The latter case is impossible, and in the former case we get $\operatorname{Re}\alpha = 1/2$, as claimed. But φ is still unique up to addition of a com-plex constant, and the imaginary part of this constant can be adjusted uniquely so that α becomes real, i.e., $\alpha = 1/2$. $\qquad\square$

DEFINITION 2.4 (Ecalle Cylinder, Ecalle Height, and Equator). *The quotient cylinder $(V \cap U)/(f^{\circ 2k})$ isomorphic to $\mathbb{C}/\mathbb{Z}$ is called the* Ecalle cylinder *of the attracting basin. Its* equator *is the unique simple closed (Euclidean) geodesic of $\mathbb{C}/\mathbb{Z}$ that is fixed by the action of f: in those coordinates in which f takes the form $w \mapsto \overline{w} + 1/2$, this equator is the projection of $\mathbb{R}$ to the quotient. Finally, the* Ecalle height *of a point $w \in \mathbb{C}/\mathbb{Z}$ in the Ecalle cylinder is defined as $\operatorname{Im} w$.*

Similarly to the Ecalle cylinders in the attracting basin, one can also define them for a local branch of f^{-1} fixing z_0; all this requires is the local parabolic dynamics in a neighbor-hood of z_0. To distinguish these, they are called incoming *and* outgoing *Ecalle cylinders (for f and f^{-1}, respectively). Both have equators and Ecalle heights.*

Note that the identification of an Ecalle cylinder with $\mathbb{C}/\mathbb{Z}$ for usual holomor-phic maps is unique only up to translation by a complex constant; in our case, with
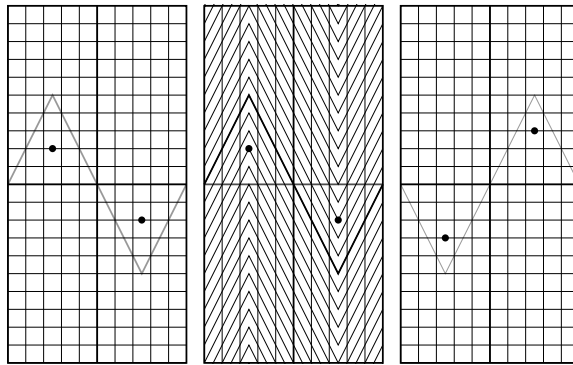

FIGURE 2.5: The Ecalle height of the critical value can be changed by putting a dif-ferent complex structure onto the Ecalle cylinder and then by pull-backs onto the entire parabolic basin. *Left*: the critical orbit (marked by heavy dots) in the Ecalle cylinder, with a square grid indicating the complex structure; the equator is high-lighted, and the critical value has Ecalle height 0.2. The grey zig-zag line will be the new equator; it is invariant under $z \mapsto \overline{z} + 1/2$. *Center*: a grid of "distorted squares" defines a new complex structure (in which each parallelogram should become a rectangle); the dynamics is the same as before, and the new equator is highlighted. *Right*: the new complex structure in the Ecalle cylinder after straight-ening; the Ecalle height of the critical value is now -0.3. The image of the old equator is indicated in grey.

an antiholomorphic intermediate iterate and thus a preferred equator, this identi-
fication is unique up to a real constant. Therefore, there is no intrinsic meaning of
$\operatorname{Re} w$ within the cylinder, or for $\operatorname{Re} \varphi(z)$ for $z \in V \cap U$. However, for two points
$z, z' \in V \cap U$, the difference $\operatorname{Re} \varphi(z) - \operatorname{Re} \varphi(z')$ has a well-defined meaning in $\mathbb{R}$
called *phase difference*; this notion actually extends to the entire attracting basin V.

PROPOSITION 2.6 (Parabolic Arcs). *Every polynomial p_c with a simple parabolic peri-*
odic point of odd period is part of a real one-dimensional family of parabolic maps $p_{c(h)}$ with
simple parabolic orbits. This family is real analytically parametrized by Ecalle height h of
the critical value; more precisely, the map $h \mapsto p_{c(h)}$ is a real-analytic bijection from $\mathbb{R}$
onto a family of parabolic maps that we call a parabolic arc.

We sketch the proof in Figure 2.5; see [MNS, Theorem 3.2] for details.

3   BIFURCATION ALONG ARCS AND THE FIXED-POINT INDEX

LEMMA 3.1 (Parabolic Arcs on Boundary of Odd-Period Components). *Near both*
ends, every limit point of every parabolic arc is a parabolic cusp.

PROOF. Each limit point of parabolic parameters of period k must be parabolic
of period k, so it could be a simple or double parabolic. But at simple parabolics,
Ecalle height is finite, while it tends to ∞ at the ends of parabolic arcs. Therefore,
each limit point of a parabolic arc is a parabolic cusp.   □

REMARK 3.2.   In fact, the number of parabolic cusps of any given (odd) period is
finite [MNS, Lemma 2.10], so each parabolic arc has two well-defined endpoints.
    As the parameter tends to the end of a parabolic arc, the Ecalle height tends to
$\pm\infty$, and the Ecalle cylinders (with first return map of period k, which permutes
the two ends) becomes pinched; in the limit, the cylinder breaks up into two cylin-
ders that are interchanged by the kth iterate, so each cylinder has a return map
of period $2k$, which is holomorphic: the double parabolic dynamics in the limit is
rigid and has no non-trivial deformations.
    In the sequel, we will need the *holomorphic fixed-point index*: if f is a local holo-
morphic map with a fixed point z_0, then the index $\iota(z_0)$ is defined as the residue of
$1/(z - f(z))$ at z_0. If the multiplier $\rho = f'(z_0)$ is different from 1, this index equals
$1/(1 - \rho)$ and tends to ∞ as $\rho \to 1$. The most interesting situation occurs if several
simple fixed points merge into one parabolic point. Each of their indices tends to
∞, but the sum of the indices tends to the index of the resulting parabolic fixed
point, which is finite. Of course, analogous properties apply for the first return
map of a periodic point.
    If z_0 is a parabolic fixed point with multiplier 1, then in local holomorphic
coordinates the map can be written as $f(w) = w + w^{q+1} + \alpha w^{2q+1} + \dots$, and α is
a conformal invariant (in fact, it is the unique formal invariant other than q: there
is a formal, not necessarily convergent, power series that formally conjugates f to
its first three terms). A simple calculation shows that α equals the parabolic fixed
point index. The quantity $1 - \alpha$ is known as "résidu itératif" [BE]; its real part
measures whether or not the parabolic fixed point of f in the given normal form
can be perturbed into q or $q + 1$ attracting fixed points; Epstein introduced the
notion "parabolic repelling" and "parabolic attracting" for these two situations,
and in the latter case he obtains an extra count in his refined Fatou-Shishikura-
inequality [E]. We will use these ideas in Theorems 3.8 and 7.1.

LEMMA 3.3 (Types of Perturbation of Odd-Period Parabolic Orbit). *Suppose p_{c_0} has a simple parabolic periodic point z_0 of odd period k. Then for any sequence $c_n \to c_0$ of parameters with $c_n \neq c_0$, the maps p_{c_n} have periodic points z_n and z'_n that both converge to z_0 as $c_n \to c_0$ and with multipliers $\rho_n := (p_{c_n}^{\circ 2k})'(z_n) \to 1$ and $\rho'_n := (p_{c_n}^{\circ 2k})'(z'_n) \to 1$, such that for large n either*

- *both z_n and z'_n have period k, we have $\rho_n, \rho'_n \in \mathbb{R}$, and one of the orbits is attracting, while the other one is repelling; or*

- *the points $z_n = z'_n$ are on a parabolic orbit of period k; or*

- *the points z_n and z'_n both have period $2k$, they are on the same orbit of p_{c_n}, and they satisfy $p_{c_n}^{\circ k}(z_n) = z'_n$ and $p_{c_n}^{\circ k}(z'_n) = z_n$. Their multipliers satisfy $\rho'_n = \overline{\rho_n} \notin \mathbb{R}$ and $\mathrm{Re}(\rho_n - 1) = O(\mathrm{Im}(\rho_n)^2)$.*

PROOF. The point z_0 is a simple parabolic fixed-point of the holomorphic map $p_{c_0}^{\circ 2k}$, so under small perturbations it must split up into exactly two fixed points of $p_{c_n}^{\circ 2k}$ (unless c_n is some other parameter on the parabolic arc, where $z_n = z'_n$ are still parabolic). As periodic points of p_{c_n}, these must both have period k or both period $2k$. They converge to the parabolic orbit, so their multipliers must tend to 1 and their fixed point indices $1/(1 - \rho_n)$ and $1/(1 - \rho'_n)$ must tend to ∞. However, the sum of these indices must tend to the finite fixed-point index of the parabolic periodic point.

If the period equals k, then the orbit of $p_c^{\circ 2k}(z_n) = z_n$ visits each of the k periodic points twice: once for an even (holomorphic) and once for an odd (antiholomorphic) iterate, and the chain rule implies that the multiplier ρ_n is real. The same argument applies to z'_n and ρ'_n. The two fixed-point indices are real and have large absolute values (once ρ and ρ' are close to 1), so their sum can be bounded only if one index is positive and the other one negative; hence one orbit must be attracting and the other one repelling.

If the period equals $2k$, then the periodic points z_n and z'_n that are near z_0 must be on the same orbit with $p_{c_n}^{\circ k}(z_n) = z'_n$ and $p_{c_n}^{\circ k}(z'_n) = z_n$. A similar argument as before shows $\rho'_n = \overline{\rho_n}$. For the sum of the fixed point indices, we obtain

$$\frac{1}{1 - \rho_n} + \frac{1}{1 - \overline{\rho_n}} = 2\,\mathrm{Re}\left(\frac{1}{1 - \rho_n}\right) = \frac{2(1 - \mathrm{Re}\,\rho_n)}{|1 - \rho_n|^2}. \tag{3.1}$$

Since his quantity must have a finite limit, the multipliers cannot be real. Writing $\varepsilon_n := 1 - \rho_n$, we have $\mathrm{Re}\,\varepsilon_n = O(\varepsilon_n^2)$; hence $\mathrm{Re}\,\varepsilon_n = O(\mathrm{Im}\,\varepsilon_n^2)$. $\qquad\square$

In the following two results, we will show that both possibilities actually occur in every neighborhood of every simple parabolic parameter of odd period.

LEMMA 3.4 (Parabolics on Boundary of Hyperbolic Components). *If a map p_c has a parabolic periodic point of period k, then c is on the boundary of a hyperbolic component of period k.*

PROOF. We will employ a classical argument by Douady and Hubbard. Consider a map p_{c_0} with a parabolic orbit of odd period k. To see that it is on the boundary of a hyperbolic component W of period k, restrict the antipolynomial to an antipolynomial-like map of equal degree and perturb it slightly so as to make the

indifferent orbit attracting: this can be achieved by adding a small complex multiple of an antipolynomial that vanishes on the periodic cycle but the derivative of which does not. Then apply the straightening theorem (Theorem 2.1) to bring it back into our family of maps p_c. This can be done with arbitrarily small Beltrami coefficients, so c is near c_0 (see also [MNS, Theorem 2.2]).   □

Of course, the indifferent orbit can also be made repelling by the same reasoning. However, one would expect that a perturbation of a simple parabolic periodic point, here of period k, creates two periodic points of period k. In our case, this is possible whenever the perturbation goes into the hyperbolic component W, and then one of the two orbits after perturbation is attracting and the other is repelling; a perturbation creating two repelling period k orbits is not possible within our family. It turns out, though, that if k is odd, then one can also perturb so that no nearby orbit of period k remains — and an orbit of period $2k$ is created. (The number of periodic orbits of given period must remain constant for perturbations of holomorphic maps such as $p_c^{\circ 2k}$, not for antiholomorphic maps such as $p_c^{\circ k}$.)

LEMMA 3.5 (Orbit Period Doubles in Bifurcation Along Arc). *Every parabolic arc with a parabolic orbit of period k (necessarily odd) is the locus of transition where two periodic orbits of period k (one attracting and one repelling near the arc) turn into one orbit of period $2k$ (attracting, repelling, or indifferent near the arc). Equivalently, every parameter c with a simple parabolic periodic orbit of odd period k is on the boundary of a hyperbolic component W of period k, and c has a neighborhood U so that for $c' \in W \cap U$, the parabolic orbit splits up into two orbits for $p_{c'}$ of period k (one attracting and one repelling), while it splits into one orbit of period $2k$ for $c' \in U \setminus \overline{W}$.*

PROOF. As in Lemma 3.4, consider a map p_{c_0} with a parabolic orbit of odd period k and restrict it to an antipolynomial-like map of equal degree. This time, we want to perturb it so as to make the period k orbit vanish altogether; therefore, we cannot just add a polynomial that takes the value zero along this orbit.

Let z_0 be one of the parabolic periodic points and change coordinates by translation so that $z_0 = 0$. By rescaling, we may assume that near 0, we have

$$p_{c_0}^{\circ k}(z) = \bar{z} + A\bar{z}^2 + o(\bar{z}^2)$$

(note that conjugation by complex scaling changes the coefficient in front of $\bar{z}$; see the following remark). The second iterate has the local form

$$p_{c_0}^{2k}(z) = z + (A + \overline{A})z^2 + o(z^2),$$

so the assumption that the parabolic orbit is simple means Re $A \neq 0$; conjugating if necessary by $z \mapsto -z$, we may assume that Re $A > 0$.

Let z_{k-1} be the periodic point with $p_{c_0}(z_{k-1}) = z_0$. Let f be an antipolynomial (presumably of large degree) that vanishes at the indifferent orbit except at z_{k-1}, where it takes the value $f(z_{k-1}) = 1$; assume further that the first and second derivatives of f vanish at the entire indifferent orbit, and that f vanishes to order d at the critical point. For sufficiently small $\varepsilon \in \mathbb{C}$, we will consider $f_\varepsilon(z) = p_{c_0}(z) + \varepsilon f(z)$; by slightly adjusting the domain boundaries, this will give an antipolynomial-like map of the same degree d as before, and it will continue to have a single critical point of maximal order. The map p_{c_0} has finitely many orbits of period k, all but one of which are repelling, and for sufficiently small ε these will remain repelling.

We claim there is a neighborhood U of $z_0 = 0$ so that for sufficiently small $\varepsilon > 0$, the map f_ε will not have a point of period k in this neighborhood. We use our local coordinates where $p_{c_0}^{\circ k}(z) = \bar{z} + A\bar{z}^2 + o(\bar{z}^2)$. We may assume that $|p_{c_0}^{\circ k}(z) - \bar{z} - A\bar{z}^2| \leq |Az^2|$ in U, and also $|y| < 1/|8A|$.

We have

$$f_\varepsilon^{\circ k}(z) = p_{c_0}^{\circ k}(z) + \varepsilon + O(|z|^3)$$

because f vanishes to second order along the parabolic orbit. By restricting U and ε, we may assume that $|f_\varepsilon^{\circ k}(z) - p_{c_0}^{\circ k}(z) - \varepsilon| < (\operatorname{Re} A)|z|^2/2$.

Now suppose $z = x + iy \in U$. If $|y| > 8|A|x^2$, then

$$|p_{c_0}^{\circ k}(z) - \bar{z}| \leq 2|Az^2| = 2|A|(x^2 + y^2) < 2|A|\left(\frac{|y|}{8|A|} + \frac{|y|}{8|A|}\right) = \frac{|y|}{2}$$

and

$$\begin{aligned} |f_\varepsilon^{\circ k}(z) - \bar{z} - \varepsilon| &\leq |f_\varepsilon^{\circ k}(z) - p_{c_0}^{\circ k}(z) - \varepsilon| + |p_{c_0}^{\circ k}(z) - \bar{z}| \\ &\leq \frac{(\operatorname{Re} A)|z|^2}{2} + 2|Az^2| \\ &\leq |y| \, , \end{aligned}$$

so on this domain $p_{c_0}^{\circ k}$ and $f_\varepsilon^{\circ k}$ behave essentially like complex conjugation (plus an added real constant) and thus have no fixed points.

However, if $|y| \leq 8|A|x^2$, that is, z is near the real axis, then

$$\operatorname{Re}(A\bar{z}^2) \geq (\operatorname{Re} A)\frac{|z|^2}{2}$$

and we have

$$\begin{aligned} \operatorname{Re} f_\varepsilon^{\circ k}(z) &\geq \operatorname{Re} p_{c_0}^{\circ k}(z) + \varepsilon - |f_\varepsilon^{\circ k}(z) - p_{c_0}^{\circ k}(z) - \varepsilon| \geq \operatorname{Re} z + \operatorname{Re}(A\bar{z}^2) + \varepsilon + O(|z|^3) \\ &\geq \operatorname{Re} z + \varepsilon > \operatorname{Re} z. \end{aligned}$$

so that $f_\varepsilon^{\circ k}$ does not have a fixed point with $|y| \leq 8|A|x^2$ either. The size of the neighborhood U is uniform for all sufficiently small ε.

Now apply the straightening theorem as in the Lemma 3.4: this yields antipolynomials p_{c_n} near c_0 with $c_n \to c_0$ for which there is one orbit of period k fewer than before perturbation, and these are all repelling, so we are outside of W. Since z_0 has period k for the *holomorphic* map $p_{c_0}^{\circ 2}$, there is a sequence (z_n) of periodic points of period k for $p_{c_n}^{\circ 2}$ with $z_n \to z_0$. These points must have period $2k$.

This shows that arbitrarily close to c_0 there are parameters for which the indifferent period k orbit has turned into an orbit of period $2k$; similarly, by Lemma 3.4 there are parameters near c_0 for which there is an attracting orbit of period k. Finally, by Lemma 3.3, any perturbation of p_{c_0} away from the parabolic arc either creates an attracting orbit of period k or an orbit of period $2k$ (which may be attracting, repelling, or indifferent; see Corollary 3.9 and Theorem 7.1). The transition between these two possibilities (attracting orbit of period k vs. orbit of period $2k$) can happen only when the period k orbit is indifferent, hence, on the boundary of a hyperbolic component of period k. This proves the claim. □

REMARK 3.6. The local behavior of an antiholomorphic map at a fixed point (or a periodic point of odd period) is quite different from the holomorphic case. If

the fixed point is at 0, such a map can be written $f(z) = a_1\bar{z} + a_2\bar{z}^2 + \dots$; we will suppose $a_1 \neq 0$. The second iterate takes the form $f^{\circ 2}(z) = |a_1|^2 z + \dots$, so indifferent orbits are always parabolic and $|a_1|$ is an invariant under conformal conjugations. However, a_1 itself is not: conjugating $u = \lambda z$ leads to

$$f_1(u) = \left(\frac{\bar{\lambda}}{\lambda}\right) a_1 \bar{u} + \left(\frac{\bar{\lambda}^2}{\lambda}\right) a_2 \bar{u}^2 + \dots,$$

so $\arg a_1$ depends on the rotation of the coordinate system (the linear approximation df has eigenvalues $|a_1|$ and $-|a_1|$ with orthogonal eigenlines, and of course their orientation depends on the rotation of the coordinate system).

Specifically if $|a_1| = 1$, we can choose λ so that $f_1(u) = \bar{u} + A\bar{u}^2 + \dots$ with $A \in \mathbb{C}$, and conjugation by scaling can change $|A|$. Note that we have

$$f_1^{\circ 2}(u) = u + (A + \overline{A})u^2 + \dots.$$

If $\operatorname{Re} A \neq 0$, then there is a local *quadratic* conjugation $v = au + bu^2$ with $a \in \mathbb{R}$ that brings our map into the form $f_2(v) = \bar{v} + \bar{v}^2 + \dots$, as can be checked easily. However, if

$$\operatorname{Re} A \neq 0,$$

there is no such change of coordinates because $f_1^{\circ 2}(u) = u + O(u^3)$, so the origin has a multiple parabolic fixed point.

PROPOSITION 3.7 (Fixed-Point Index on Parabolic Arc). *Along any parabolic arc of odd period, the fixed-point index is a real-valued real-analytic function that tends to $+\infty$ at both ends.*

PROOF. The fact that the fixed-point index is real valued follows for instance from (3.1). The Ecalle height parametrizes the arc real-analytically (Proposition 2.6), and as the Ecalle height is changed by a quasiconformal deformation, the residue integral that defines the fixed-point index depends analytically on the height (the integrand as well as the integration path). Therefore, the index depends real-analytically on Ecalle height.

The parabolic periodic point is simple for all parameters along the arc, but towards the end of a cusp the parabolic orbit merges with another repelling periodic point so as to form a double parabolic (Lemma 3.1). The orbit with which it merges is repelling, say with multiplier ρ, so its index $1/(1 - \rho)$ tends to ∞ in $\mathbb{C}$. In order for the limiting double parabolic to have finite index, the index $\iota(z_n)$ of the parabolic orbit of p_{c_n} must tend to ∞ as well as c_n tends to the end of a parabolic arc.

Note that the index $\iota(z_n)$ is real by (3.1), so it tends to $+\infty$ or to $-\infty$. Since $|\rho| > 1$, the index $1/(1 - \rho)$ always has real part less than $+1/2$. This implies that $\iota(z_n) \to +\infty$ (or the sum in the limit would not be finite). □

THEOREM 3.8 (Odd-Even Bifurcation and Fixed-Point Index). *Every parabolic arc of period k intersects the boundary of a hyperbolic component of period $2k$ at the set of points where the fixed-point index is at least 1, except possibly at (necessarily isolated) points where the index has an isolated local maximum with value 1.*

PROOF. Consider a parameter p_{c_0} on a parabolic arc, and a sequence $c_n \to c_0$ so that all p_{c_n} have all periodic orbits of period k repelling. As in Lemma 3.5, let z_0 be a parabolic periodic point for p_{c_0}, let z_n be a periodic point of period $2k$ for p_{c_n} with $z_n \to z_0$, and let $z'_n := p_c^{\circ k}(z_n)$. Let ρ_n and $\rho'_n = \overline{\rho_n}$ be the multipliers of z_n and z'_n. The sum of the two fixed-point indices equals $2 \operatorname{Re} \left(1 / \left(1 - \rho_n \right) \right)$. We have $|\rho_n| < 1$ if and only if $2 \operatorname{Re} \left(1 / \left(1 - \rho_n \right) \right) > 1$.

Therefore, if c_0 is on the boundary of a period $2k$ component, we can choose c_n so that $|\rho_n| < 1$ and the fixed-point index at c_0 is at least 1. Conversely, if the fixed-point index is greater than 1, then we must have $|\rho_n| < 1$ for all large n, and the limit is on the boundary of a period $2k$ component. If the index equals 1, by real-analyticity of the index, either the index has an isolated local maximum there or the point is a limit point of points with index greater than 1 (note that the set of points with index 1 is isolated as the index is real-analytic and tends to ∞ at the ends). $\qquad\square$

COROLLARY 3.9 (Bifurcation Along Arcs). *Every parabolic arc has, at both ends, an interval of positive length at which a bifurcation from a hyperbolic component of odd period k to a hyperbolic component of period $2k$ occurs.* $\qquad\square$

COROLLARY 3.10 (Boundary of Bifurcating Component Lands). *Let W be a hyperbolic component of period $2k$ bifurcating from a hyperbolic component W_0 of odd period k, the set $\partial W \setminus \overline{W_0}$ accumulates only at isolated points in ∂W_0.* $\qquad\square$

This rules out the possibility that the boundary curve of W accumulates at ∂W_0, like a topologist's sine curve. The reason is that the set of limit points must have fixed point index 1, and the set of such points is discrete.

4   PARABOLIC PERTURBATIONS

In this section, we fix a hyperbolic component W of odd period k and a parameter $c_0 \in \partial W$ with a simple parabolic orbit $z_0, \ldots, z_{k-1}$. It is well known that there exists a neighborhood V of z_0 and a local coordinate $\varphi_{c_0} \colon V \to \mathbb{C}$ such that

$$\varphi_{c_0} \circ p_{c_0}^{\circ 2k} \circ \varphi_{c_0}^{-1}(\zeta) = \zeta + \zeta^2 h(\zeta),$$

with $\varphi_{c_0}(z_0) = 0$, $h(0) = 1$ and $|h(\zeta) - 1| < \varepsilon$ on V. We need to establish similar local coordinates for the local dynamics after perturbation.

PROPOSITION 4.1 (Perturbed Parabolic Dynamics). *For every $\varepsilon > 0$, one can choose neighborhoods V of c_0 and U of z_0 so that there is a $\varphi_c \colon V \to \mathbb{C}$ that satisfies*

$$f_c(\zeta) := \varphi_c \circ p_c^{\circ 2k} \circ \varphi_c^{-1}(\zeta) = \zeta + (\zeta^2 - a_c^2) h_c(\zeta),$$

with $|h_c(\zeta) - 1| < \varepsilon$ on V and $a_c \in \mathbb{C}$.

PROOF. Since $p_c^{\circ 2k}$ is holomorphic, after perturbation the parabolic fixed point splits up into two simple fixed points in the domain of φ_{c_0}. Let $\varphi_c := \varphi_{c_0} + b$, where b is chosen so the images of these fixed points are symmetric to 0, i.e., at some $\pm a_c \in \mathbb{C}$ (note that a_c may not be a continuous function in a neighborhood of c, but a_c^2 is). Then

$$h_c(\zeta) := \frac{\left(\varphi_c \circ p_c^{\circ 2k} \circ \varphi_c^{-1} \right)(\zeta) - \zeta}{(\zeta - a_c)(\zeta + a_c)}$$

must be holomorphic on V, and the map h_c is close to h at least for $\zeta \neq 0$, hence, also near 0.   □

Write $U^+ := U \cap W$ and $U^- := U \setminus \overline{W}$ (the parts inside and outside of the hyperbolic component W). By Lemma 3.3, for parameters $c \in U^-$ the parabolic orbit splits up into one orbit of period $2k$; denote it by $w_0(c), \ldots, w_{2k-1}(c)$. By restricting U, we may assume that $|w_0(c) - z_0(c_0)| < \varepsilon$ and $|w_k(c) - z_0(c_0)| < \varepsilon$. Moreover, the multipliers $\rho_c := (p_c^{\circ 2k})'(z_0)$ and $\rho_c' := (p_c^{\circ 2k})'(z_k)$ are complex conjugate and $|\operatorname{Re}(\rho_c - 1)| \in O(|\operatorname{Im}(\rho_c)|^2)$. Since $f_c'(a_c) = 1 + 2a_c h_c(a_c) \in \{\rho_c, \rho_c'\}$ and h_c is close to 1, we see that a_c is almost purely imaginary.

Let L_c be the straight line through a_c and $-a_c$ when $c \in U^-$; when $c \in \partial W$, let L_c be the eigenline for eigenvalue -1 for the parabolic fixed point (every antiholomorphic fixed point with multiplier 1 has eigenvalues $+1$ and -1). This family of lines is continuous in c for $c \in \overline{U^-}$ (a one-sided neighborhood of c_0), and L_c is vertical (in ζ-coordinates) when $c \in \partial W$. For $c \in U^-$, let ℓ_c be the segment $[a_c, -a_c] \subset L_c$.

Choose $r > 0$, and consider the arc of circle K_c^{in} connecting a_c to $-a_c$ going through r, and K_c^{out} connecting the same two points through $-r$. If U is chosen so small that a_c is almost purely imaginary for $c \in U^-$, these arcs are well defined, and as $a_c \to 0$, each of these arcs has a limit, which is the circle through 0 and centered at $\pm r/2$; see Figure 4.2.
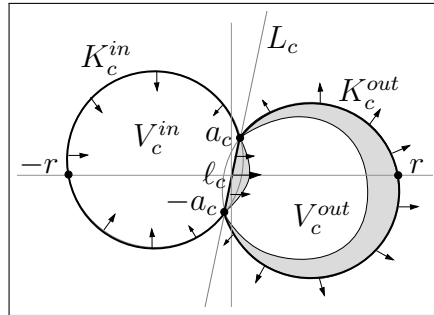

FIGURE 4.2: The line segment ℓ_c joining a_c to $-a_c$ with its image under f_c; the region between these (shaded) is a fundamental domain for the dynamics. The arcs K_c^{in} and K_c^{out} are also shown, as well as an inverse image of K_c^{out} and another fundamental domain bounded by K_c^{out} and its inverse image. This latter fundamental domain has a non-vanishing limit as $a_c \to 0$.

Denote by V_c^{in} the region bounded by K_c^{in} and ℓ_c, and let V_c^{out} denote the region bounded by K_c^{out} and ℓ_c. There is a fixed choice of $r > 0$ so that for all $c \in U^-$, iterates of $\zeta \in V_c^{\text{in}}$ under f_c will remain in V_c^{in} until they exit to V_c^{out} through ℓ_c; similarly iterates of $\zeta \in V_c^{\text{out}}$ under f_c^{-1} will remain in V_c^{out} (for $c \in U \cap \partial W$ these iterates never exit at all). The details of this argument are somewhat tedious but not difficult: as soon as $|a_c| \ll r$, the iterates in the first quadrant of K_c^{out} move upwards and to the right, while the iterates in the second quadrant (up to the point a_c) move upwards and to the left — except on a short piece of arc near the top of the circle, where the iteration moves essentially upwards. The argument is similar for the other parts of K_c^{out} and K_c^{in}.

PROPOSITION 4.3 (Ecalle Cylinders After Perturbation). *For every $c \in \overline{U^-}$, the quotients $C_c^{\text{in}} := V_c^{\text{in}}/p_c^{\circ 2k}$ and $C_c^{\text{out}} := V_c^{\text{out}}/p_c^{\circ 2k}$ (the quotients of V_c^{in} and V_c^{out} by the dynamics, identifying points that are on the same finite orbits entirely in V_c^{in} or in V_c^{out}) are complex annuli isomorphic to $\mathbb{C}/\mathbb{Z}$.*

PROOF. In the parabolic case of $c \in \partial W$, this is a standard result, proved using Fatou coordinates (see Milnor [M4, Sec. 10]). We will thus focus on the case $c \in U^-$.

Since the points $\pm a_c$ are almost purely imaginary and h_c is almost 1, it is easy to understand the dynamics of f_c, represented in Figure 4.2. In particular, the line segment $\ell = [-a_c, a_c]$ is sent by f_c to the arc $f_c(\ell) \subset \overline{V_c^{\text{out}}}$, still joining a_c to $-a_c$ but disjoint from ℓ (except at the endpoints). Let A^{out} be the domain bounded by ℓ and $f_c(\ell)$. Identifying the two boundary edges of A^{out} by f_c, we obtain a complex annulus that represents $C_c^{\text{out}} = A^{\text{out}}/p_c^{\circ 2k}$: every finite orbit in V^{out} enters $A^{\text{out}} \cup f(\ell)$ exactly once.

Finally, we must see that A^{out}/f_c is a bi-infinite annulus, i.e., isomorphic to $\mathbb{C}/\mathbb{Z}$ so that its ends are punctures.

This follows from the following lemma.

LEMMA 4.4. *Let λ be a non-real complex number, and let $g \colon u \mapsto \lambda u + O(u^2)$ be an analytic map defined in some neighborhood of 0. Let $\widetilde{Q} \subset \mathbb{C} \setminus \{0\}$ be the region bounded by $[0, r]$, $g([0, r])$ and $[r, g(r)]$, that we will take to include $(0, r)$ and $g((0, r))$ but not $(r, g(r))$. Then for r sufficiently small, $(0, r)$ and $g((0, r])$ are disjoint, so that the quotient of $\widetilde{Q}$ by the equivalence relation identifying $x \in (0, r)$ to $g(x)$ is homeomorphic to an annulus, and and it has infinite modulus.*

PROOF. The proof consists of passing to $\log u$ coordinates, where the corresponding annulus is bounded by $(-\infty, \log r) \subset \mathbb{R}$ and the image curve

$$\log g(u) = \log u + \log \lambda + O(u) \, ;$$

when setting $t = \log u$, the boundary identification relates $t \in \mathbb{R}^-$ to the image curve $t + \log \lambda + e^t$ (for $t \ll -1$). The claim follows. $\square$

This lemma clearly applies to both ends of A^{out}, and the argument about A^{in} is the same. This proves the proposition. $\square$

Recall the definition of Ecalle cylinders in the parabolic attracting and repelling petals of the holomorphic map $p_c^{\circ 2k}$ (Lemma 2.3), and the statement that the antiholomorphic iterate $p_c^{\circ k}$ introduces an antiholomorphic self-map of the Ecalle cylinders that interchanges the two ends. The situation is similar here: since the map $p_c^{\circ k}$ commutes with $p_c^{\circ 2k}$, it induces antiholomorphic self-maps from C_c^{in} to itself and from C_c^{out} to itself. As $p_c^{\circ k}$ interchanges the two periodic points at the end of the cylinders, it interchanges the ends of the cylinders, so it must fix a geodesic in the cylinders $\mathbb{C}/\mathbb{Z}$ that we call again the *equator*. Choosing complex coordinates in the cylinders for which the equator is at imaginary part 0, we can again define *Ecalle height* as the imaginary part in these coordinates. We will denote the Ecalle height of a point $z \in C_c^{\text{in/out}}$ by $E(z)$.

Since our arcs of circle $K_c^{\text{in/out}}$ depend continuously on c and have a finite non-zero limit as a_c tends to 0, the construction of the perturbed Ecalle cylinders depends continuously on $c \in \overline{U^-}$. We summarize this in the following proposition.

PROPOSITION 4.5 (Bundle of Ecalle Cylinders). *The disjoint unions*

$$\mathcal{C}^{\mathrm{in}} := \bigsqcup_{c \in \overline{U^-}} C_c^{\mathrm{in}} \quad and \quad \mathcal{C}^{\mathrm{out}} := \bigsqcup_{c \in \overline{U^-}} C_c^{\mathrm{out}}$$

form two-dimensional complex manifolds with boundary, and the natural maps

$$\mathcal{C}^{\mathrm{in}} \to \overline{U^-} \quad and \quad \mathcal{C}^{\mathrm{out}} \to \overline{U^-}$$

are smooth morphisms that make $\mathcal{C}^{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{out}}$ into topologically trivial bundles with fibers isomorphic to $\mathbb{C}/\mathbb{Z}$.

The equators form subbundles of circles, and the Ecalle height of fixed points in $\mathbb{C}$ near 0 depends continuously on c.

REMARK 4.6. Here "smooth morphism" means that $\mathcal{C}^{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{out}}$ are families of complex manifolds parametrized by $\overline{U^-}$ and that the fibers have analytic local coordinates that depend continuously on the parameter.

Of central importance to us is that above U^- (not the closure $\overline{U^-}$), the two bundles $\mathcal{C}^{\mathrm{in}}$ and $\mathcal{C}^{\mathrm{out}}$ are canonically isomorphic as follows.

DEFINITION 4.7 (The Transit Map). *The* transit map *is the conformal isomorphism*

$$T_c \colon C_c^{\mathrm{in}} \to C_c^{\mathrm{out}}$$

induced by the conformal isomorphism $p_c^{\circ 2k} \colon A^{\mathrm{in}} \to A^{\mathrm{out}}$.

This transit map clearly depends continuously on the parameter $c \in U^-$ and preserves the equators, hence Ecalle heights.

Finally, choose a smooth real curve $s \mapsto c(s)$ in $\overline{U^-}$ (in parameter space), parametrized by $s \in [0, \delta]$ for some $\delta > 0$, with $c(s) \in U^-$ for $s > 0$. Choose a smooth curve $s \mapsto \zeta(s)$ (in the dynamical planes, typically the critical value), also defined for $s \in [0, \delta]$ such that

$$\zeta(s) \in V^{\mathrm{in}}(c(s))$$

for all $s \in [0, \delta]$. Then $s \mapsto \zeta(s)$ induces a map $\sigma \colon [0, \delta] \to \mathcal{C}^{\mathrm{in}}$ with $\sigma(s) \in C_{c(s)}^{\mathrm{in}}$.

PROPOSITION 4.8 (Limit of Perturbed Fatou Coordinates). *The curve*

$$\gamma := s \mapsto T_{c(s)}(\sigma(s))$$

in $\mathcal{C}^{\mathrm{out}}$, parametrized by $s \in (0, \delta]$, spirals as $s \searrow 0$ towards the circle on $C_{c_0}^{\mathrm{out}}$ at Ecalle height $E(\sigma(0))$.

Before proving this we need to say exactly what "spirals" means. We know that $\mathcal{C}^{\mathrm{out}}$ is a trivial topological bundle of bi-infinite annuli $\mathbb{C}/\mathbb{Z}$ over $\overline{U^-}$; we can choose a trivialization

$$\Phi \colon \mathcal{C}^{\mathrm{out}} \to \overline{U^-} \times \mathbb{C}/\mathbb{Z}$$

by deciding that the point r (see Figure 4.2) corresponds for all $c \in \overline{U^-}$ to the origin of $\mathbb{C}/\mathbb{Z}$. That allows us to define an *Ecalle phase* $\arg(\gamma(s))$ to be a continuous lift φ of

$$s \mapsto \mathrm{Re}(pr_2(\Phi(\gamma(s)))) \in \mathbb{R}/\mathbb{Z}.$$

Spiralling will mean that the image of γ accumulates exactly on the circle on $C_{c_0}^{\mathrm{out}}$ at Ecalle height $E(\sigma(0))$, and that in the process the Ecalle phase tends to infinity.

PROOF. Since the transit map preserves Ecalle heights, the curve $t \mapsto \gamma(s)$ can only accumulate on the circle on $C_{c_0}^{\mathrm{out}}$ at Ecalle height $E(\sigma(0))$. It remains to show that the Ecalle phase tends to infinity. The magnitude of the Ecalle phase essentially measures how many iterates of f_c it takes for $\zeta(s)$ to reach the fundamental domain in V_c^{out} shown in Figure 4.2.

This more or less obviously tends to infinity as $a_c \to 0$; to get from V_c^{in} to V_c^{out}, the orbit must cross ℓ_c, and near ℓ_c the map $f_c^{\circ k}$ moves points less and less as $a_c \to 0$. (In the language of Douady "it takes longer and longer to go through the egg-beater.") In fact, in the dynamics of the limit c_0 it takes infinitely many iterations of f_c^{-1} for r to get to the origin and, thus, arbitrarily many iterations to reach any small neighborhood X of the origin, and for sufficiently small perturbations the number of backwards iterations to go from r into X is almost the same.   □

REMARK 4.9. A computation in logarithmic coordinates shows that the Ecalle phase $\arg \gamma(s)$ (essentially the number or iterations required to get from $\zeta(s)$ to the fundamental domain in V_c^{out} containing r) satisfies

$$\arg \gamma(s) = \frac{\pi}{|a_c(s)|}(1 + o(1))$$

as $a_c \to 0$.

5   PARABOLIC TREES AND COMBINATORICS

DEFINITION 5.1 (Characteristic Parabolic Point and Principal Parabolic). *The characteristic point on a parabolic orbit is the unique parabolic periodic point on the boundary of the critical value Fatou component.*

A map with a parabolic orbit is called a principal parabolic *if the parabolic orbit is simple and each point on the parabolic orbit is the landing point of at least two periodic dynamic rays.*

REMARK 5.2. As proved in [MNS], every hyperbolic component W of odd period k in the multicorn $\mathcal{M}_d^*$ has a Jordan curve boundary consisting of exactly $d + 1$ parabolic arcs and $d + 1$ parabolic cusps where the arcs meet in pairs. Suppose $k \geq 3$. For each $c \in W$, each periodic bounded Fatou component has exactly $d + 1$ boundary points that are fixed under the first return map of the component, and these points together are the landing points of $d + 2$ periodic dynamic rays: one boundary fixed point is the landing point of two rays, both of period $2k$ and called the *dynamic root* of the component, and the other boundary fixed points are the *dynamic co-roots* and landing points of one ray each, of period k. Specifically for the critical value Fatou component, the two rays landing at the dynamic root separate this Fatou component and its co-roots from the entire critical orbit except the critical value (see Figure 5.3).

For each $c \in W$, the set of dynamic root and co-roots of the Fatou component containing the critical value is in natural bijection to the parabolic boundary arcs of W: at each of the $d + 1$ boundary arcs of W, a different one of the dynamic roots or co-roots becomes parabolic. The *parabolic root arc* is the arc at which the dynamic root becomes parabolic, while the *d co-root arcs* are those where one the d co-roots becomes parabolic. Principal parabolic maps p_c are thus maps from the root arc, and they exist on the boundary of each odd period component. At a parabolic
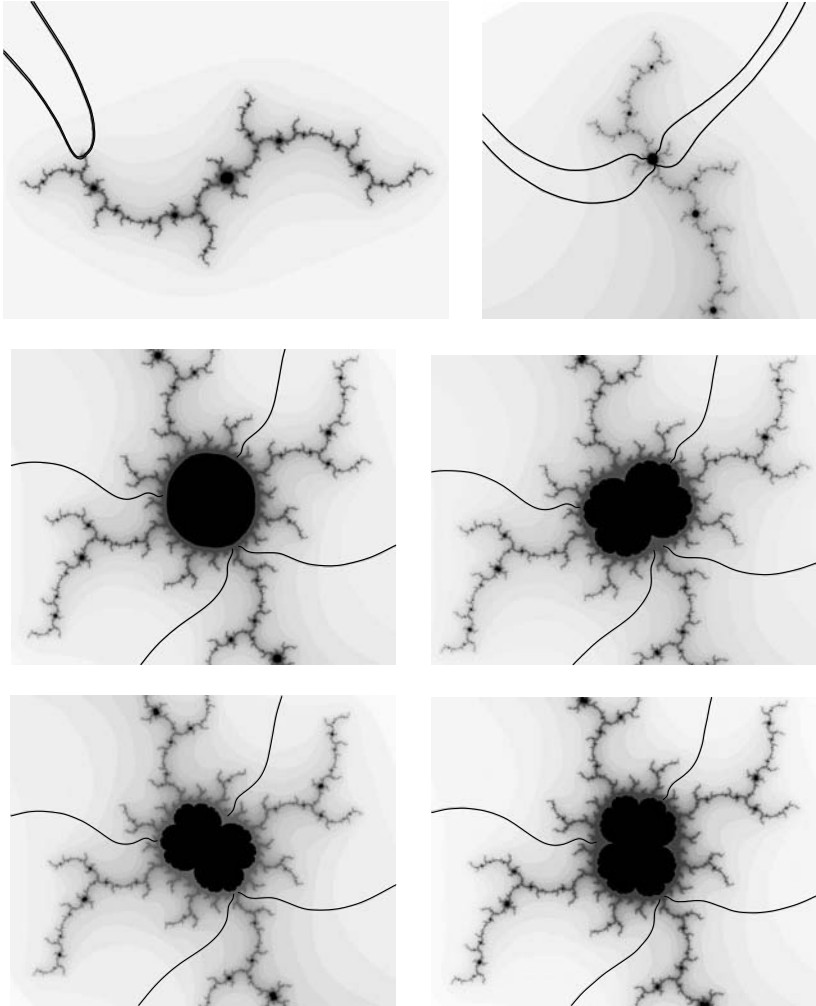
FIGURE 5.3: An antiholomorphic map $p_c(z) = \bar{z}^2 + c$ of degree $d = 2$ with an attracting cycle of period 5. The Fatou component containing the critical value has $d + 1 = 3$ boundary points that are fixed under $p_c^{\circ 5}$, and together these are the landing points of $d + 2 = 4$ dynamic rays: the dynamic root is the landing point of 2 rays (here, at angles $371/1023$ and $404/1023$ of period 10, and the two dynamic 2-roots are the landing points of one ray each (at angles $12/33$ and $13/33$ of period 5). *Upper left*: the entire Julia set with the four rays indicated. *Upper right*: blow-up of a neighborhood of the critical value Fatou component where the four rays can be distinguished. The hyperbolic component containing the parameter c is bounded by $d + 1$ parabolic arcs (see Figure 1.3): one arc contains the accumulation set of the parameter rays at angles $12/33$ and $13/33$ (the root arc), and the other two arcs contain the accumulation sets of one parameter ray each (at angles $12/33$ and $13/33$ respectively). The four remaining pictures show further blow-ups near the critical value, for parameters at the center (middle row, left), from the parabolic root arc (middle right) and from the two parabolic co-root arcs (bottom row). (See also Plates 7 and 8.)

cusp, a dynamic root or co-root merges with one of its adjacent dynamic (co-)roots. Specifically at a cusp at the end of the root arc, the dynamic root merges with a co-root: at such parameters, each parabolic periodic point is the landing point of two rays of period $2k$ and one ray of period k.

DEFINITION 5.4 (Parabolic Tree). *If p_c has a principal parabolic orbit of odd period k, we define its* parabolic tree *as the unique minimal tree within the filled-in Julia set that connects the parabolic orbit and the critical orbit, so that it intersects the critical value Fatou component along a simple $p_c^{\circ k}$-invariant curve connecting the critical value to the characteristic parabolic point, and it intersects any other Fatou component along a simple curve that is an iterated preimage of the curve in the critical value Fatou component. A* loose parabolic tree *is a tree that is homotopic to the parabolic tree, by a homotopy that fixes the Julia set (so it acts separately on bounded Fatou components). It is easy to see that the parabolic tree intersects the Julia set in a Cantor set, and these points of intersection are the same for any loose tree (not that for simple parabolics, any two periodic Fatou components have disjoint closures).*

This tree is defined in analogy to the Hubbard tree for postcritically finite polynomials. In our case, note first that the filled Julia set is locally connected hence path connected, so any minimal tree connecting the parabolic orbit is uniquely defined up to homotopies within bounded Fatou components. The parabolic tree is p_c-invariant (this is clear by construction separately in the Julia set and in the Fatou set). A simple standard argument (analogous to the postcritically finite case) shows that the critical value Fatou component has exactly one boundary point on the tree (the characteristic parabolic point), and all other bounded Fatou components have at most d such points (the preimages of the characteristic parabolic point). The critical value is an endpoint of the parabolic tree. All branch points of the parabolic tree are either in bounded Fatou components or repelling (pre)periodic points; in particular, no parabolic point (of odd period) is a branch point.

DEFINITION 5.5 (OPPPP: Odd Period Prime Principal Parabolic). *A principal parabolic map p_c with a parabolic orbit of odd period $k \geq 3$ is called* prime *if the parabolic tree does not have any proper connected subtree that connects at least two Fatou components and that is invariant under some iterate of p_c.*

Parabolics with these properties will be called OPPPP-parabolics, and these are the ones that we will work with.

REMARK 5.6. The condition of "prime" can be motivated informally as follows. Just like the Mandelbrot set contains countably many "little Mandelbrot set," it is experimentally observed (but not yet formally proved) that the multicorn $\mathcal{M}_d^*$ contains countably many "little multicorns," finitely many for each odd period $k \geq 3$ (these are combinatorial copies, not homeomorphic copies, for reasons we mentioned in the introduction); the period n means that periods of hyperbolic components in the combinatorial copy are n times the original periods. There is a natural map from the little multicorn onto $\mathcal{M}_d^*$ that is given by an antiholomorphic version of the straightening theorem, but it is not necessarily continuous. Each little multicorn, say, of period k, contains in turn countably many little multicorns, and all their periods are multiples of k. Under tuning (the inverse of straightening), the little multicorns thus form a semi-group (see [M1]), and a "prime" multicorn is one that cannot be written as a composition of other small multicorns. A map p_c

with an attracting or parabolic orbit of odd period k is prime if the parameter c is from the closure of the main hyperbolic component of a prime multicorn.

Formally speaking, any map p_c with a parabolic orbit of odd period $k \geq 3$ is clearly prime if the period k is prime (it may or many not be prime otherwise). This establishes the existence of infinitely many OPPPP parabolics, using the existence of hyperbolic components of all periods.

Concerning the latter, define a sequence $s_{d,k} := d^{k-1} - \sum_{m|k,\, m<n} s_{d,m}$ for each $d \geq 2$. Then $s_{d,k}$ is the number of hyperbolic components of period k for the "multi-brot sets" of period d: each hyperbolic component of period k has a center parameter that satisfies $((c^d + c)^d + c + \ldots + c) = 0$, and dividing out solutions for periods k strictly dividing n we obtain the given recursive formula. It turns out that the number of hyperbolic components of the multicorns $\mathcal{M}_d^*$ of period k also equals $s_{d,k}$, except if k is twice an odd number: in the latter case, the number of hyperbolic components equals $s_{d,k} + 2s_{d,k/2}$ [MNS].

In order to reassure readers concerned that we might be talking about the empty set, here is a simple existence argument.

LEMMA 5.7 (Existence of Hyperbolic Components). *Every multicorn $\mathcal{M}_d^*$ has hyperbolic components of all odd periods.*

PROOF. Let k be an odd number and let $\varphi \in \mathbb{R}/\mathbb{Z}$ be an angle with period k under multiplication by $-d$ modulo 1; i.e., $\varphi = s/(d^k + 1)$ for some $s \in \mathbb{Z}$. The parameter ray $R(\varphi)$ at angle φ is defined as the set of parameters $c \in \mathbb{C} \setminus \mathcal{M}_d^*$ for which the critical value is on the dynamic ray at angle φ (and escapes to ∞). In [Na] it was shown that $\mathcal{M}_d^*$ is connected, and in particular that $R(\varphi)$ is a curve in $\mathbb{C} \setminus \mathcal{M}_d^*$ that converges to ∞ in one direction, and that accumulates at $\partial \mathcal{M}_d^*$ in the other direction. Let $c \in \partial \mathcal{M}_d^*$ be any accumulation point of $R(\varphi)$; note that we do *not* claim that $R(\varphi)$ has a well-defined limit point in $\partial \mathcal{M}_d^*$ (it usually will not), but its accumulation set is non-empty.

In the dynamics of p_c, the filled-in Julia set is connected, and the dynamic ray at angle φ lands at a periodic point that is repelling or parabolic. If the landing point is repelling, then stability under small perturbations assures that for parameters c' near c, the dynamic ray at angle φ lands at a repelling periodic point, and ray and landing point depend continuously on c'. But since the critical value has positive distance from ray and endpoint, this will remain so under perturbations, and this is a contradiction (compare [GM, Lemma B.1]).

Therefore, for p_c, the dynamic ray at angle φ lands at a parabolic periodic point. Let k' be the period of the parabolic orbit. Since k' is odd, the rays landing at this orbit must have period k' or $2k'$ [NS, Lemma 3.1]. This implies $k = k'$, so c is on the boundary of a hyperbolic component of period k (Lemma 3.4). □

LEMMA 5.8 (Analytic Arc Only for Real Parameters). *Suppose the filled-in Julia set of an OPPPP parabolic map p_c contains a simple analytic arc that connects two bounded Fatou components. If the critical value has Ecalle height zero, then p_c is conformally conjugate to a real map $p_{c'}$ (i.e., $c' \in \mathbb{R}$).*

PROOF. Let k be the period of the parabolic orbit, and let z_1 be the characteristic point on this orbit. Since the parabolic orbit is simple, any two bounded Fatou components have disjoint closures, so the analytic arc must traverse infinitely many bounded Fatou components. Iterating the analytic arc forward finitely many times and cutting at the critical point if necessary, we obtain a simple analytic arc

connecting z_1 to some other bounded Fatou component that intersects the parabolic tree. Truncate if necessary so that the arc does not meet any branch point of the parabolic tree, nor any bounded Fatou component that contains a branch point, but so that it still connects z_1 to some other bounded Fatou component and so that all Fatou components that this arc intersects take more than k iterations to reach the critical value Fatou component. Call this piece of analytic arc J_1. Then $p_c^{\circ k}\colon J_1 \rightarrow p_c^{\circ k}(J_1) =: J_2$ is an analytic diffeomorphism between simple analytic arcs.

The arcs J_1 and J_2 are parts of the parabolic tree, except for homeomorphisms within bounded Fatou components (so they are part of a loose parabolic tree). They both start at z_1, which is not a branch point of the parabolic tree, so they must coincide at a Cantor set of points in the Julia set. As analytic arcs, they must thus coincide (except for truncation). It follows that one of the two arcs J_1 and J_2 is a sub-arc of the other. If $J_2 \subset J_1$, then p_c is not prime, so $J_2 \supset J_1$ and hence $J_{n+1} := p_c^{\circ k}(J_n) \supset J_n$ for all n. Again by definition of being prime, there is some N so that J_N covers the entire parabolic tree.

As long as $p_c^{\circ nk}\colon J_1 \rightarrow J_{n+1}$ is a homeomorphism, the image is a simple analytic arc. If during the iteration, the critical point is covered, the p_c-image will contain the critical value, but this cannot introduce any branching: suppose $J = p_c^{\circ m}(J_1)$ is a simple analytic arc that contains the critical point 0 in the interior and let J' and J'' be the components of $J \setminus \{0\}$. Then $p_c(J')$ and $p_c(J'')$ both start at the critical value and have z_1 as an interior point, so as above they must coincide in a neighborhood of z_1; hence $p_c(J') \cup p_c(J'') = p_c(J)$ is again a simple analytic arc. Therefore, all J_n are simple analytic arcs, and the same holds for the parabolic tree, which equals J_N. The parabolic tree thus is unbranched.

Now we claim that p_c is conformally conjugate to its complex conjugate $p_{\bar{c}}$ (they are obviously conjugate by an *anti*conformal homeomorphism, but we want a conformal conjugation). The condition of Ecalle height zero implies that $p_c^{\circ k}$ and $p_{\bar{c}}^{\circ k}$ are conformally conjugate on their incoming Ecalle cylinders, respecting the critical orbits. This conjugation can be pulled back to the incoming petal of the parabolic orbit and thus to their periodic Fatou components. It follows from local connectivity that this conformal conjugation on each individual Fatou component extends homeomorphically to the closure of the component.

The next step is to extend this conjugation homeomorphically to the filled-in Julia set, again using local connectivity. This is possible because the parabolic trees are unbranched, so their combinatorial structure is unaffected by complex conjugation.

Finally, we extend the conjugation to the basin of infinity, using the Riemann map between the basins of infinity so that ∞ is fixed. There are $d + 1$ choices for this conjugation near ∞, and one of them maps the rays landing at the parabolic orbit to the rays landing at the parabolic orbit (this is possible because we already know that the dynamics on the Julia set is conjugate); for this map, the extension to the boundary coincides with the conjugation on the Julia set we already have.

This way, we obtain a topological conjugation $h\colon \mathbb{C} \rightarrow \mathbb{C}$ between p_c and $p_{\bar{c}}$ that is conformal away from the Julia set. If we know that the Julia set is holomorphically removable, then we have a conformal conjugation on $\mathbb{C}$. This fact can be established directly without too much effort. Consider the equipotential E of p_c at some positive potential, and let $E_1, \ldots, E_k$ be piecewise analytic simple closed curves, one in each bounded periodic Fatou component of p_c, that surround the postcritical set in their Fatou components and that intersect the boundary of their

Fatou components in one point, which is on the parabolic orbit. Let V_0 be the domain bounded on the outside by E and on the inside by the E_i. Then there is a quasiconformal homeomorphism $h_0 \colon \mathbb{C} \to \mathbb{C}$ with $h_0 = h$ on $\mathbb{C} \setminus V$ (i.e., the homeomorphism h is modified on V_0 so as to become quasiconformal, possibly giving up on the condition that h_0 is a conjugation on V_0).

Now construct a sequence of quasiconformal homeomorphisms $h_n \colon \mathbb{C} \to \mathbb{C}$ as a sequence of pull-backs, satisfying $p_{\bar{c}} \circ h_{n+1} = h_n \circ p_c$: the construction assures that this is possible, and all h_n satisfy the same bounds on the quasiconformal dilatation as h_0. Moreover, the support of the quasiconformal dilatation shrinks to the Julia set, which has measure zero. By compactness of the space of quasiconformal maps with given dilatation, the h_n converge to a conformal conjugation between p_c and $p_{\bar{c}}$. This limiting conjugation must coincide with h on the Fatou set, so the Julia set is holomorphically removable, as claimed.

Finally, since p_c and $p_{\bar{c}}$ are conformally conjugate, we have $c = \zeta^s \bar{c}$, where ζ is a complex $(d+1)$st root of unity and $s \in \mathbb{Z}$, so writing $c = re^{2\pi i\varphi}$ it follows that $\varphi = -\varphi + s/(d+1)$ or $\varphi \in \mathbb{Z}/2(d+1)$. Since p_c is conformally conjugate to $p_{c'}$ with $c' = c\zeta^{s'}$ for $s' \in \mathbb{Z}$, we may add $s'/(d+1)$ to φ, so $\varphi \in \{0, 1/2\}$, and this means that p_c is conformally conjugate to a real map.   □

REMARK 5.9. From the statement that the parabolic tree is unbranched, there is an alternative argument that p_c is conformally conjugate to $p_{\bar{c}}$.

We can modify the parabolic tree topologically into a superattracting tree. The map on this tree extends to a postcritically finite orientation-reversing branched mapping whose combinatorial equivalence class is well defined and that is obviously combinatorially equivalent to its complex conjugate.

We can then apply Thurston's theorem (in fact, just Thurston rigidity) to claim that the corresponding superattracting antipolynomial $p_{c'}$ is conformally conjugate on $\mathbb{C}$ to its complex conjugate. This says that up to conjugacy, the tricorn with $p_{c'}$ at its center can be taken to be on the real axis, hence also the point on the principal boundary arc at Ecalle height 0.

We give this more elementary but somewhat tedious argument to avoid using Thurston's theorem for orientation-reversing branched maps, though the result is true and requires almost no modifications in the proof. Similarly, one can use the methods of "Posdronasvili" [DH1] or of Poirier [P] to prove that any two postcritically finite polynomials with unbranched Hubbard trees having identical combinatorics are conformally conjugate.

LEMMA 5.10 (Approximating Ray Pairs). *Every OPPPP parabolic map p_c with Ecalle height zero is either conformally conjugate to a map $p_{c'}$ with real parameter c', or the characteristic parabolic point z_1 is the limit of repelling preperiodic points w_n and $\tilde{w}_n$ on the parabolic tree so that all w_n have Ecalle heights $h > 0$ and all $\tilde{w}_n$ have Ecalle heights $-h$, with the following property: if φ and φ' are the external angles of the dynamic rays landing at z_1, then dynamic rays at angles φ_n and φ'_n land at w_n so that $\varphi_n \to \varphi$ and $\varphi'_n \to \varphi'$; similarly, dynamic rays at angles $\tilde{\varphi}_n$ and $\tilde{\varphi}'_n$ land at $\tilde{w}_n$ with $\tilde{\varphi}_n \to \varphi$ and $\tilde{\varphi}'_n \to \varphi'$.*

PROOF. Repelling preperiodic points are dense on the Cantor set where the parabolic tree intersects the Julia set (for instance by the condition of being prime), so choose one such point, say w_0, in the repelling petal of z_1 (repelling periodic points must accumulate at z_1 and cannot do this within the attracting petal; and some

neighborhood of z_1 is covered by the union of attracting and repelling petals). If all repelling periodic points on the parabolic tree and near z_1 have Ecalle height 0, then the parabolic tree must intersect the Julia set entirely at Ecalle height zero, and then one can construct an analytic arc that satisfies the hypotheses of Lemma 5.8, so p_c is conformally conjugate to a real map.

If this is not the case, then we may assume that w_0 has non-zero Ecalle height h; to fix ideas, say, $h > 0$. Construct a sequence (w_n) so that $w_{n+1} := p_c^{\circ(-2k)}(w_n)$, choosing a local branch that fixes z_1 and so that all w_n are in the repelling petal of z_1; hence $w_n \to z_1$ as $k \to \infty$. All w_n have the same Ecalle height h.

As w_0 is on the parabolic tree, which is invariant, it follows that w_0 is accessible from outside of the filled Julia set on both sides of the tree, so w_0 is the landing point of (at least) two dynamic rays, "above" and "below" the tree. If φ_n and φ_n' are the corresponding angles of rays landing at w_n, then it follows that these sequences of angles converge to angles of rays landing at z_1 on both sides of the tree, and the claim follows.

Now let $w_n' := p_c^{\circ k}(w_n)$; then $w_n' \to z_1$ and all these points have Ecalle heights $-h$. The rays landing at z_1 have angles φ and φ' and their period is $2k$, so $p_c^{\circ k}$ permutes these and the claim about w_n' and its rays follows. $\square$

6   NON-PATHWISE CONNECTIVITY

We denote the dynamic ray at angle φ for the map p_c by $R_c(\varphi)$ and, as before, the parameter ray at angle φ by $R(\varphi)$.

THEOREM 6.1 (Rays Approximating at OPPPP Arc). *Let $\mathcal{A}$ be a prime parabolic root arc of odd period $k \geq 3$ that does not intersect the real axis or its images by a symmetry rotation of $\mathcal{M}_d^*$, and let $c \in \mathcal{A}$ be the parameter with Ecalle height zero. Let φ and φ' be the characteristic angles of the parabolic orbit for parameters $c \in \mathcal{A}$. Then there is a sub-arc $\mathcal{A}_\tau \subset \mathcal{A}$ of positive length and there are angles $\tilde{\varphi}_n \to \tilde{\varphi}$ and $\varphi_n' \to \varphi'$ so that $\mathcal{A}_\tau$ is contained in the limit of the parameter rays $R(\tilde{\varphi}_n)$, and also of $R(\varphi_n')$ (this is the limit of the sequence of rays, not necessarily of individual rays: each $c \in \mathcal{A}_\tau$ is the limit of a sequence of points on the parameter rays $R(\tilde{\varphi}_n)$, and of another sequence on $R(\varphi_n')$.)*

PROOF.   In the dynamics of p_c, let z_1 be the characteristic parabolic point. The angles φ and φ' have period $2k$, so the rays $R_c(\varphi)$ and $R_c(\varphi')$ are interchanged by the first return map of z_1. In the outgoing Ecalle cylinders at z_1 of the holomorphic map $p_c^{\circ 2k}$, the rays $R_c(\varphi)$ and $R_c(\varphi')$ project to disjoint simple closed curves, not necessarily at constant Ecalle heights, but it makes sense to say which of the two rays has greater Ecalle heights (removing the projection of one ray from the Ecalle cylinder, the other ray is in the component with arbitrarily large positive or negative Ecalle heights). Without loss of generality, suppose that $R_c(\varphi)$ has greater heights than $R_c(\varphi')$. Let h^+ be the maximum of Ecalle heights of φ, and h^- be the minimum of Ecalle heights of φ'; since $p_c^{\circ k}$ interchanges $R_c(\varphi)$ and $R_c(\varphi')$, we have $h^+ = -h^- > 0$.

Consider the sequences of repelling preperiodic points w_n and $\tilde{w}_n$ converging to z_1 as provided by Lemma 5.10, and let $h > 0$ and $-h$ be their Ecalle heights. Then clearly $h < h^+$ (the points w_k are in the part of the Ecalle cylinder bounded by the rays $R_c(\varphi)$ and $R_c(\varphi')$). The rays $R_c(\tilde{\varphi}_n)$ terminate at the points $\tilde{w}_n$ with Ecalle heights $-h$, while they all project to the same ray in the Ecalle cylinder, in which they spiral upwards and converge towards the projection of the ray at
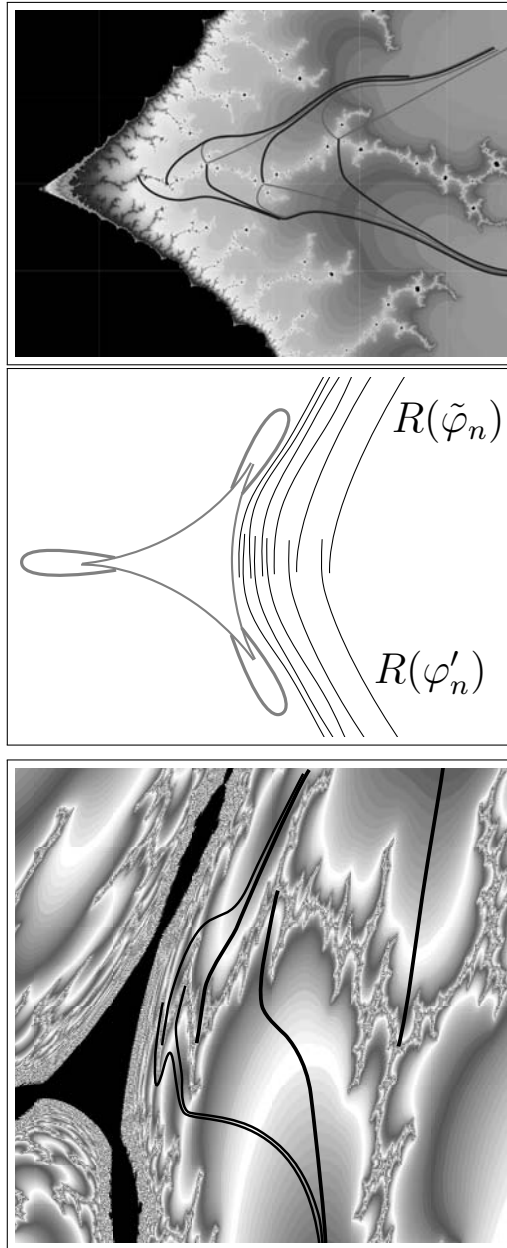
FIGURE 6.2: Loss of pathwise connectivity because of approximating overlapping parameter rays. *Top*: Approximating preperiodic dynamic rays in the dynamic plane with a parabolic orbit. Only the rays drawn by heavy lines are used in the argument below; other rays landing at the same points are drawn in grey. *Middle*: Symbolic sketch of the situation in the parameter space. *Bottom*: Actual parameter rays accumulating in the same pattern, producing a double-comb-like structure. (See also Plate 6.)

angle φ. Therefore, for any compact subinterval of $(-h, h)$, the rays $R_c(\tilde{\varphi}_n)$ have Ecalle heights in this entire compact interval. Similarly, the rays $R_c(\varphi'_n)$ terminate at the w_n with Ecalle height h and also have Ecalle heights within any compact subinterval of (h^-, h); see Figure 6.2.

Now let $c_t \in \mathcal{A}$ be the parameter where the critical value has Ecalle height $t \in \mathbb{R}$ (see Proposition 2.6). The points w_k depend real-analytically on t (like the entire Julia set); let $h(t)$ be their Ecalle heights; these too depend analytically on t. Therefore, there is a $\tau \in (0, h)$ so that $h(t) > t$ for all $t \in (-\tau, \tau)$. Choose $\varepsilon \in (0, \tau)$. Let $\mathcal{A}_\tau \subset \mathcal{A}$ be the sub-arc with Ecalle heights in $(-\tau + \varepsilon, \tau - \varepsilon)$.

To transfer these dynamic rays from the Ecalle cylinders to parameter space, we employ Proposition 4.8. Choose any smooth path $c \colon [0, \delta] \to \mathbb{C}$ with $c(0) = c_t \in \mathcal{A}_\tau$ but so that, except for $c(0)$, the path avoids closures of hyperbolic components of period k and so that the path is transverse to $\mathcal{A}$ at c_t.

In the outgoing cylinder of $c(0) = c_t \in \mathcal{A}$, all $R_c(\tilde{\varphi}_n)$ traverse Ecalle heights in $[-h + \varepsilon/2, h - \varepsilon/2]$. Since each ray $R_c(\tilde{\varphi}_n)$ and its landing point depend uniformly continuously on c, and since the projection into Ecalle cylinders is also continuous, there is a $\delta_\varepsilon > 0$ so that for all $c(s)$ with $s < \delta$ the projection of the rays $R_c(\tilde{\varphi}_n)$ into the Ecalle cylinders traverses heights $[-h + \varepsilon, h - \varepsilon]$, while the phase is uniformly continuous in s.

For $s \in [0, \delta]$, let $z(s)$ be the critical value. For $s > 0$, the critical orbit "transits" from the incoming Ecalle cylinder to the outgoing cylinder; as $s \searrow 0$, the image of the critical orbit in the outgoing Ecalle cylinder has Ecalle height tending to $t \in (-\tau + \varepsilon, \tau - \varepsilon) \subset (-h + \varepsilon, h - \varepsilon)$, while the phase tends to infinity. Therefore, there are $s \in (0, \delta_\varepsilon)$ arbitrarily close to 0 at which the critical value, projected into the incoming cylinder and sent by the transfer map to the outgoing cylinder, lands on the projection of the rays $R_c(\tilde{\varphi}_n)$. But in the dynamics of $p_{c(s)}$, this means that the critical value is on one of the dynamic rays $R_c(\tilde{\varphi}_n)$, so $c(s)$ is on the parameter ray $R(\tilde{\varphi}_n)$.

The analogous statement holds for φ'_n.   □

Now the proof of our main result is simple.

THEOREM 6.3 (Multicorns Are Not Path Connected). *For each $d \geq 2$, the multicorn $\mathcal{M}^*_d$ is not path connected.*

PROOF. Let W be any hyperbolic component of odd period not intersecting the real axis, let $\mathcal{A} \subset W$ be the parabolic root arc and suppose it is prime. By Theorem 6.1, there is a sub-arc $\mathcal{A}_\tau$ of positive length and there are two sequences of angles $\tilde{\varphi}_n$ and φ'_n converging to limits $\tilde{\varphi} \neq \varphi'$ so that the set

$$\bigcup_n R(\tilde{\varphi}_n) \cup \bigcup_n R(\varphi'_n) \cup \mathcal{A}_\tau$$

disconnects $\mathbb{C}$ into at least 2 path-components. If the angles are oriented so that $\varphi < \varphi'$, then W is in a different component from $R(0)$ and any hyperbolic component W' in the limit set of the angle $R(1/(2^n \pm 1))$ for sufficiently large n: any path connecting W to W' must accumulate at all points in $\mathcal{A}_\tau$, and this is impossible.   □

REMARK 6.4. We believe that the only hyperbolic components for which the umbilical cord lands are on the real axis, or symmetric to such components by a rota-

tional symmetry of $\mathcal{M}_d^*$. For individual components, this can be verified numerically: all one needs to know is that the parabolic tree does not contain analytic arcs, even in the non-prime situation; for this it is good enough to know that the subtree of renormalizable points contains periodic points with non-real multipliers.

7 FURTHER RESULTS

The following proposition and its proof are inspired by more general results due to Bergweiler [Be] as well as Buff and Epstein [BE].

THEOREM 7.1 (No Bifurcation at Ecalle Height Zero). *On every parabolic arc of period k, the point with Ecalle height zero has a neighborhood (along the arc) that does not intersect the boundary of a hyperbolic component of period 2k.*

PROOF. Suppose p_c is the center point of the parabolic arc (at Ecalle height 0). We will now discuss the local dynamics of the holomorphic first return map, i.e., the $(2k)$th iterate of p_c. The upper and lower endpoints of the Ecalle cylinders correspond to fixed points of $p_c^{\circ 2k}$; but they are interchanged by $p_c^{\circ k}$, so they are simultaneously attracting or repelling with complex conjugate multipliers.

Points in the outgoing petal at sufficiently large positive Ecalle heights will return to the incoming petal; this map is called the "horn map." This induces a conformal map from the upper end of the outgoing cylinder to the incoming cylinder which, by the Koebe compactness theorem, is close to a translation by a complex constant (writing the cylinders as $\mathbb{C}^*$, it is close to multiplication by a constant). Let η be the imaginary part of the translation constant. Therefore, for every $\varepsilon > 0$ there is an $H > 0$ so that points at Ecalle heights $h_o > H$ in the outgoing cylinder return to points in the incoming cylinder with height h_i so that $|h_i - h_o + \eta| < \varepsilon$, i.e. $h_i \in (h_o - \eta - \varepsilon, h_o - \eta + \varepsilon)$.

Cut the outgoing and incoming Ecalle cylinders at the equators into upper and lower half-cylinders and label the upper halves C_o and C_i, where i and o stand for "incoming" and "outgoing" (the whole discussion can be done analogously in the lower halves, with negative Ecalle heights, just as well; the antiholomorphic iteration assures that they are completely symmetric). Let $C_o' \subset C_o$ be the restriction to the parabolic basin. Let $f : C_o' \to C_o$ be a conformal isomorphism; it is unique up to addition of a real constant (a phase). There is a number $\delta \in \mathbb{C}$ so that asymptotically near the end, $f(z) = z - i\delta$. By adjusting the freedom in f, we can make δ purely real. The Schwarz lemma, together with the fact that $C_o \setminus C_o'$ contains open sets in the basin of infinity, implies $\delta > 0$. This number is called the *Grötzsch defect* of the outgoing cylinder (with respect to the parabolic basin).

For $h > 0$, let $C_i(h)$ and $C_o'(h)$ be C_i and C_o' restricted to Ecalle heights in $(0, h)$. Then $h = \mathrm{mod}(C_o'(h))$ and δ can be viewed as the limit, as $h \to \infty$, of $h - \mathrm{mod}(C_o'(h))$. The Grötzsch inequality implies that $\mathrm{mod}(C_o'(h)) \leq h - \delta$ for all h.

Choose $\varepsilon \in (0, \delta)$ and H depending on ε as before. Since by hypothesis the critical orbit is at Ecalle height 0, one can pull back $C_i(H)$ conformally into C_o; it must land within $C_o'(H + \eta + \varepsilon)$ (it must have Ecalle height less than $H + \eta + \varepsilon$ and it must be in the part within the attracting basin). Since $\mathrm{mod}(C_i(H)) = H$, while $\mathrm{mod}(C_o'(H + \eta + \varepsilon)) \leq H + \eta + \varepsilon - \delta$, this implies $\eta + \varepsilon - \delta \geq 0$ and thus $\eta > 0$.

By Proposition 4.8, the outgoing and incoming cylinders exist after small perturbations of the parameter outside of its hyperbolic component, the Ecalle heights

depend continuously on the perturbation, and the same holds for the "horn maps" from the ends of the outgoing into the incoming cylinders.

If $\varepsilon < |\eta|$, then $\eta < 0$ means that points with great Ecalle heights the outgoing cylinder return into the incoming cylinder at greater heights. For such sufficiently small perturbations, $\eta < 0$ thus implies that points near the end of the cylinder will, after perturbation, converge to the end of the cylinder; the endpoint of the cylinder thus becomes an attracting fixed point. Similarly, if $\eta > 0$, then the endpoints become repelling. Since these are the periodic points that bifurcate from the period k orbit, this shows that parameters c with $\eta > 0$ are not on the boundary of a period $2k$ hyperbolic component, and this is the case when the Ecalle height h is zero or sufficiently close to zero.   □

REMARK 7.2. This result can be strengthened in various ways. One can give an explicit lower bound on the Grötzsch defect δ: the basin at infinity alone occupies an annulus of modulus at least $1/(2k \log d)$, so $\delta > 1/(2k \log d)$ (compare [BE], Theorem B, and especially the first half of the proof). Moreover, one can deduce an inequality between the Ecalle height of the critical orbit and the fixed-point index: if the critical value has Ecalle height h, one can estimate the conformal modulus of the largest embedded annulus in $C_i(H)$ that separates the critical value from the upper boundary (this is a classical extremal length estimate; the modulus is $H - |h| + o(1)$), and this gives a correspondingly greater upper bound on the fixed-point index (after all, we know that for large Ecalle heights h, the fixed-point index must become greater than 1). Combining both facts, this implies a definite interval of Ecalle heights around 0, depending only on d, for which the parabolic arc does not meet bifurcating components.

THEOREM 7.3 (Decorations Along Parabolic Arc). *Every parabolic arc on a hyperbolic component of odd period k has Ecalle heights $h_1, h_2, h_3, h'_1, h'_2, h'_3 \in \mathbb{R}$ so that $h_3 > 0 > h'_3$, $h_3 > h_2$, $h'_2 > h'_3$, satisfying the following properties:*

- *the sub-arc with Ecalle heights $h > h_3$ is an arc of bifurcation to a component of period $2k$; and also for Ecalle heights $h < h'_3$;*

- *the sub-arc with Ecalle heights $h \in (h_2, h_3)$ is the limit of decorations (attached to the period $2k$ components bifurcating for large positive Ecalle heights); and also for Ecalle heights $h \in (h'_3, h'_2)$;*

- *if the arc is a root arc, then the sub-arc with Ecalle heights $h \in [h'_1, h_1]$ is the limit of the "umbilical cord."*

Note that by Theorem 7.1 we have $h_3 > 0 > h'_3$ (the two hyperbolic components near the end of the parabolic arc are disjoint), but we do not know whether it is always true that $h_2 > 0 > h'_2$ (if $h_2 < h'_2$, this would mean that the decorations from the period $2k$ components at both ends of the arc would overlap; this would imply $h_1 \geq |h'_2|$). We clearly have $h_3 > \max\{h_2, |h'_2|, h_1\}$. Loss of pathwise connectivity of the umbilical cord occurs whenever $h_1 > 0$.

If $h_1 < h_2$, then we have an "open beach," where the boundary of the multicorn locally equals just the parabolic arc without further decorations. We do not know whether there are infinitely many parabolic arcs for which this occurs.
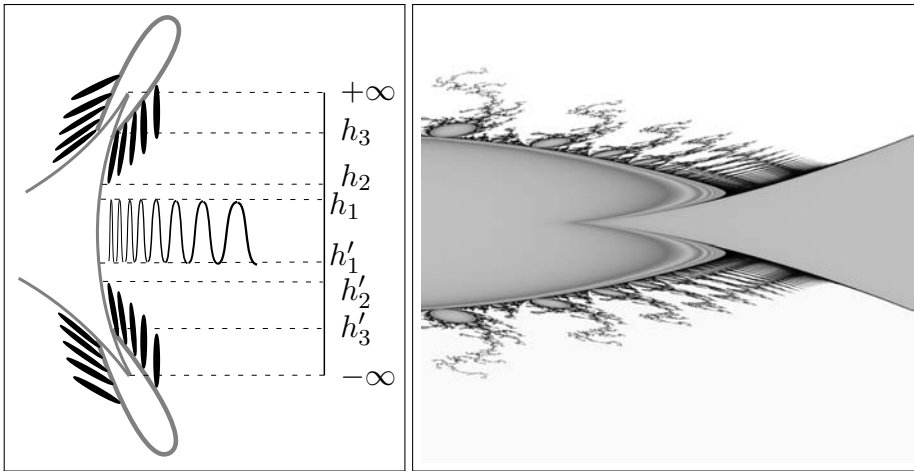
FIGURE 7.4: Illustration of Theorem 7.3. *Left*: Schematic illustration of the decorations along parabolic arcs, together with their threshold heights. *Right*: Decorations at a period 2 component that accumulate along arcs on the boundary of the period 1 component. (See also Plate 3.)

SKETCH. For sufficiently large Ecalle heights, the parabolic arc is on the locus of bifurcation from odd period k to period $2k$ (Corollary 3.9). Let $h_3 \in \mathbb{R}$ be the infimum of such Ecalle heights, and consider p_c for c on this parabolic arc with Ecalle height h_3. The boundary of the Fatou component, projected into the outgoing Ecalle cylinder, is not a geodesic in this cylinder (because this boundary is not an analytic curve). It will thus project into the cylinder at an interval (h', h'') of Ecalle heights with h'' strictly greater than h'. In fact, we have $h'' = h_3$.

For parameters $c(h)$ with h slightly less than h_3, the lower Ecalle height h' depends on h, but there is an interval (h_2, h_3) when $h'(h) < h$. For these, both escaping points and the Julia set intersect the outgoing Ecalle cylinder at Ecalle height in a neighborhood of h, so the parameter $c(h)$ can be approximated by parameters inside and outside of $\mathcal{M}_d^*$. □

REMARK 7.5. Each parabolic arc contains the accumulation set of one or two periodic parameter rays (two rays for root arcs, one for co-root arcs). By symmetry, the rays accumulating at the boundary arcs of the period 1 hyperbolic component actually land. We believe that all the other periodic parameter rays do not land, at least those that accumulate at root arcs. Instead, we believe that they accumulate at a sub-arc of positive length.

The reason is as follows. For a parameter c on a root arc of period $n \geq 3$, the parabolic periodic point is the landing point of 2 dynamic rays. These form hyperbolic geodesics in the access that is bounded on one side by a periodic Fatou component and on the other side by the parabolic tree, decorated by various structures of the Julia set. Even though the ray is an analytic curve, it would seem to require a miracle that the boundaries of the access at the two sides are symmetric enough so that the ray projects to an equator in the Ecalle cylinder. But if it does not project to an equator, but has varying Ecalle height instead, then these wiggles
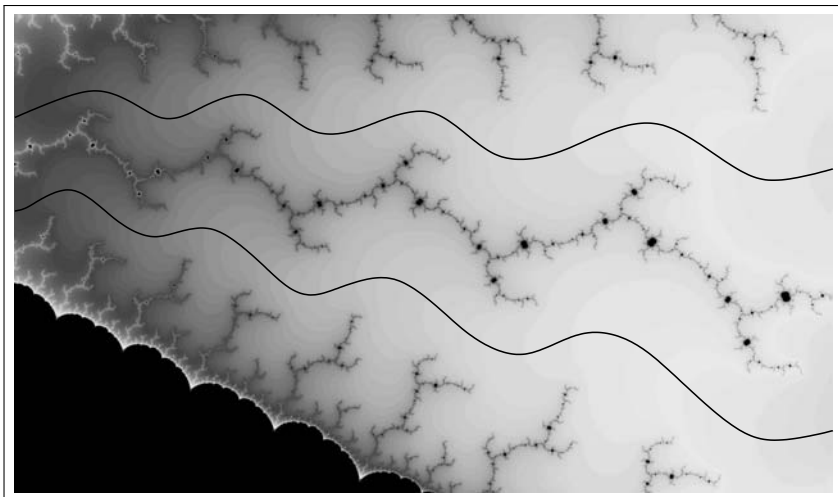
FIGURE 7.6: Heuristic argument why parameter rays should not land at parabolic root arcs but rather accumulate at a sub-arc of positive length. Sketch of the situation in the parabolic dynamics: near the top and bottom, there is the parabolic Fatou component (black, visible only in the bottom); in the middle there is the Julia set around the parabolic tree, and the two black curves are the two dynamic rays landing at the parabolic periodic point. They have no reason to have constant Ecalle height between two very different structures. While the parabolic basin is not stable under perturbations, the rays move continuously and keep their wiggles.

will transfer into parameter space to a ray that accumulates on the parabolic arc like a topologist's sine curve (see Figure 7.6).

REMARK 7.7. It is tempting to try to show for hyperbolic components of odd period k that the bifurcating period $2k$ component has wiggly boundary near the parabolic arc, transferring the wiggly boundary of the Fatou component to parameter space (using the fact that these Fatou components do not have analytic boundary arcs). However, this Fatou component is not stable under perturbation away from the parabolic arc, and we do not obtain wiggles in parameter space. In fact, it follows from Theorem 3.8 that the boundary of the period $2k$ component has a well-defined limit point on the parabolic arc: a simple parabolic with fixed point index $+1$, and those are isolated. What can be transferred into parameter space are the repelling periodic points on the boundary of the Fatou component, and the rays landing at them. These yield the decorations of the period $2k$ components that accumulate at parabolic arcs in a comb-like manner, as described in Theorem 7.3.

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