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Short communication

Stability of strongly nonlinear normal modes

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Abstract

It is shown that a transformation of time can allow the periodic solution of a strongly nonlinear oscillator to be written as a simple cosine function. This enables the stability of strongly nonlinear normal modes in multidegree of freedom systems to be investigated by standard procedures such as harmonic balance.
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1. Introduction

This work is concerned with the stability of periodic motions in multidegree of freedom conservative dynamical systems. As an example, consider the system with kinetic energy T and potential energy V, where

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2, \tag{1}$$

$$V = \frac{1}{2}(\omega^2 + x^2)y^2 + \frac{1}{4}x^4 + \frac{1}{5}x^5. \tag{2}$$

This system is governed by the following equations of motion:

$$\ddot{x} + x^3 + x^4 + xy^2 = 0, \quad \ddot{y} + \omega^2 y + x^2 y = 0. \tag{3}$$

These equations possess an invariant manifold $y = 0$, on which lies a family of nonlinear normal modes (NNMs) which satisfy the following equations:

$$\ddot{x} + x^3 + x^4 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{4}$$

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Let us suppose that the NNM which satisfies Eq. (4) is written in the form

$$x = f(t), \tag{5}$$

where $f(t)$ is a periodic function, $f(t+T) = f(t)$. In order to investigate the stability of this solution, we set

$$x = f(t) + u(t), \quad y = 0 + v(t). \tag{6}$$

Substituting Eq. (6) into (3) and linearizing in u and v, we obtain

$$\ddot{u} + (3f^2 + 4f^3)u = 0, \quad \ddot{v} + \omega^2 v + f^2 v = 0. \tag{7}$$

The first of Eq. (7) determines the stability of the motion (5) in the invariant manifold $y = 0$, that is, in the x–$\dot{x}$ phase plane. This is well-known to be Liapunov unstable due to phase shear, that is, due to the change in period associated with a change in amplitude, but is orbitally stable [1]. This effect is well understood and is of no interest to us here.

We are rather interested in the boundedness of solutions to the second of Eq. (7), the v-equation, which determines the stability of the invariant manifold $y = 0$. The NNM (5) will be said to be stable if all solutions of the v-equation are bounded, and unstable if an unbounded solution exists. The stability will be determined by two parameters, ω and A, and we ask which regions in the ω–A parameter plane correspond to stable motions, and which to unstable motions.

The stability of the v-equation may be investigated by appealing to Floquet theory [2], which states that on transition curves in the ω–A plane separating stable regions from unstable regions, there exists periodic solutions to the v-equation with period T or $2T$, where the NNM $x = f(t)$ has period T. This property may be implemented by writing v in the form of a Fourier series with period $2T$ (which includes period T motions as a special case in which the odd coefficients vanish)

$$v = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T}. \tag{8}$$

Substituting Eq. (8) into the v-equation in (7), trigonometrically simplifying and collecting terms, we obtain an infinite set of coupled linear algebraic equations on the coefficients a_n, b_n. By truncating this system and requiring the determinant to vanish, we may obtain an analytic approximation for the transition curves.

Of course in order for this standard procedure to work [3], the function $f(t)$ must be expanded in a Fourier series. Herein lies the problem, and the point of this paper. If the NNM $f(t)$ were to satisfy a simpler ODE than Eq. (4), for example $\ddot{x} + x = 0$, then $f(t)$ could be trivially written as $A\cos t$. Even if $f(t)$ were to satisfy a weakly nonlinear ODE, such as $\ddot{x} + x + x^3 = 0$, perturbation methods could be used to obtain $f(t)$ in the form of a truncated Fourier series. However, if the equation governing the NNM is strongly nonlinear, as in the case of Eq. (4), then the previously described algorithm for determining stability is stymied.

In this work, we provide a method for dealing with such situations. The idea of the method is to transform time in the original system (3) so that the NNM is given by $x = A_0 + A_1\cos 2\tau$, where τ is transformed time. Once this simplified representation of the NNM is achieved, the rest of the analysis proceeds as above.

2. Time transformation

For the purposes of clarity of presentation, we will explain the method by applying it to the foregoing example. Our goal is to transform time from t to τ so that the NNM $x = f(t)$ which satisfies Eq. (4) can be written in the simplified form $x = A_0 + A_1\cos 2\tau$. We set

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{\sqrt{g(\tau)}}, \tag{9}$$

where $g(\tau)$ is to be found. Applying the chain rule, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}\tau}\sqrt{g(\tau)}, \tag{10}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}g(\tau) + \frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{\mathrm{d}g}{\mathrm{d}\tau}, \tag{11}$$

so that Eq. (4) becomes

$$x''g + \frac{1}{2}x'g' + x^3 + x^4 = 0, \tag{12}$$

where primes represent differentiation with respect to τ. We wish to select $g(\tau)$ such that Eq. (12) has the solution

$$x(\tau) = A_0 + A_1\cos 2\tau. \tag{13}$$

Substituting the latter into Eq. (12), we obtain the following equation on $g(\tau)$:

$$-A_1\sin 2\tau\ g' - 4A_1\cos 2\tau\ g + (A_0 + A_1\cos 2\tau)^3 + (A_0 + A_1\cos 2\tau)^4 = 0. \tag{14}$$

We seek a solution to Eq. (14) in the form

$$g(\tau) = \frac{K(\tau)}{\sin^2 2\tau}. \tag{15}$$

Substituting Eq. (15) into Eq. (14), we obtain an expression for $K'(\tau)$

$$K'(\tau) = \frac{\sin 2\tau}{A_1}((A_0 + A_1\cos 2\tau)^3 + (A_0 + A_1\cos 2\tau)^4). \tag{16}$$

Eq. (16) may be easily integrated by using the substitution $u = \cos 2\tau$, giving

$$K(\tau) = \frac{1}{2A_1^2}\left(C - \frac{1}{4}(A_0 + A_1\cos 2\tau)^4 - \frac{1}{5}(A_0 + A_1\cos 2\tau)^5\right), \tag{17}$$

where C is an arbitrary constant. Combining Eqs. (17) and (15), we obtain an expression for $g(\tau)$

$$g(\tau) = \frac{1}{2A_1^2\sin^2 2\tau}\left(C - \frac{1}{4}(A_0 + A_1\cos 2\tau)^4 - \frac{1}{5}(A_0 + A_1\cos 2\tau)^5\right). \tag{18}$$

Eq. (18) may be written in a more convenient form by using the identity $\cos^2 2\tau = 1 - \sin^2 2\tau$ and collecting terms

$$g(\tau) = \frac{1}{2A_1^2\sin^2 2\tau}\left(C + q_1 + p_1\cos 2\tau + \sin^2 2\tau(q_2 + p_2\cos 2\tau) + \sin^4 2\tau(q_3 + p_3\cos 2\tau)\right), \tag{19}$$

where

$$q_1 = -A_0A_1^4 - \frac{A_1^4}{4} - 2A_0^3A_1^2 - \frac{3A_0^2A_1^2}{2} - \frac{A_0^5}{5} - \frac{A_0^4}{4}, \tag{20}$$

$$p_1 = -\frac{A_1^5}{5} - 2A_0^2A_1^3 - A_0A_1^3 - A_0^4A_1 - A_0^3A_1, \tag{21}$$

$$q_2 = 2A_0A_1^4 + \frac{A_1^4}{2} + 2A_0^3A_1^2 + \frac{3A_0^2A_1^2}{2}, \tag{22}$$

$$p_2 = \frac{2A_1^5}{5} + 2A_0^2A_1^3 + A_0A_1^3, \tag{23}$$

$$q_3 = -A_0A_1^4 - \frac{A_1^4}{4}, \tag{24}$$

$$p_3 = -\frac{A_1^5}{5}. \tag{25}$$

The method involves selecting the arbitrary constant C and the amplitudes A_0 and A_1 so that there are no singularities in $g(\tau)$ due to the vanishing of $\sin 2\tau$ in the denominator of Eq. (19). This requires that $C = -q_1$ and $p_1 = 0$. The latter condition prescribes a relationship between A_0 and A_1

$$A_1^4 + 10A_0^2A_1^2 + 5A_0A_1^2 + 5A_0^4 + 5A_0^3 = 0. \tag{26}$$

Thus we obtain the following expression for $g(\tau)$:

$$g(\tau) = \frac{1}{2A_1^2}\left(q_2 + p_2\cos 2\tau + \sin^2 2\tau(q_3 + p_3\cos 2\tau)\right) \tag{27}$$

which gives, using Eqs. (22)–(25),

$$g(\tau) = A_0A_1^2 + \frac{A_1^2}{4} + A_0^3 + \frac{3A_0^2}{4} + \frac{A_1^3\cos(2\tau)}{5} + A_0^2A_1\cos(2\tau) + \frac{A_0A_1\cos(2\tau)}{2} - \frac{A_0A_1^2\sin^2(2\tau)}{2} - \frac{A_1^2\sin^2(2\tau)}{8} - \frac{A_1^3\cos(2\tau)\sin^2(2\tau)}{10}. \tag{28}$$

This choice of $g(\tau)$ defines the time transformation via Eq. (9), $\mathrm{d}t = \mathrm{d}\tau/\sqrt{g(\tau)}$, which allows us to represent the NNM solution $x = f(t)$ of Eq. (5) in the form $x = A_0 + A_1\cos 2\tau$. The initial condition of Eq. (4) requires that

$$x(0) = A, \quad x'(0) = 0 \Rightarrow A = A_0 + A_1. \tag{29}$$

3. Discussion

The idea of this paper is that by transforming time so that the solution to a strongly nonlinear equation such as Eq. (4) is able to be written in the simplified form $x = A_0 + A_1\cos 2\tau$, the stability analysis which is based on the v-equation of Eq. (7) can be handled by standard methods. In the case of the example based on Eq. (3), the simplified v-equation becomes, transforming time from t to τ by using Eq. (11)

$$v''g + \frac{1}{2}v'g' + \omega^2 v + f^2 v = 0. \tag{30}$$

Substituting the expression for $g(\tau)$ obtained in Eq. (27) and trigonometrically simplifying, we obtain

$$h_1(\tau)v'' + h_2(\tau)v' + (\omega^2 + h_3(\tau))v = 0, \tag{31}$$

where

$$h_1(\tau) = -\frac{1}{8A_1^2}(p_3\cos 6\tau + 2q_3\cos 4\tau - p_3\cos 2\tau - 4p_2\cos 2\tau - 2q_3 - 4q_2), \tag{32}$$

$$h_2(\tau) = \frac{1}{8A_1^2}(3p_3\sin 6\tau + 4q_3\sin 4\tau - p_3\sin 2\tau - 4p_2\sin 2\tau), \tag{33}$$

$$h_3(\tau) = \frac{A_1^2\cos 4\tau}{2} + 2A_0A_1\cos 2\tau + \frac{A_1^2}{2} + A_0^2. \tag{34}$$

Eq. (31) may be described as a generalized Ince equation [4]. Since the period in time τ of the variable coefficients is $T = \pi$, Floquet theory tells us that we may look for transition curves by seeking a periodic solution of period π or 2π. Using Eq. (8), this involves writing

$$v = \sum_{n=0}^{\infty} a_n\cos n\tau + b_n\sin n\tau. \tag{35}$$

Substituting Eq. (35) into Eq. (31) and collecting terms ("the method of harmonic balance") we obtain an infinite Hill's determinant which relates ω, A_0 and A_1. We omit the details of this procedure here since it is well-known [2,3]. The resulting equation, together with the relations (26) and (29) may be used to obtain stability transition curves in the A–ω plane.

4. Conclusion

The method presented in this paper for studying the stability of a strongly nonlinear NNM applies to the general class of problems in which the NNM is given by the ODE:

$$\ddot{x} + F(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{36}$$

where $F(x)$ is an analytic function. For such an equation, the function $g(\tau)$ in the time transformation given by Eq. (9) is able to be chosen so that the periodic solutions to Eq. (36) can be written in the form of Eq. (13), $x = A_0 + A_1\cos 2\tau$.

We note that in the important special case that $F(x) = -F(-x)$, that is when $F(x)$ is an odd function, the foregoing procedure will produce periodic functions of the form $x = A\cos 2\tau$. For example if the x^4 term is omitted in Eq. (4), then the condition $p_1 = 0$, which was required to remove the singularity from $g(\tau)$ in Eq. (19), will be satisfied by taking $A_0 = 0$. This can be seen from Eq. (21) by omitting the terms of the fifth power, which come from the x^4 term in Eq. (4). Setting $A_0 = 0$ and omitting cubic terms in Eq. (28), we are left with the result that the strongly NNM defined by

$$\ddot{x} + x^3 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0 \tag{37}$$

can be written in the form $x = A\cos 2\tau$ by transforming time as follows:

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{\sqrt{g(\tau)}}, \quad g(\tau) = \frac{A^2}{4} - \frac{A^2\sin^2(2\tau)}{8}. \tag{38}$$

This result can be checked by using the elliptic integral solution of Eq. (37):

$$x = A\ \mathrm{cn}(At, k), \tag{39}$$

where the modulus $k = 1/\sqrt{2}$ [5]. Now it is well-known that the elliptic function cn can be transformed to a cosine by stretching its argument [6]:

$$\mathrm{d}v = \frac{\mathrm{d}u}{\sqrt{1 - k^2\sin^2 u}} \Rightarrow \mathrm{cn}(v, k) = \cos u \tag{40}$$

Taking $v = At$ and $u = 2\tau$ in Eq. (40) gives Eq. (38).

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