CANNON–THURSTON MAPS DO NOT ALWAYS EXIST

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Abstract. We construct a hyperbolic group with a hyperbolic subgroup for which inclusion does not induce a continuous map of the boundaries.

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1. Introduction

Hyperbolic groups are the finitely generated groups whose Cayley graphs display characteristics of negative curvature. Their systematic study was initiated by Gromov in [12] and, mimicking the study of Riemannian manifolds, pays particular attention to large–scale and asymptotic geometric features such as boundaries.

One of the many equivalent definitions of the *Gromov boundary* ∂G of an infinite hyperbolic group G with word metric d is as the set of equivalence classes of sequences (a_n) in G such that

$$(a_m \cdot a_n)_e \;:=\; (d(a_m, e) + d(a_n, e) - d(a_m, a_n))/2 \;\to\; \infty$$

as $m, n \to \infty$, where two such sequences (a_n) and (b_n) are equivalent when $(a_m \cdot b_n)_e \to \infty$ as $m, n \to \infty$. It is independent of the choice of finite generating set defining d and of the choice of basepoint. See [6] and [14] for surveys.

When H is an infinite hyperbolic subgroup of G, one can seek to induce a map $\partial H \to \partial G$ from the inclusion map. In [21] and [23] Mitra (or Mj, as he is now known) asks whether this is always well–defined, the concern being that ∂H is defined via the word metric on H and ∂G via that on G, and these may differ. He cites Bonahon [5] for similar questions and Bonahon [4], Floyd [8] and Minsky [19] for related work on Kleinian groups. The question is also raised by I. Kapovich & Benakli [14] and appears in the problem lists [3] and [15]. When the map exists, it is known as the Cannon–Thurston map.

The Cannon–Thurston map exists for many families of examples. The most straightforward is when H is quasi–convex (that is, undistorted). Cannon & Thurston [7] gave the first distorted example: they showed the map exists for $\pi_1 S \hookrightarrow \pi_1 M$ where M is a closed hyperbolic 3–manifold fibering over the circle with fiber a hyperbolic surface S (and, strikingly, the Cannon–Thurston map is a group–equivariant space–filling Peano curve $S^1 \twoheadrightarrow S^2$). Mitra generalized this widely. He showed the Cannon–Thurston map exists when H is an infinite normal subgroup of a hyperbolic group G [21], and he developed a theory of ending laminations for this context (inspired by [7]) to describe it [20]. He also showed

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the Cannon–Thurston map exists when H is one of the infinite vertex– or edge–groups of a finite graph of groups G in which G and all of the vertex– and edge–groups are hyperbolic, and all the defining monomorphisms from edge–groups to vertex–groups are quasi–isometric embeddings [22].

Recently, Mj established that Cannon–Thurston maps exist for surface Kleinian groups [25] (answering a question of Cannon & Thurston from [7] and Question 14 from Thurston's celebrated 1982 Bulletin AMS paper [28]) and then for arbitrary Kleinian groups [24] (proving a conjecture of McMullen from [18]). Mitra's question can be viewed as asking whether the natural generalization of these results in the setting of Geometric Group Theory holds. We answer it negatively:

Theorem 1. *There are positive words C, C_1, C_2 on c_1, c_2 and $D_1, D_2, D_{11}, D_{12}, D_{21}, D_{22}$ on d_1, d_2 so that*

$$G = \left\langle a,\, b,\, c_1, c_2,\, d_1, d_2 \;\middle|\; \begin{array}{cc} a^{-1}b^{-1}ab = C, & b^{-1}c_i b = C_i, \\ (ab)^{-1}d_j(ab) = D_j, & c_i^{-1}d_j c_i = D_{ij},\ \ 1 \le i, j \le 2 \end{array} \right\rangle$$

is hyperbolic, the subgroup

$$H = \langle b, d_1, d_2 \rangle$$

is free of rank 3*, and there is no Cannon–Thurston map $\partial H \to \partial G$.*

A *positive* word is one in which inverses of generators never appear.

At the expense of complicating the construction, G can be made CAT(−1), as we will outline in Remark 9.

That H is free is not pertinent to the nonexistence of the Cannon–Thurston map. Theorem 1 is the starting point for a proof by Matsuda and Oguni [17] that for every non–elementary hyperbolic group there is an embedding in some other hyperbolic group for which there is no Cannon–Thurston map. Implications of Theorem 1 have also been explored by Gerasimov and Potyagailo in a recent paper [9] on convergence actions.

Given that Cannon–Thurston maps do not always exist, the question arises as to what bearing subgroup distortion has. Heavy distortion appears to be no obstacle to the map's existence: we showed in [1] that Cannon–Thurston maps exist for highly distorted free subgroups of hyperbolic hydra groups; these examples exhibit the maximum distortion known among hyperbolic subgroups of hyperbolic groups. As for small distortion, if a subgroup of a hyperbolic group is subexponentially distorted, then the subgroup is quasi–convex by Proposition 2.6 of [13] and so the Cannon–Thurston Map exists. The natural open question, then, (which Ilya Kapovich asked us) is whether there is an exponentially distorted hyperbolic subgroup of a hyperbolic group for which the Cannon–Thurston map does not exist.[1] It will be apparent from our proof of Theorem 1 that the subgroup $H \le G$ is at least doubly–exponentially distorted.

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[1] An earlier version of this article claimed that the subgroup $G_{bcd} \le G$ (defined before Lemma 5) is such an example. Although $\partial G_{bcd} \to \partial G$ is not well-defined, we realized that the distortion is at least doubly exponential, so Kapovich's question remains open.

2. Proof of the theorem

Denote the free group on a set S by $F(S)$. If $S = \{s_1, \ldots, s_n\}$, write $F(S) = F(s_1, \ldots, s_n)$. If F is a group and $X \subseteq F$ a subset so that the natural map $F(X) \to F$ is an isomorphism, then X is called a *free basis* for F and F is said to be a *free group of rank* $\mathrm{card}(X)$.

We begin by showing that when C, C_i, D_j and D_{ij} are chosen suitably, the group G of Theorem 1 is hyperbolic.

A finite presentation for a group satisfies the $C'(\lambda)$ small–cancellation condition when, after cyclically reducing all the defining relations, the set S of all their cyclic permutations and those of their inverses, has the property that every common prefix between two distinct $r_1, r_2 \in S$ has length less than λ times the lengths of each of r_1 and r_2 [16, page 240].

Following Rips [26], we take

$$\begin{aligned} C &= c_1c_2c_1c_2^2c_1c_2^3\cdots c_1c_2^r,\\ C_i &= c_1c_2^{ri+1}c_1c_2^{ri+2}c_1c_2^{ri+3}\cdots c_1c_2^{ri+r},\\ D_j &= d_1d_2^{rj+1}d_1d_2^{rj+2}d_1d_2^{rj+3}\cdots d_1d_2^{rj+r},\\ D_{ij} &= d_1d_2^{r(il+j)+1}d_1d_2^{r(il+j)+2}d_1d_2^{r(il+j)+3}\cdots d_1d_2^{r(il+j)+r}, \end{aligned}$$

where r is sufficiently large that the presentation for G of Theorem 1 satisfies the $C'(1/6)$ condition, and so G is hyperbolic. (All $C'(1/6)$ groups admit linear isoperimetric functions and so are hyperbolic [10].)

Next we analyze the construction of G to show (via Lemmas 2 and 5 (iv), (v)) that H is free of rank 3 for such C, C_i, D_j and D_{ij}. Specifically, we will view G as being built from the free group $F(d_1, d_2)$ by *HNN-extensions*.

Suppose $B \leq A$ are groups and $\phi : B \to A$ is any injective homomorphism (not necessarily the subgroup inclusion map). The HNN-extension $A_{*\phi}$ of A with *defining homomorphism* ϕ and *stable letter* t is the group presented by

$$A_{*\phi} := \langle A, t \mid t^{-1}bt = \phi(b) \text{ for all } b \in B \rangle,$$

where t is a new generator. (We may instead present $A_{*\phi}$ by only including the relations $t^{-1}bt = \phi(b)$ for b in some particular generating set for B.) The groups B and $\phi(B)$ are called *associated subgroups* of the HNN-extension.

Lemma 2 (Britton's Lemma; e.g. [6, 16, 27])**.** *Suppose a non-empty word w on the alphabet $\{A \smallsetminus \{e\}\} \sqcup \{t, t^{-1}\}$ contains no two consecutive letters from $A \smallsetminus \{e\}$ and no subword tt^{-1} or $t^{-1}t$. Then $w \neq 1$ in $A_{*\phi}$ unless w contains a subword $t^{-1}bt$ where $b \in B$ or tct^{-1} where $c \in \phi(B)$.*

In particular, the natural map $A \to A_{*\phi}$ is injective, so A can be regarded as a subgroup of $A_{*\phi}$ (hence "extension"), and t generates an infinite cyclic subgroup of $A_{*\phi}$ trivially intersecting A.

We will need to recognize when a map between free groups is injective in order to show that it gives rise to an HNN-extension. To this end, we will want to be able to recognize free bases. Nielsen showed that a set of words represents a free basis for a subgroup of $F(X)$ when certain small-cancellation conditions are satisfied.

Proposition 3 (Nielsen, see [16, pages 6–7])**.** *A set U of words on an alphabet X represents a free basis for a subgroup of $F(X)$ if for every $v_1, v_2, v_3 \in U^{\pm 1}$,*

N0. $v_1 \neq e$,
N1. $v_1 v_2 \neq e \implies |v_1 v_2| \geq |v_1|, |v_2|$,
N2. $v_1 v_2 \neq e$ *and* $v_2 v_3 \neq e \implies |v_1 v_2 v_3| > |v_1| - |v_2| + |v_3|$.

Corollary 4. *C, C_1, C_2 span a rank–3 free subgroup of $F(c_1, c_2)$ and D_1, D_2, D_{11}, D_{12}, D_{21}, D_{22} span a rank–6 free subgroup of $F(d_1, d_2)$.*

(Indeed, N0–N2 are satisfied if U satisfies the $C'(1/2)$ property.)

Define

$$G_{cd} := \langle\, c_1, c_2,\, d_1, d_2 \mid c_i^{-1} d_j c_i = D_{ij},\ \ 1 \leq i, j \leq 2 \,\rangle,$$
$$G_{bcd} := \langle\, G_{cd}, b \mid b^{-1} c_i b = C_i,\ \ 1 \leq i \leq 2 \,\rangle.$$

Lemma 5. *The groups defined above have the following properties.*

(i) *$F(d_1, d_2)$ is a subgroup of G_{cd}.*
(ii) *$F(c_1, c_2)$ is also a subgroup of G_{cd} and $F(c_1, c_2) \cap F(d_1, d_2) = \{1\}$.*
(iii) *G_{bcd} is an HNN-extension of G_{cd} with stable letter b and defining homomorphism $\phi : F(c_1, c_2) \to G_{bcd}$ mapping $c_i \mapsto C_i$.*
(iv) *$H = \langle b, d_1, d_2 \rangle \leq G_{bcd}$ is free of rank 3.*
(v) *G of Theorem 1 is an HNN-extension of G_{bcd} with stable letter a:*

$$G = \langle\, G_{bcd}, a \mid a^{-1} b a = b C^{-1},\ a^{-1} d_j a = b D_j b^{-1},\ \ 1 \leq j \leq 2 \,\rangle.$$

Proof. (i) By Corollary 4, the map $\phi_1 : F(d_1, d_2) \to F(d_1, d_2)$ given by $d_j \mapsto D_{1j}$ is injective. So ϕ_1 defines an HNN-extension of $F(d_1, d_2)$. Calling the stable letter c_1, this HNN-extension has presentation

$$G_{c_1 d} := \langle\, c_1,\, d_1, d_2 \mid c_1^{-1} d_j c_1 = D_{1j},\ \ 1 \leq j \leq 2 \,\rangle.$$

By Britton's Lemma, $F(d_1, d_2) \leq G_{c_1 d}$. Similarly, G_{cd} is an HNN-extension of $G_{c_1 d}$ with stable letter c_2 and defining homomorphism $\phi_2 : F(d_1, d_2) \to G_{c_1 d}$ given by $d_j \mapsto D_{2j}$. Note that ϕ_2 has image contained in $F(d_1, d_2) \leq G_{c_1 d}$. Again, ϕ_2 is injective by Corollary 4. So $F(d_1, d_2) \leq G_{c_1 d} \leq G_{cd}$ by Britton's Lemma.

(ii) To show that $\langle c_1, c_2 \rangle$ is a free subgroup $F(c_1, c_2)$ of G_{cd} trivially intersecting $F(d_1, d_2)$, we prove the following claim. For any $n \geq 1$, any non-identity $d \in F(d_1, d_2)$, and any integers $r_0, \ldots, r_{n+1}, s_1, \ldots, s_n$:

$$(d c_1^{r_0}) c_2^{s_1} c_1^{r_1} c_2^{s_2} \cdots c_1^{r_n} c_2^{s_n} c_1^{r_{n+1}} \neq 1 \text{ in } G_{cd}$$

whenever $r_i, s_i \neq 0$ for all $1 \leq i \leq n$. This, in turn, follows from Britton's Lemma applied to the extension G_{cd} (which has stable letter c_2) once we show $d c_1^{r_0}$ and $c_1^{r_i}$ are not in the associated subgroups of G_{cd}. As these associated subgroups are $F(d_1, d_2)$ and $F(D_{21}, D_{22}) \leq F(d_1, d_2)$, the observation that $\langle c_1 \rangle$ is an infinite cyclic subgroup of $G_{c_1 d}$ trivially intersecting $F(d_1, d_2)$ by Britton's Lemma completes the proof.

(iii) By Corollary 4, $\{C_1, C_2\}$ is a free basis of a subgroup of $F(c_1, c_2) \leq G_{bcd}$. So the defining homomorphism $\phi : F(c_1, c_2) \to G_{bcd}$, $c_i \mapsto C_i$ is injective.

(iv) By Britton's Lemma applied to the HNN-extension G_{bcd} of G_{cd}, any freely reduced word w on b, d_1, d_2 representing the identity would contain a subword $b^{\pm1}ub^{\mp1}$ where u is a nonempty reduced word on d_1, d_2 representing an element of the associated subgroup $F(c_1, c_2)$ or of the associated subgroup $\phi(F(c_1, c_2)) \leq F(c_1, c_2)$. By (ii), this is impossible. So $H := \langle b, d_1, d_2\rangle$ is free of rank 3.

(v) The given presentation for G arises from that in Theorem 1 by rewriting the defining relations involving a. We must show that $\langle bC^{-1}, bD_1b^{-1}, bD_2b^{-1}\rangle \leq G_{bcd}$ is free of rank 3. It suffices to show the same of the conjugate subgroup $\langle b^{-1}C, D_1, D_2\rangle \leq G_{bcd}$. We do this by proving that if $i_1, \dots i_{r-1} \neq 0$ and $W_1, ..., W_r$ are nontrivial elements of the rank–2 free group $F(D_1, D_2) \leq F(d_1, d_2) \leq G_{bcd}$, then

$$w \;:=\; (b^{-1}C)^{i_0}W_1(b^{-1}C)^{i_1}\cdots W_r(b^{-1}C)^{i_r}$$

does not represent the identity in G_{bcd}. This is achieved by writing w in a way so that Britton's Lemma applies.

The relations $b^{-1}c_ib = C_i$ imply that $(b^{-1}C)^{i_k} \in \langle c_1, c_2\rangle\, b^{-i_k}\, \langle c_1, c_2\rangle$, so:

$$w \;\in\; \langle c_1, c_2\rangle\, b^{-i_0}\, \langle c_1, c_2\rangle\, W_1\, \langle c_1, c_2\rangle\, b^{-i_1}\, \langle c_1, c_2\rangle \;\cdots\; W_r\, \langle c_1, c_2\rangle\, b^{-i_r}\, \langle c_1, c_2\rangle.$$

If $b^{\pm1}$ does not appear in w, then $r = 1, i_0 = i_1 = 0$, and $w = W_1$ does not represent the identity in G_{bcd}. So we may assume b appears. To apply Britton's Lemma, we must show that w has no subword $b^{\pm1}Yb^{\mp1}$ where Y is a word on c_1, c_2, d_1, d_2 representing an element of $F(c_1, c_2)$. This is so because $F(c_1, c_2)\cap F(d_1, d_2) = \{1\}$ by (ii) and $W_k \in F(d_1, d_2) \leq G_{bcd}$ does not represent the identity. □

We will use the following lemma of Mitra to show the absence of a Cannon–Thurston map $\partial H \to \partial G$. We give our own account of this lemma in [1].

Lemma 6 (Mitra [21, 22])**.** *Suppose H is a hyperbolic subgroup of a hyperbolic group G and X_H and X_G are their Cayley graphs with respect to finite generating sets where that for H is a subset of that for G. (So X_H is a subgraph of X_G.) Let $M(N)$ be the infimal number such that if λ is a geodesic in X_H outside the ball of radius N about e in X_H, then every geodesic in X_G connecting the end–points of λ lies outside the ball of radius $M(N)$ about e in X_H. The Cannon–Thurston map $\partial H \to \partial G$ exists if and only if $M(N) \to \infty$ as $N \to \infty$.*

We will apply this to G and H of Theorem 1, using the generating sets a, b, c_1, c_2, d_1, d_2 and b, d_1, d_2, respectively.

The next lemma identifies some geodesics in Cayley graphs of small–cancellation groups. We learnt it from Ilya Kapovich and Hamish Short. It can be extracted from Strebel's appendix to [11] as we will explain. For a finite presentation $\langle A \mid R\rangle$, a word w on A is *Dehn–reduced* if every subword α of w that is a prefix of a cyclic conjugate of some $\rho \in R^{\pm1}$ satisfies $|\alpha| \leq |\rho|/2$, and is *strongly* Dehn–reduced if $|\alpha| \leq |\rho|/6$.

Lemma 7. *If $\langle A \mid R\rangle$ is a $C'(1/6)$–presentation, then every strongly Dehn–reduced word on A is geodesic. (Indeed, it is the unique geodesic word and also the unique Dehn–reduced word for the group element it represents.)*

Proof. Suppose u and v are freely reduced words which represent the same group element, and u is strongly Dehn–reduced and v is geodesic. In his proof of Proposition 39(i) in his

appendix to [11], Strebel explains that there is a van Kampen diagram Δ for uv^{-1} whose 2–dimensional portions are *ladder–like* disc–diagrams. (See the figure within Theorem 35.)

Suppose there is a 2–cell in Δ and that ρ is the defining relation one reads around its boundary. That 2–cell's boundary cycle is assembled from four paths: two run along the boundaries of adjacent 2–cells and have lengths less than $|\rho|/6$ (by the $C'(1/6)$ condition); one runs along u and has length at most $|\rho|/6$ by the strongly Dehn–reduced condition; but then the final path, which runs along v, has length more than $|\rho|/2$ contrary to v being a geodesic word. (Indeed, if we only required v to be Dehn–reduced we would get the same contradiction.) So Δ has no 2–cells and $u = v$ as words. □

Proof of Theorem 1. Recall that

$$G = \left\langle a, b, c_1, c_2, d_1, d_2 \;\middle|\; \begin{array}{ll} a^{-1}b^{-1}ab = C, & b^{-1}c_ib = C_i, \\ (ab)^{-1}d_j(ab) = D_j, & c_i^{-1}d_jc_i = D_{ij}, \quad 1 \le i, j \le 2 \end{array} \right\rangle$$

where

$$\begin{aligned} C &= c_1c_2c_1c_2^2c_1c_2^3\cdots c_1c_2^r, \\ C_i &= c_1c_2^{ri+1}c_1c_2^{ri+2}c_1c_2^{ri+3}\cdots c_1c_2^{ri+r}, \\ D_j &= d_1d_2^{rj+1}d_1d_2^{rj+2}d_1d_2^{rj+3}\cdots d_1d_2^{rj+r}, \\ D_{ij} &= d_1d_2^{r(il+j)+1}d_1d_2^{r(il+j)+2}d_1d_2^{r(il+j)+3}\cdots d_1d_2^{r(il+j)+r}. \end{aligned}$$

We must show that for sufficiently large r, G is hyperbolic, $H = \langle b, d_1, d_2\rangle$ is free of rank 3, and there is no Cannon–Thurston map $\partial H \to \partial G$.

As we observed at the start of this section, G can be made hyperbolic by choosing r large enough to make G satisfy $C'(1/6)$. Britton's Lemma and Lemma 5 (iv), (v) together show that H is a rank 3 free subgroup for the same r. We may assume $r > 17$. It remains to show the Cannon–Thurston map does not exist.

The longest subword of $b^{-n}a^{-n}d_1a^nb^n$ that is a prefix of a cyclic conjugate of a defining relation or the inverse of a defining relation is $a^{-1}d_1a$. Since $r > 17$, the length of $a^{-1}d_1a$ is a small fraction (less than $1/6$) of the length of the shortest of the relators. So $b^{-n}a^{-n}d_1a^nb^n$ is *strongly* Dehn–reduced. So, by Lemma 7, the path γ_n it labels, passing through the identity e as shown in Figure 1, is geodesic in the Cayley graph of G.

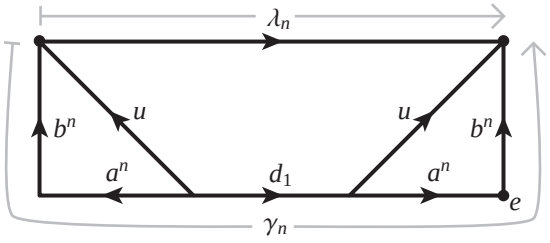

FIGURE 1. Paths in the Cayley graph of G illustrating our proof of Theorem 1.

We now wish to express $b^{-n}a^{-n}d_1a^nb^n$ as a word in d_1, d_2. To begin, we prove by induction on n that

$$ab^n = b^{n-1}ab\phi(C)\cdots\phi^{n-2}(C)\phi^{n-1}(C) \tag{1}$$

in G, where $\phi : F(c_1, c_2) \to G_{bcd}$, $c_i \mapsto C_i$ is the defining homomorphism of the HNN-extension G_{bcd} with stable letter b—see Lemma 5 (iii). The base case $n = 1$ is the equation $ab = ab$. The induction step follows from the relation $a^{-1}b^{-1}ab = C$ (which rearranges to $ab = baC$):

$$\begin{aligned} ab^{n+1} = (ab)b^n = (baC)b^n &= b(ab^n)(b^{-n}Cb^n) \\ &= b(ab^n)\phi^n(C) \\ &= b(b^{n-1}ab\phi(C)\cdots\phi^{n-2}(C))\phi^{n-1}(C))\phi^n(C), \end{aligned}$$

where the last equality uses the induction hypothesis. Left-multiplying (1) by a^{n-1} yields:

$$a^n b^n = (a^{n-1}b^{n-1})ab\phi(C)\cdots\phi^{n-2}(C))\phi^{n-1}(C) \tag{2}$$

Another induction then shows that $a^n b^n$ can be written as a positive word u in the alphabet $\{ab, c_1, c_2\}$. So $b^{-n}a^{-n}d_1a^nb^n = u^{-1}d_1u$ in G, which equals a positive word on d_1, d_2 since $(ab)^{-1}d_j(ab) = D_j$ and $c_i^{-1}d_jc_i = D_{ij}$ in G.

So the endpoints of γ_n are in H, and the geodesic λ_n joining them in the Cayley graph of H (which is a tree) is labelled by a word on d_1, d_2. The distance (along the path labelled b^n) from e to λ_n in H is n.

As the distance from γ_n to e in the Cayley graph of G is zero and the distance from λ_n to e in the Cayley graph of H is n, there is no Cannon–Thurston map $\partial H \to \partial G$ by Lemma 6. □

3. Remarks

Remark 8. The inclusion $H \hookrightarrow G$ factors through G_{bcd}, which is also hyperbolic as its presentation is also $C'(1/6)$. So Theorem 1 implies the absence of at least one Cannon–Thurston map $\partial H \to \partial G_{bcd}$ or $\partial G_{bcd} \to \partial G$. In fact, more elaborate versions of the argument given above establish that both fail to exist. As an HNN–extension is an example of a graph of groups, the latter example also shows that the quasi–isometric embedding hypothesis in Mitra's theorem from [22] is necessary.

Remark 9. With a similar construction, one can obtain a CAT(−1) group G with a free subgroup H with no Cannon-Thurston map. Wise's modification in [29] of the Rips construction [26] is used in [2] to construct CAT(−1) groups. Each relator is realized on the boundary of the unions of $n = 5$ congruent right–angled regular hyperbolic pentagons, arranged as row houses atop a geodesic segment. Each edge of the boundary corresponds to a generator. The vertices of the boundary are either right angles or straight angles, but the base geodesic gives $n - 1$ consecutive straight angles, bounding a segment of length $n - 2$. Wise shows that the Gromov link condition is satisfied when this straight segment is a freely reduced word and when the length–$(2n + 4)$ remainder of the boundary is obtained from the *Wise word*:

$$c_1(c_1c_2c_1c_3\cdots c_1c_r)c_2(c_2c_3c_2c_4\cdots c_2c_r)c_3(c_3c_4\cdots c_3c_r)\cdots c_{r-1}(c_{r-1}c_r)c_r$$

by chopping it into consecutive length $2n + 4$ segments (one for each defining relator). The argument works just as well for any n, so we take $n = 7$ and fit the $(ab)^{-1}d_j(ab)$, $a^{-1}b^{-1}ab$, $b^{-1}c_ib$, and $c_i^{-1}d_jc_i$ portions of our relators along the straight segment. We form one Wise word of c's and one of d's. To get sufficiently many length–18 subwords of the Wise words, we increase the number of c_i and d_j in the generating set for G. Then $H = \langle b, d_1, d_2, \ldots\rangle$ is a free subgroup of the CAT(−1) group G by the same argument as before.

Remark 10. H has *infinite height* in G. That is, for all n, there exist $g_1, \dots, g_n \in G$ such that $\bigcap_{i=1}^{n} g_i^{-1} H g_i$ is infinite and $Hg_i \neq Hg_j$ for all $i \neq j$. Specifically, take $g_i = c_1^i$. Then, if $\phi_1 : F(d_1, d_2) \to F(d_1, d_2)$ is the map $d_j \mapsto D_{1j}$ for $j = 1, 2$, then $\phi_1^n(F(d_1, d_2))$ is an infinite subgroup inside $g_i^{-1} H g_i$ for $1 \leq i \leq n$, and $Hg_i \neq Hg_j$ for all $i \neq j$ since $c_1^k \in H$ only for $k = 0$ by Lemma 5. Likewise, G_{bcd} has infinite height in G: instead of taking $g_i = c_1^i$, take $g_i = (ab)^i$ and apply the same argument as above. So our examples do not resolve the question attributed to Swarup in [23]: if H is a finitely presented subgroup of a hyperbolic group G and H has *finite height* in G, is H quasiconvex in G?

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