Lifting of Characters for Nonlinear Simply Laced Groups

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September 5, 2008

Abstract

One aspect of the Langlands program for linear groups is lifting of characters, which relates virtual representations on a group G with those on an endoscopic group for G. The goal of this paper is to extend this theory to nonlinear two-fold covers of real groups in the simply laced case. Suppose $\widetilde{G}$ is a two-fold cover of a real reductive group G. The main result is that there is an operation, denoted $\mathrm{Lift}_G^{\widetilde{G}}$, taking a stable virtual character of G to 0 or a virtual genuine character of $\widetilde{G}$, and $\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)$ may be explicitly computed if π is a stable sum of standard modules.

1 Introduction

The Langlands program is concerned with representation theory and automorphic forms of algebraic (linear) groups. Suppose $\widetilde{G}$ is a nonlinear group, for example the metaplectic group $Mp(2n)$, the nontrivial twofold cover of $Sp(2n)$. It is well known that such groups play an important role in automorphic forms, although they don't fit into the formalism of the Langlands

2000 Mathematics Subject Classification: Primary 22E50, Secondary 05E99

The first author was supported in part by National Science Foundation Grant #DMS-0554278.

program. The goal of this paper and of [3] is to extend some of the results of the Langlands program to certain nonlinear groups.

Let G be the real points of a connected reductive algebraic group. Characters of admissible representations of G have been studied extensively, and the theory is reasonably complete. In particular if π is a discrete series representation the character Θ_π of π is known explicitly [14]. The formula for Θ_π implicitly comes from the theory of transfer and endoscopy, originating in work of Shelstad and Langlands on the trace formula and automorphic forms. We would like to extend the theory of transfer of characters to nonlinear groups.

Suppose $\widetilde{G}$ is a nonlinear two-fold cover of G, i.e. $\widetilde{G}$ can not be realized as a subgroup of $GL(n,\mathbb{C})$. Examples include the metaplectic group $Mp(2n,\mathbb{R})$, the unique connected two-fold cover of $Sp(2n,\mathbb{R})$, and the twofold cover of $GL(n,\mathbb{R})$ of [18], [19]. There is substantial evidence that character theory for genuine representations of $\widetilde{G}$ (those which do not factor to G) may be reduced to that of a linear group.

The case of $GL(n)$ over any local field of characteristic zero has been studied by Kazhdan, Patterson and Flicker ([9], [8], [20]). The general philosophy is spelled out in [21]. Some discussion of how to extend these ideas to general groups is given in [5], and this is carried out for $G = Sp(2n,\mathbb{R})$ in [4].

Here is an outline of the approach. For now suppose G is the $\mathbb{F}$-points of a connected, reductive algebraic group defined over a local field of characteristic 0, and $\widetilde{G}$ is a nonlinear two-fold cover of G. Identify the kernel of the covering map $p : \widetilde{G} \to G$ with ± 1. We would like to relate genuine characters of $\widetilde{G}$ to characters of G. By character we mean the character of a representation, viewed as a function on the regular semisimple elements.

The theory of endoscopy for G relates characters of G to characters of *endoscopic groups* of smaller semisimple rank. A key part of the theory is a relationship between semisimple conjugacy classes in G and H.

Our theory is modelled on this, with $\widetilde{G}$ in place of G, and G playing the role of H. We define a relation on conjugacy classes as follows. For $g \in G$ define $\psi(g) = s(g)^2$ where $s(g) \in \widetilde{G}$ satisfies $p(\widetilde{g}) = g$. Then $\psi(g)$ is independent of the choice of $s(g)$, and this induces a map from conjugacy classes of G to those of $\widetilde{G}$. This idea goes back to [9].

Now suppose π is an admissible representation of G, with character Θ_π viewed as a function on the (strongly) regular semisimple elements G' of G

(see Section 3). For $\widetilde{g} \in \widetilde{G}'$ define

$$\psi^*(\Theta_\pi)(\widetilde{g}) = \sum_{\{h \in G \,|\, \psi(h) = \widetilde{g}\}} \Theta_\pi(h). \tag{1.1}$$

This is a conjugation invariant function on $\widetilde{G}'$. One can ask whether it is the character of a genuine representation $\widetilde{\pi}$ of $\widetilde{G}$. More generally it may be a virtual character, i.e. the character of a finite sum of irreducible representations with (possibly negative) integral coefficients.

Although this definition is too naive for several reasons, we describe a basic property of characters which suggest it is at least a good first approximation. Suppose H is a Cartan subgroup of G, with inverse image $\widetilde{H}$ in $\widetilde{G}$. Then $\widetilde{H}$ is not necessarily abelian, and we let $Z(\widetilde{H})$ be its center. The fact that $\widetilde{H}$ is not abelian is a major reason why the representation theory of $\widetilde{G}$ at least appears to be more complicated than that of G. However, at least as far as character theory goes, the situation is actually very simple: the character $\Theta_{\widetilde{\pi}}$ of an admissible genuine representation of $\widetilde{G}$ satisfies $\Theta_{\widetilde{\pi}}(\widetilde{g}) = 0$ for $\widetilde{g} \notin Z(\widetilde{H})$ (Lemma 9.3). It is easy to see that $\psi(H) \subset Z(\widetilde{H})$, of finite index, so at least from this point of view (1.1) is similar to the character of a genuine representation of $\widetilde{G}$.

An obvious shortcoming of(1.1) is that it is not necessarily the case that $\psi^*(\Theta_\pi)(-\widetilde{g}) = -\psi^*(\Theta_\pi)(\widetilde{g})$, an obvious requirement if $\psi^*(\Theta_\pi)$ is to be the character of a genuine representation. It would be better to define $\phi(h) = h^2$, sum over h satisfying $\phi(h) = p(\widetilde{g})$, and modify the definition by some genuine function μ of $\widetilde{G}$:

$$\phi^*(\Theta_\pi)(\widetilde{g}) = \sum_{\{h \in G \,|\, \phi(h) = p(\widetilde{g})\}} \mu(\widetilde{g})\Theta_\pi(h). \tag{1.2}$$

We then need to choose μ appropriately.

There are also some Weyl denominators to take into account. Suppose Φ^+ is a set of positive roots of H in G. For $h \in H$ one version of the Weyl denominator is $|D(h)|^{\frac{1}{2}} = |\prod_{\alpha \in \Phi^+}(1 - \alpha^{-1}(h))||e^\rho(h)|$ (cf. (5.1)). This is independent of Φ^+ and is well defined. The character of any representation has a particular form when muliplied by $|D(h)|^{\frac{1}{2}}$; for the left hand side of (1.1) to have a chance of having this form we should multiply each term on the right by the quotient of Weyl denominators $|D(h)|^{\frac{1}{2}}/|D(\widetilde{g})|^{\frac{1}{2}}$.

Putting these two considerations together we look for a function Δ on $G' \times \widetilde{G}'$ satisfying the following conditions:

$$\begin{aligned}
&\Delta(h,\widetilde{g}) = 0 \text{ unless } p(\widetilde{g}) = \phi(h)\\
&|\Delta(h,\widetilde{g})| = |D(h)|^{\frac{1}{2}}/|D(\widetilde{g})|^{\frac{1}{2}}\\
&\Delta(xhx^{-1}, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Delta(h,\widetilde{g}) \quad (\widetilde{x}\in\widetilde{G}, x = p(\widetilde{g}))\\
&\Delta(h,-\widetilde{g}) = -\Delta(h,\widetilde{g}).
\end{aligned} \tag{1.3}$$

Given such a function we define

$$\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)(\widetilde{g}) = \sum_{\{h\in G\,|\,\phi(h)=p(\widetilde{g})\}} \Delta(h,\widetilde{g})\Theta_\pi(h). \tag{1.4}$$

Now $\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)$ is a conjugation invariant, genuine function on $\widetilde{G}$, and is a reasonable candidate for the character of a virtual representation. The latter condition amounts to further conditions on Δ. If this holds, one can *define* $\mathrm{Lift}_G^{\widetilde{G}}(\pi)$ to be the virtual representation of $\widetilde{G}$ whose character is equal to $\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)$: $\Theta_{\mathrm{Lift}_G^{\widetilde{G}}(\pi)} = \mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)$.

For a version of this for covering groups of $GL(n)$ see [20].

Formula (1.3) is analogous with the main character identity in the theory of endoscopy. For example see [25]. In that setting Δ is a *transfer factor*, and by analogy we use the same terminology here. As in the theory of endoscopy, correctly defining transfer factors is a difficult part of the theory.

We now discuss another less obvious consideration which arises. Recall Θ_π is conjugation invariant: $\Theta_\pi(g) = \Theta_\pi(g')$ if $g, g' \in G'$ and $g' = xgx^{-1}$ for some $x \in G$. Following Langlands and Shelstad we say π and Θ_π are *stable* if the following stronger condition holds: $\Theta_\pi(g) = \Theta_\pi(g')$ if $g, g' \in G'$ and $g' = xgx^{-1}$ for some $x \in G(\overline{\mathbb{F}})$.

For $GL(n)$ every conjugation invariant function is automatically stable, so stability plays no role in [8], [9] and [20].

For guidance we consider the case of $G = Sp(2n,\mathbb{R})$, $\widetilde{G} = Mp(2n,\mathbb{R})$, as discussed in [4]. Let $G' = SO(n+1,n)$. In this case the map ϕ above is replaced by a bijection between (strongly) regular semisimple conjugacy classes in G and G'. Suppose π is a *stable* representation of G'. The main result of [4] is that, for appropriate definition of the transfer factor Δ, if we define

$$\mathrm{Lift}_{G'}^{\widetilde{G}}(\Theta_\pi)(\widetilde{g}) = \Delta(h,\widetilde{g})\Theta_\pi(h) \quad (\phi(h) = p(\widetilde{g})) \tag{1.5}$$

then $\mathrm{Lift}_{G'}^{\widetilde{G}}(\Theta_\pi)$ is the character of a genuine virtual representation of $\widetilde{G}$.

This suggests we should relate characters of $\widetilde{G}$ with *stable* characters of real forms of the *dual group* $G^\vee(\mathbb{C})$. We expect the general theory to have this form. In this paper we restrict ourselves to simply laced groups. This avoids a number of technical complications, and the distinction between $G(\mathbb{C})$ and $G^\vee(\mathbb{C})$ is less critical. We may now state a special case of the main result.

Theorem 1.6 (Theorem 19.1 and Corollary 19.11) *Suppose* $G(\mathbb{C})$ *is a connected, reductive, simply laced complex group, with real points* G*. We assume the derived group of* $G(\mathbb{C})$ *is acceptable. Suppose* $\widetilde{G}$ *is a admissible cover of* G *(cf. Section 2 and Definition 3.3). Then we can define the transfer factor*$\Delta(h,\widetilde{g})$*, satisfying* (1.3)*, such that for all stable admissible representations* π *of* G*,*

$$\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi)(\widetilde{g}) = \sum_{\{h\in G\,|\,\phi(h)=p(\widetilde{g})\}} \Delta(h,\widetilde{g})\Theta_\pi(h) \tag{1.7}$$

is the character of a genuine virtual representation $\widetilde{\pi}$ *of* $\widetilde{G}$*, or* 0*. We say* $\widetilde{\pi}$ *is the lift of* π*, and write* $\widetilde{\pi} = \mathrm{Lift}_G^{\widetilde{G}}(\pi)$*.*

If π *is a stable sum of standard modules we compute* $\mathrm{Lift}_G^{\widetilde{G}}(\pi)$ *explicitly.*

We note that unlike the case of $Mp(2n,\mathbb{R})$, the role of stability is very subtle in the simply laced case. It appears in the proof of Theorem 14.1; see Remark 14.32.

Theorem 1.6 is formally similar to transfer in the setting of endoscopic groups. For example see [25, Lemma 4.2.4]. More precisely $\mathrm{Lift}_G^{\widetilde{G}}$ is analogous to the simplest case of endoscopy: transfer from the quasisplit form G_{qs} of G to G [24]. As we will see below it is often possible to obtain a single irreducible representation of $\widetilde{G}$ as a lift, and there is no natural notion of stability for $\widetilde{G}$. This suggests that in the simply laced case this is the only notion of lifting which is needed. We note that there *is* a useful notion of stability for $Mp(2n,R)$ [4], and David Renard has defined a family of "endoscopic groups" for $Mp(2n,\mathbb{R})$ and proved lifting results for them [23]. We believe this is the only situation in which this is either necessary or possible. We plan to return to the two root length case in another paper.

Example 1.8 Let $G = SL(2,\mathbb{R})$ and let $\widetilde{G}$ be the unique non-trivial two folder cover of G. Let B be a Borel subgroup of G and write $B = AN$ with

$A \simeq \mathbb{R}^\times$. For $\delta = \pm 1$ and $\nu \in \mathbb{C}$ define a character χ of $\mathbb{R}^\times$ by $\chi(-1) = \delta$ and $\chi(x) = x^\nu$ for $x \in \mathbb{R}^+$. Let $\pi(\delta, \nu)$ be the corresponding principal series representation of G, i.e. $\mathrm{Ind}_{AN}^G(\chi \otimes 1)$ (normalized induction).

Now let $\widetilde{A}$ be the inverse image of A in $\widetilde{G}$. Then $\widetilde{A} \simeq \mathbb{R}^\times \cup i\mathbb{R}^\times$. We may identify the inverse image of N with N, and let $\widetilde{B} = \widetilde{A}N$. For $\epsilon \in \pm 1$ and $\gamma \in \mathbb{C}$ define $\widetilde{\chi}(i) = \epsilon i$ and $\widetilde{\chi}(x) = x^\gamma$ for $x \in \mathbb{R}^+$. Let $\widetilde{\pi}(\epsilon, \gamma)$ be the genuine principal series representation $\mathrm{Ind}_{\widetilde{B}}^{\widetilde{G}}(\widetilde{\chi} \otimes 1)$ of $\widetilde{G}$.

Let $\pi = \pi(\delta, \nu)$. It is easy to see that $\mathrm{Lift}_G^{\widetilde{G}}(\Theta_\pi) = 0$ if $\delta = -1$. If $\delta = 1$ then by an easy calculation (see Section 11), using the transfer factor of Section 5 we see

$$\mathrm{Lift}_G^{\widetilde{G}}(\pi(1, \nu)) = \widetilde{\pi}(1, \nu/2) \oplus \widetilde{\pi}(-1, \nu/2). \tag{1.9}$$

Note that the image of ϕ restricted to A is A^0, and the character of this Lift is 0 on $p^{-1}(-A^0) = i\mathbb{R}^\times$.

This example illustrates several features. In this example computing $\mathrm{Lift}_G^{\widetilde{G}}(\pi)$ essentially reduces to computing $\mathrm{Lift}_A^{\widetilde{A}}(\chi)$. This is a special case of the general situation.

Note that $\widetilde{\pi}(\pm 1, \nu/2)$ have different central characters (the center of $\widetilde{G}$ is $\widetilde{Z(G)} \simeq \mathbb{Z}/4\mathbb{Z}$), so $\mathrm{Lift}_G^{\widetilde{G}}(\pi(1, \nu))$ does not have a central character. Also note that this character is supported on $\widetilde{A^0}$, which is a proper subgroup of $Z(\widetilde{A})$.

A hint that Theorem 1.6 is not the best result possible is seen by considering the preceding example from another point of view. View $SL(2, \mathbb{R})$ as $Sp(2, \mathbb{R})$, and apply [4]. In this case $G' = SO(2, 1)$. There are two principal series representations $\pi(\pm 1, \nu)$ of $SO(2, 1)$ analogous to those described for $SL(2, \mathbb{R})$. Writing $\mathcal{L}ift$ for the lifting of [4] we see

$$\mathcal{L}ift_{SO(2,1)}^{Mp(2,\mathbb{R})}(\pi(\pm 1, \nu)) = \widetilde{\pi}(\pm 1, \nu). \tag{1.10}$$

See [4, Proposition 15.10]. Thus we may obtain each principal series representation $\widetilde{\pi}(\pm 1, \nu)$ of $\widetilde{SL}(2, \mathbb{R})$, rather than just their sum as in (1.9).

Note that $SO(2, 1)$ is isomorphic to $PSL(2, \mathbb{R})$, the real points of $PSL(2, \mathbb{C})$ (also denoted $PGL(2, \mathbb{R})$). This is the real form of the adjoint group, and suggests it should be possible to generalize lifting of (1.5) to allow G to be replaced with a real form of a quotient of $G(\mathbb{C})$ (without changing $\widetilde{G}$). We revisit the previous example from this point of view.

Example 1.11 Let $G(\mathbb{C}) = SL(2, \mathbb{C})$ and

$$\overline{G}(\mathbb{C}) = PSL(2, \mathbb{C}) = SL(2, \mathbb{C}) / \pm I.$$

Let $G = SL(2, \mathbb{R})$ and let $\overline{G} = PSL(2, \mathbb{R})$ be the real points of $\overline{G}(\mathbb{C})$. Recall $\overline{G} \simeq PGL(2, \mathbb{R}) \simeq SO(2, 1)$.

We define an orbit correspondence between $\overline{G}$ and G as follows. For $g \in \overline{G}$ choose an inverse image $s(g)$ of g in $G(\mathbb{C})$, and let $\phi(g) = s(g)^2$. This is independent of the choice of $s(g)$, and $\phi(g) \in SL(2, \mathbb{R})$.

Let $A(\mathbb{C}) = \{\mathrm{diag}(z, 1/z)\} \subset G(\mathbb{C})$, and let $\overline{A}(\mathbb{C})$ be its image in $\overline{G}(\mathbb{C})$. Then $A(\mathbb{C})$ and $\overline{A}(\mathbb{C})$ are defined over $\mathbb{R}$, and let $A = A(\mathbb{R}), \overline{A} = \overline{A}(\mathbb{R})$. Both A and $\overline{A}$ are isomorphic to $\mathbb{R}^\times$. Note that while the map $A(\mathbb{C}) \to \overline{A}(\mathbb{C})$ is surjective, the restriction $A(\mathbb{R}) \to \overline{A}(\mathbb{R})$ is not.

Write $\overline{\mathrm{diag}}(z, \frac{1}{z})$ for the image of $\mathrm{diag}(z, \frac{1}{z})$ in $\overline{A}(\mathbb{C})$. Let $g = \overline{\mathrm{diag}}(x, \frac{1}{x}) \subset \overline{A}$ with $x \in \mathbb{R}^\times$. Then $\phi(g) = \mathrm{diag}(x^2, \frac{1}{x^2}) \in A^0$. However suppose $y \in \mathbb{R}^\times$ and let $g = \overline{\mathrm{diag}}(iy, \frac{1}{iy}) \in \overline{A}$. Then $\phi(g) = \mathrm{diag}(-y^2, -\frac{1}{y^2})$. Therefore (unlike in Example 1.8) ϕ maps $\overline{A}$ *onto* A.

This suggests that if we develop a similar lifting theory from $\overline{G}$ to $\widetilde{G}$, then the lift of a principal series representation will have support on all of $\widetilde{A}$. In fact this is the case: we can define $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}$. We recover (1.10) from this point of view (the difference between ν and $\nu/2$ on the right hand side is an issue of normalization):

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\pi}(\pm 1, \nu)) = \widetilde{\pi}(\pm 1, \nu/2). \tag{1.12}$$

Motivated by this example, we generalize (1.5) as follows. Let $G(\mathbb{C})$ be our given complex group, with real points G and nonlinear cover $\widetilde{G}$. Suppose $C \subset Z(G)$ is a two group and let $\overline{G}(\mathbb{C}) = G(\mathbb{C})/C$, with real points $\overline{G}$. For $g \in \overline{G}$ define $\phi(g) = s(g)^2$ where $s(g)$ is an inverse image of g in $G(\mathbb{C})$. The fact that C is a two-group implies $\phi(g) \in G$, and this is independent of the choice of $s(g)$. It is not necessarily the case that ϕ induces a map on conjugacy classes; however it does define a map on *stable* conjugacy classes, which is consistent with our application.

We need to make several technical assumptions on C (Definition 3.14); $C = 1$ is allowed. Under these assumptions we can define the transfer factor $\Delta(h, \widetilde{g})$ on $\overline{G}' \times \widetilde{G}$ and define the lift $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\pi)$ of a stable representation of $\overline{G}$ as in (1.5). See Definition 9.10.

Now suppose H is a Cartan subgroup of G, with inverse image $\widetilde{H}$ in $\widetilde{G}$ and corresponding subgroup $\overline{H} \subset \overline{G}$. Our assumptions on C imply $\phi(\overline{H}) \subset p(Z(\widetilde{H}))$, which is necessary to have a meaningful theory. As in the preceding examples, if this image is large then $\text{Lift}_{\overline{G}}^{\widetilde{G}}(\pi)$ will have fewer terms in its sum. This is desirable; it would be nice to have only one term in the sum. Taking C larger makes this image larger, so we would like to take C as large as possible. An optimal choice would be to choose C so that $\phi(\overline{H}) = p(Z(\widetilde{H}))$ for all Cartan subgroups, in which case each Lift would consist of a single term. This is not possible in general. See Example 3.16.

The proofs follow the same general outline as those of [4]. A hard part of the theory is the definition of transfer factors. Once transfer factors and lifting are defined we first show that the lift of a stable invariant eigendistribution is an invariant eigendistribution. This requires checking that lifting respects the Hirai matching conditions, and it is here that stability plays a crucial role. Once this is done it is fairly easy to compute the lift of a stable sum of discrete series representations. It is also straightforward to prove that lifting commutes with parabolic induction. This enables us to compute the lift of any stable sum of standard modules. In principal, giving the Kazhdan-Lusztig-Vogan polynomials for $\overline{G}$ and $\widetilde{G}$, we may then compute the Lift of any stable virtual representation of $\overline{G}$.

Here is an outline of the contents of the paper.

In Section 2 we make some basic definitions and establish some notation. Admissible triples $(\widetilde{G}, G, \overline{G})$ are defined in Section 3. Such a set consists of a nonlinear group $\widetilde{G}$, a linear group G, and the real form $\overline{G}$ of a quotient of $G(\mathbb{C})$, satisfying certain assumptions. Section 3 also contains a discussion of some basic structural facts about Cartan subgroups, and defines the orbit correspondence ϕ. Section 4 recalls some standard facts about Cartan subgroups and Cayley transforms, and generalizes them to nonlinear groups. Cayley transforms are an important tool in the theory.

The basic, and most important case, that of the real points of a semisimple, simply connected complex group, is discussed in Section 5. The transfer factors are canonical in this case, have a simple form, and this case provides guidance for the general theory. We recommmend the reader restrict to this case the first time through.

Sections 6 and 7 are technically the most difficult. Section 6 defines certain characters of Cartan subgroups and their application to transfer factors.

Transfer factors are defined in Section 7. Some constants associated to Cartan subgroups are defined and studied in Section 8.

With transfer factors in place we define lifting and study its basic properties in Section 9. The case of tori, which is both a good example and an important special case, is discussed in Section 10. Minimal principal series of split groups are covered in Section 11, and discrete series on the compact Cartan in Section 12. Section 13 summarizes some material about invariant eigendistributions which we will need. Some details of an extension of Hirai's results which we need are in the Appendix (Section 20); this is work of the second author.

In Section 14 we use the results of Section 13 to prove that the lift of a stable invariant eigendistribution in an invariant eigendistribution.

We prove that lifting commutes with parabolic induction in Section 15. *Modified character data* appropriate to our setting is defined in Section 16. This is a (mild) modification of character data and the Langlands classification due to Vogan [26]. Formal lifting of (modified) character data is defined in Section 17. This essentially comes down to lifting applied to a Cartan subgroup.

We compute the lifting of stable discrete series representations in Section 18, and of general standard modules in Section 19.

These results are closely related to *duality* of representations, also known as Vogan duality, introduced in [27]. Duality for nonlinear covers of simply laced groups is discussed in a paper by Peter Trapa and the first author [3]. These two results grew up together, and we thank Peter Trapa for many helpful discussions.

2 Some Notation

A simple root system is said to be *simply laced* if all roots have the same length, and an arbitrary root system is simply laced if this holds for each simple factor. More succinctly a root system is simply laced if whenever α, β are non-proportional roots then $\langle \alpha, \beta^\vee \rangle = 0, \pm 1$. We adopt the convention that in this case all roots are long.

We say a root system is *oddly laced* if whenever α, β are non-proportional roots then $\langle \alpha, \beta^\vee \rangle = 0$ or is *odd*. Thus *oddly laced* is shorthand for *each simple factor is simply laced or of type G_2*. We also adopt the convention that in type G_2 all roots are long. The reason for these conventions is Lemma 3.2.

The main results in this paper hold for oddly laced groups, although (with the general case in mind) we will only make this assumption when necessary.

Suppose $G(\mathbb{C})$ is a connected, reductive complex Lie group. Let $G_{\mathrm{ad}}(\mathbb{C})$ be the adjoint group. If $G(\mathbb{C})$ is defined over $\mathbb{R}$ let G be its real points, and let G_{ad} be the real points of $G_{\mathrm{ad}}(\mathbb{C})$. Write $\mathrm{int}(g)$ for the action of $G_{\mathrm{ad}}(\mathbb{C})$ on $G(\mathbb{C})$, or G_{ad} on G.

Let $G_d(\mathbb{C})$ be the derived group of $G(\mathbb{C})$, and let G_d be the derived group of G. Note that by (2.8)(b) G_d is the identity component of the real points of $G_d(\mathbb{C})$.

We denote real Lie algebras by Gothic letters $\mathfrak{h}, \mathfrak{g}, \mathfrak{t}, \dots$, and their complexifications by $\mathfrak{h}(\mathbb{C}), \mathfrak{g}(\mathbb{C}), \mathfrak{t}(\mathbb{C}), \dots$. We write σ for the action of the nontrivial element of the Galois group on $\mathfrak{g}(\mathbb{C})$ and $G(\mathbb{C})$, so $\mathfrak{g} = \mathfrak{g}(\mathbb{C})^\sigma$ and $G(\mathbb{R}) = G(\mathbb{C})^\sigma$.

Fix a Cartan involution θ of $G(\mathbb{C})$, i.e. $K = G^\theta$ is a maximal compact subgroup of G. Unless otherwise noted all Cartan subgroups of G or $G(\mathbb{C})$ are assumed to be θ-stable. Let H be a (θ-stable) Cartan subgroup of G, with complexification $H(\mathbb{C})$. Write $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ as usual, and $H = TA$ with $T = H \cap K$ and $A = \exp(\mathfrak{a})$. Let $\Phi = \Phi(G, H)$ be the root system of $H(\mathbb{C})$ in $G(\mathbb{C})$. Let $W(G, H) = \mathrm{Norm}_G(H)/H$; this is the *real* Weyl group. It is a subgroup of the absolute Weyl group $W = W(G(\mathbb{C}), H(\mathbb{C}))$ which is isomorphic to the Weyl group of Φ. The Cartan involution θ acts on W, and $W(G, H) \subset W^\theta$.

Note $\sigma(\alpha) = -\theta(\alpha)$ for all $\alpha \in \Phi$. Roots are classified as real, imaginary, complex, or compact as in [26]. Write $\Phi = \Phi_r \cup \Phi_i \cup \Phi_{cx}$ accordingly; note that Φ_r and Φ_i are root systems. We also have the decomposition of $\Phi_i = \Phi_{i,c} \cup \Phi_{i,n}$ into compact and noncompact roots; $\Phi_{i,c}$ is a root system. If Φ^+ is a set of positive roots, write $\Phi_r^+ = \Phi^+ \cap \Phi_r$, and Φ_i^+, Φ_{cx}^+ similarly. Let $\rho = \rho(\Phi^+) = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ as usual, and define ρ_r, ρ_i and ρ_{cx} similarly, so $\rho = \rho_r + \rho_i + \rho_{cx}$.

We say $G(\mathbb{C})$ is *acceptable* if ρ exponentiates to a character of $H(\mathbb{C})$.

An important role is played by certain sets of positive roots:

Definition 2.1 *A set Φ^+ of positive roots is said to be special if $\sigma(\Phi^+_{cx}) = \Phi^+_{cx}$, or equivalently $\sigma(\alpha) > 0$ for all positive non-imaginary roots.*

Let

(2.2) $$\Gamma(H) = \exp(i\mathfrak{a}) \cap H = \{\exp(iX) \mid X \in \mathfrak{a}, \exp(2iX) = 1\}.$$

An important role is played by a certain character of $\Gamma(H)$. Let S be a set of complex roots such that the set of all complex roots is $\{\pm\alpha, \pm\sigma\alpha \,|\, \alpha \in S\}$. Define

$$\zeta_{\mathrm{cx}}(G,H)(h) = \prod_S e^{\alpha}(h) \quad (h \in \Gamma(H)). \tag{2.3}$$

If G, H are understood we let $\zeta_{\mathrm{cx}} = \zeta_{\mathrm{cx}}(G,H)$. It is elementary to see that ζ_{cx} is independent of the choice of S, factors to G_{ad}, and for all $h \in \Gamma(H)$ satisfies

(2.4)(a) $$\zeta_{\mathrm{cx}}(h) = \zeta_{\mathrm{cx}}(wh) \quad (w \in W(G,H))$$

(2.4)(b) $$\zeta_{\mathrm{cx}}(h) = e^{\rho_{cx}}(h) = e^{\rho-\rho_r}(h)$$

for any special set of positive roots. For (b) note that if Φ^+ is a special set of positive roots and $S \subset \Phi^+_{cx}$ is as above, then for $X \in \mathfrak{a}(\mathbb{C})$

$$\rho_{cx}(X) = (\rho - \rho_r)(X) = \sum_{\alpha \in S} \alpha(X), \tag{2.5}$$

and it follows that $e^{\rho_{cx}}(\exp(X)) = e^{\rho-\rho_r}(\exp(X)) = e^{\rho_{cx}(X)}$ is a well-defined character of $A(\mathbb{C})$.

For $\alpha \in \Phi_r$ let $m_\alpha = \alpha^\vee(-1) = \exp(\pi i \alpha^\vee)$ and define

$$\Gamma_r(H) = \langle m_\alpha \,|\, \alpha \in \Phi_r \rangle \subset \Gamma(H) \cap G^0_d. \tag{2.6}$$

It is well known that

$$H = \Gamma(H)H^0, \quad H \cap G^0 = \Gamma_r(H)H^0. \tag{2.7}$$

It is also well known that if H_s is a maximally split Cartan subgroup of G then

(2.8)(a) $$G = H_s G^0,$$

and this implies

(2.8)(b) $$G_d = G^0_d.$$

3 Admissible Triples

Fix G as in Section 2 and suppose $p : \widetilde{G} \to G$ is a two-fold cover. We identify the kernel of p with ± 1.

If H is a subgroup of G we let $\widetilde{H} = p^{-1}(H)$. Let $Z(\widetilde{H})$ be the center of $\widetilde{H}$ and let $Z_0(H) = p(Z(\widetilde{H}))$. It is immediate that $Z(\widetilde{H}) = p^{-1}(Z_0(H))$, so to describe $Z(\widetilde{H})$ it is enough to describe $Z_0(H)$. In particular $Z_0(G) \subset Z(G)$ plays an important role; it is immediate that $Z_0(G) = Z(G)$ if G is connected (cf. (4.4)).

Suppose H is a Cartan subgroup of G. Typically $\widetilde{H}$ is not abelian and $Z_0(H)$ plays an important role. It is easy to see

$$H^0 \subset Z_0(H) \subset H. \tag{3.1}$$

Fix a real or imaginary root α. Associated to α is the root subgroup M_α, which is locally isomorphic to $SL(2,\mathbb{R})$ or $SU(2)$. As in [3, Definition 3.2] we say α is *metaplectic* if $p^{-1}(M_\alpha)$ is a nonlinear group. If M_α is compact it has no such cover, nor does $SL(2,\mathbb{R})/\pm 1$, so if α is metaplectic then $M_\alpha \simeq SL(2,\mathbb{R})$ and α is either real or noncompact imaginary.

Let $\widetilde{m}_\alpha$ be an inverse image of m_α in $\widetilde{G}$. It is easy to see that $\widetilde{m}_\alpha$ has order 4 if α is metaplectic, and 1 or 2 otherwise. For the next Lemma see [6] or [3, Lemma 3.3].

Lemma 3.2 *Assume $G(\mathbb{C})$ is simple and simply connected and that $\widetilde{G}$ is nonlinear. Fix a Cartan subgroup H of G. Then a real or imaginary root α of $H(\mathbb{C})$ in $G(\mathbb{C})$ is metaplectic if and only if α is long and is real or noncompact imaginary. Furthermore G admits a nonlinear cover if and only if there is a Cartan subgroup H with a long real or long noncompact imaginary root. If this condition holds the nonlinear two-fold cover is unique up to isomorphism.*

It is enough to check this condition on a fundamental or maximally split Cartan subgroup. If G is oddly laced it is enough to check this condition on any Cartan subgroup.

Definition 3.3 *We say that $p : \widetilde{G} \to G$ is an admissible two-fold cover if for every θ-stable Cartan subgroup H, every long real or long noncompact imaginary root is metaplectic.*

Equivalently $\widetilde{G}$ is admissible if and only if $\widetilde{G_i}$ is nonlinear for every simple factor G_i of G which admits such a cover.

See [3, Definition 3.4]).

Example 3.4 Let $G = U(1,1)$. This has three non-trivial two-fold covers, described by their restriction to $T \simeq U(1)$, the diagonal maximal compact subgroup. The cover of T is a connected torus $\widetilde{T}$, and is best described by its character lattice. Write the character lattice $X^*(T)$ as $\mathbb{Z}^2$ in the usual coordinates. The three covers are given by $X^*(\widetilde{T}) = \frac{1}{2}\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$, or $\mathbb{Z}^2 \cup (\mathbb{Z} + \frac{1}{2})^2$.

The derived group is $SU(1,1) \simeq SL(2,\mathbb{R})$, which has a unique non-trivial two-fold cover. The first two covers of $U(1,1)$ restrict non-trivially to $SU(1,1)$, and are therefore admissible. The third one is trivial when restricted to $SU(1,1)$ (it is the $\sqrt{\det}$ cover), and is not admissible.

As discussed in the Introduction we are going to lift characters from a real form of a quotient of $G(\mathbb{C})$. We need to impose some conditions on this quotient. This will take up the remainder of this section.

Suppose C is a finite subgroup of $Z(G)$. Then $\overline{G}(\mathbb{C}) = G(\mathbb{C})/C$ is defined over $\mathbb{R}$, and let $\overline{G}$ be its real points. Write $\overline{p} : G(\mathbb{C}) \to \overline{G}(\mathbb{C})$ for the projection map. Let $\mathrm{Orb}(G)$ be the conjugacy classes of G. If $g, g' \in G$ are conjugate by $G(\mathbb{C})$ we say g, g' are stably conjugate, and let $\mathrm{Orb}^{st}(G)$ be the set of stable conjugacy classes. (This is a naive definition, which agrees with the usual one for strongly regular semisimple elements.) Similar notation applies to other groups.

For $g \in G$ let $\mathcal{O}(G,g)$ be the conjugacy class of g, and $\mathcal{O}^{st}(G,g) = \{xgx^{-1} \mid x \in G(\mathbb{C}), xgx^{-1} \in G\}$ the stable conjugacy class. Similar notation applies to other groups.

Definition 3.5 *Assume C is a two-group. For $h \in \overline{G}$ let $\phi(h) = s(h)^2$ where $s : \overline{G}(\mathbb{C}) \to G(\mathbb{C})$ is any section.*

Lemma 3.6 *The map ϕ is well defined, and satisfies:*

1. *$\phi(\overline{G}) \subset G$ and $\phi(\overline{G}^0) \subset G^0$,*
2. *ϕ induces a map $Orb(\overline{G}^0) \to Orb(G^0)$,*
3. *ϕ induces a map $Orb^{st}(\overline{G}) \to Orb^{st}(G)$.*

Proof. Since C is a two-group it is immediate that $\phi(h)$ is independent of the choice of s.

Suppose $h \in \overline{G}, g \in G(\mathbb{C})$, and $\overline{p}(g) = h$. Then $\overline{p}(\sigma(g)) = \sigma(\overline{p}(g)) = \overline{p}(g)$, so $\sigma(g) = zg$ for some $z \in C$. Since C is a two-group $\sigma(g)^2 = g^2$, so $\phi(h) = g^2 \in G$. Furthermore since $\overline{p} : G^0 \to \overline{G}^0$ is surjective the second assertion in (1) is clear.

Define

$$\phi(\mathcal{O}(\overline{G}^0, g)) = \mathcal{O}(G^0, \phi(g)) \quad (g \in \overline{G}^0). \tag{3.7}$$

If $x \in \overline{G}^0$ choose $y \in G^0$ with $\overline{p}(y) = x$. Then $\phi(xgx^{-1}) = y\phi(g)y^{-1}$ so this is well defined, proving (2). Similarly define

$$\phi(\mathcal{O}^{\text{st}}(\overline{G}, g)) = \mathcal{O}^{\text{st}}(G, \phi(g)) \quad (g \in \overline{G}). \tag{3.8}$$

Suppose $x \in \overline{G}(\mathbb{C})$ and $xgx^{-1} \in \overline{G}$. Choose $y \in G(\mathbb{C})$ with $\overline{p}(y) = x$. Then $\phi(xgx^{-1}) = y\phi(g)y^{-1}$, so again ϕ is well-defined. □

It is worth noting that ϕ does *not* define a map $\text{Orb}(\overline{G}) \to \text{Orb}(G)$. Suppose we try to define $\phi(\mathcal{O}(\overline{G}, g)) = \mathcal{O}(G, \phi(g))$ as in (3.7). For this to be well defined we need to know that for $x \in \overline{G}$, $\phi(xgx^{-1})$ is G-conjugate to $\phi(g)$. This is true if $x \in \overline{p}(G)$, but not in general. (By (3) $\phi(g)$ is *stably* conjugate to $\phi(xgx^{-1})$).

Example 3.9 Let $G = SL(2,\mathbb{R})$ and $\overline{G} = PSL(2,\mathbb{R}) \simeq SO(2,1)$ (cf. Example 1.11). Let $t(\theta) = \text{diag}(t'(\theta), 1) \in SO(2,1)$ (with the appropriate form) where $t'(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in SL(2,\mathbb{R})$. Then just as in Example 1.11 $\phi(t(\theta)) = t'(\theta)$. However while $t(\theta)$ is conjugate to $t(-\theta)$, $t'(\theta)$ is not conjugate to $t'(-\theta)$ (for generic θ). Therefore the map $\phi(\mathcal{O}(SO(2,1), t(\theta)) = \mathcal{O}(SL(2,\mathbb{R}), t'(\theta))$ is not well defined.

We assume throughout this paper that C is a two-group (and therefore finite).

Now fix a Cartan subgroup H of G, and let $\overline{H}(\mathbb{C}) = H(\mathbb{C})/C \subset \overline{G}(\mathbb{C})$, with real points $\overline{H}$. We define $\phi : \overline{H}(\mathbb{C}) \to H(\mathbb{C})$ by the same formula as in Definition 3.5. Equivalently, if we write $\exp : \mathfrak{h}(\mathbb{C}) \to H(\mathbb{C})$, $\overline{\exp} : \mathfrak{h}(\mathbb{C}) \to \overline{H}(\mathbb{C})$, then for $X \in \mathfrak{h}(\mathbb{C})$, $\phi(\overline{\exp}(X)) = \exp(2X)$. It is immediate that

$$\alpha(\phi(h)) = \alpha(h)^2 \quad (\text{for all } \alpha \in \Phi). \tag{3.10}$$

Lemma 3.11 *The homomorphism $\phi : \overline{H}(\mathbb{C}) \to H(\mathbb{C})$ has the following properties.*

1. $\phi(wh) = w\phi(h)$ *for all* $w \in W(\Phi), h \in \overline{H}(\mathbb{C})$,
2. $\phi(\overline{H}^0) = H^0$,
3. $\phi(h) = 1$ *for all* $h \in \Gamma_r(\overline{H})$,
4. $\phi(\overline{H} \cap \overline{G}^0) = H^0$,
5. $\phi(\Gamma(\overline{H})) = \Gamma(H) \cap C$,
6. $\phi(\overline{H}) = (\Gamma(H) \cap C)H^0$.

Proof. The first two are clear from the definition. For (3) note that $\phi(\overline{\exp}(\pi i\alpha^\vee)) = \exp(2\pi i\alpha^\vee) = 1$ for all $\alpha \in \Phi_r$. Then (4) follows since $\overline{H} \cap \overline{G}^0 = \Gamma_r(\overline{H})\overline{H}^0$ (cf. 2.7).

Let $h = \overline{\exp}(iX) \in \Gamma(\overline{H})$ where $X \in \mathfrak{a}$ with $\overline{\exp}(2iX) = 1$. Now $\phi(h) = \exp(2iX) \in \exp(i\mathfrak{a})$. But $\overline{p}(\phi(h)) = \overline{\exp}(2iX) = 1$ so that $\phi(h) \in C \cap \exp(i\mathfrak{a}) \subset \Gamma(H) \cap C$. Conversely, let $z_G = \exp(iY) \in C \cap \Gamma(H)$ where $Y \in \mathfrak{a}$ with $\exp(2iY) = 1$. Since $z_G \in C$, $\overline{p}(z_G) = \overline{\exp}(iY) = 1$. Let $z = \overline{\exp}(iY/2)$. Then $z \in \Gamma(\overline{H})$ and $\phi(z) = z_G$. Finally (6) follows since $\overline{H} = \Gamma(\overline{H})\overline{H}^0$. $\square$

It is a standard fact that the character of an irreducible genuine representation of $\widetilde{G}$, considered as a function on the regular semisimple elements, vanishes off of $Z(\widetilde{H})$ (Lemma 9.3). Our character identities involve the image of ϕ in H, so we would like to know that $\phi(\overline{H}) \subset Z_0(H)$, with image as large as possible.

Recall $C \subset Z(G)$. We first show that we may as well assume $C \subset Z_0(G)$.

Let H_s be a maximally split Cartan subgroup of G, and assume $\phi(\overline{H}_s) \subset Z_0(H_s)$. Let $C_s = \Gamma(H_s) \cap C$ so that by Lemma 3.11 $C_s H_s^0 = \phi(\overline{H}_s) \subset Z_0(H_s)$. Therefore $\widetilde{C_s}$ centralizes $\widetilde{H_s}$, and by (4.3) $\widetilde{C_s}$ centralizes $\widetilde{G^0}$. Therefore $\widetilde{C_s}$ centralizes $\widetilde{H_s}\widetilde{G^0}$ which equals $\widetilde{G}$ by (2.8)(a). Therefore $C_s \subset Z_0(G)$.

Now for a general Cartan subgroup $H = TA$, there is a maximally split Cartan $H_s = T_sA_s$ so that $A \subset A_s$. Then $\Gamma(H) \subset \Gamma(H_s)$, so that $\phi(\overline{H}) = (C \cap \Gamma(H))H^0 = (C_s \cap \Gamma(H))H^0$. That is, if we use C_s in place of C we have the same images $\phi(\overline{H})$ for every Cartan subgroup H. Thus we may as well assume that $C \subset Z_0(G)$.

Lemma 3.12 *Assume that $C \subset Z_0(G)$. Then for every Cartan subgroup $\phi(\Gamma(\overline{H})) \subset Z_0(G)$ and $\phi(\overline{H}) \subset Z_0(H)$. Furthermore $\phi(Z(\overline{G})) \subset Z_0(G)$.*

Proof. The first part follows from the above discussion. Let H_s be a maximally split Cartan subgroup with roots Φ. Let $z \in Z(\overline{G}) \subset \overline{H}_s, \widetilde{z} \in Z(\widetilde{H}_s)$ such that $p(\widetilde{z}) = \phi(z)$. Then for every $\alpha \in \Phi$,

$$\alpha(\widetilde{z}) = \alpha(\phi(z)) = \alpha(z)^2 = 1$$

by (3.10). Thus $\widetilde{z} \in \mathrm{Cent}_{\widetilde{G}}(\widetilde{G}^0) \cap Z(\widetilde{H}_s) = Z(\widetilde{G})$ as above. □

To reiterate, so far we have assumed C is a (finite) subgroup of $Z_0(G)$, and $z^2 = 1$ for all $z \in C$. For the definition of transfer factors we need to impose a further technical condition on C.

Suppose $\widetilde{\chi}$ is a genuine character of $Z(\widetilde{G})$. Then $\widetilde{\chi}^2$ factors to a character of $Z_0(G)$. Suppose $z \in Z_0(G)$ has order 2 and $p(\widetilde{z}) = z$. Then $\widetilde{z}$ has order 2 or 4 and $\widetilde{\chi}^2(z) = 1$ or -1 accordingly, independent of $\widetilde{\chi}$. Assume that $G_d(\mathbb{C})$ is acceptable.

Let

$$\zeta_2(z) = \widetilde{\chi}^2(z)e^{\rho}(z) \quad z \in Z_0(G) \cap G_d, z^2 = 1. \tag{3.13}$$

This is well defined since G_d is acceptable, and is independent of the choices of $\widetilde{\chi}$, H and Φ^+.

Definition 3.14 *An admissible triple is a set $(\widetilde{G}, G, \overline{G})$ where:*

1. *G is the set of real points of a connected reductive complex Lie group $G(\mathbb{C})$ such that the derived group $G_d(\mathbb{C})$ is acceptable, with oddly laced root system i.e. each simple factor is simply laced or G_2 (cf. Section 2).*

2. *$p : \widetilde{G} \to G$ is an admissible two-fold cover (Definition 3.3).*

3. *$\overline{G}$ is the set of real points of $\overline{G}(\mathbb{C}) = G(\mathbb{C})/C$ where C is a finite subgroup of $Z(G)$ satisfying the following conditions:*

   (a) $c^2 = 1$ for all $c \in C$,

   (b) $C \subset Z_0(G) = p(Z(\widetilde{G}))$,

   (c) $\zeta_2(z) = 1$ for all $z \in C \cap G_d$ (cf. 3.13).

For example if $\widetilde{G}$ and G satisfy (1) and (2) then $(\widetilde{G}, G, G)$ is an admissible triple.

More generally, suppose $\widetilde{G}$ and G satisfy conditions (1) and (2). Then $(\widetilde{G}, G, \overline{G})$ is an admissible triple for $\overline{G}(\mathbb{C}) = G(\mathbb{C})/C$ for any subgroup C satisfying conditions (3)(a-c). We may take $C = 1$; by Lemma 3.11(6) taking C bigger makes $\phi(\overline{H}) \subset Z_0(H)$ bigger, and in general, we get the best lifting results when we take C as large as possible. However, even if we take C as large as possible for G, it may not be the maximal choice for Levi subgroups of G. Thus we do not specify C beyond the requirements of Definition 3.14.

Assume that $(\widetilde{G}, G, \overline{G})$ is an admissible triple. Let $H = TA$ be a Cartan subgroup of G. Let $M = \mathrm{Cent}_G(A)$ be a cuspidal Levi subgroup. Then $\widetilde{M} \subset \widetilde{G}$ and $\overline{M} = \mathrm{Cent}_{\overline{G}}(\overline{A}) \subset \overline{G}$ are also cuspidal Levi subgroups. We will show in Section 15 that $(\widetilde{M}, M, \overline{M})$ is also an admissible triple. Thus our constructions are compatible with parabolic induction.

The following lemma, which is an immediate consequence of (6.21) is useful for producing admissible triples.

Lemma 3.15 *Let H be a maximally split Cartan subgroup of G. Then $\zeta_2(c) = 1$ for all $c \in Z_0(G) \cap \Gamma_r(H)$.*

Lemma 3.16 *Suppose that $G(\mathbb{C})$ is simple and simply connected, and let H be a maximally split Cartan subgroup of G. Then we can choose C so that $\phi(\overline{H}) = Z_0(H)$ except when $G = SU(n,n)$ where n is even, $G = Spin(p,q)$ where p and q are even with $p > q \geq 2$ or when $G = Spin^*(2n)$ where $n \equiv 0$ mod 4.*

Proof.

Clearly if $Z_0(H)$ is connected we can take $C = \{1\}$. We will prove in (4.4) and Proposition 4.7 that $Z_0(H) = Z_0(G)H^0 = Z(G)H^0$ since G is connected. But since $H = \Gamma_r(H)H^0$ where $\Gamma_r(H)$ is a two group, it is easy to see that $Z(G)H^0 = Z_e(G)H^0$ where $Z_e(G)$ is the group of elements in $Z(G)$ with even order. Thus $Z_0(H)$ is connected when $Z_e(G) = \{1\}$. This will be the case for all real forms when Φ is of type A_{2n}, E_6, E_8, or G_2. Thus we may as well assume that Φ is type A_{2n-1}, D_n, or E_7. In these cases $Z_e(G) = Z(G)$. We may also assume that G is a real form such that $Z_0(H) = Z(G)H^0$ is not connected. In particular, we can assume that G is not compact or complex. Suppose that $Z(G) \subset \Gamma_r(H)$. Then $Z(G)$ is a two-group, and by Lemma 3.15 we can take $C = Z(G)$, giving us $\phi(\overline{H}) = Z(G)H^0 = Z_0(H)$ as desired. This will always be the case if G is the split real form.

Suppose that Φ is of type A_{2n-1}. Then G is split or H is connected unless $G = SU(n,n)$. In this case $Z_0(H) = H$ has two components. Now $Z(G) =< z >$ is cyclic of order $2n$ and $z_0 = z^n \in \Gamma_r(H)$ is the unique element of order two. Thus the biggest C we can take is $C = \{1, z_0\}$. When n is odd, $z_0 \notin H^0$ so that $\phi(\overline{H}) = CH^0 = H$, but when n is even, $z_0 \in H^0$ so that $\phi(\overline{H}) = CH^0 = H^0$ is a proper subgroup of $Z_0(H)$.

Suppose that Φ is of type D_n. Let $G = Spin(p,q)$ with $p \geq q$. Then G is split or H is connected unless $p > q \geq 2$. If p, q are odd, then $Z(G) = Z_2 \subset \Gamma_r(H)$. If p, q are even, then $Z(G) = \{1, z_0, z_1, z_0 z_1\}$ where $z_0 \in \Gamma_r(H)$, but $z_1 \notin \Gamma(H)$. Thus the biggest C we can take is $C = \{1, z_0\}$. But $z_1 \notin CH^0$ so that CH^0 is a proper subgroup of $Z_0(H)$.

Let $G = Spin^*(2n)$. If n is odd H is connected, so assume $n = 2k$. Then $Z(G) = \{1, z_0, z_1, z_0 z_1\}$ as above where $z_1 \in \Gamma_r(H)$, but $z_0 \notin \Gamma(H)$ so the biggest C we can take is $C = \{1, z_1\}$. If k is even, then $z_0 \notin CH^0$ while $z_0 \in CH^0$ when k is odd.

Finally, if $\Phi = E_7$, then either G is split or $Z_0(H)$ is connected. □

Example 3.17 Let $H = TA$ be a maximally split Cartan subgroup of G. By Lemma 3.16 it is not always possible to pick C satisfying the conditions of Definition 3.14 such that $\phi(\overline{H}) = Z_0(H)$. Let $M = \mathrm{Cent}_G(A)$. Here we show it is possible to pick a finite central subgroup C_M of M satisfying the conditions of Definition 3.14 for M such that $\phi(\overline{H}) = Z_0(H)$ where ϕ is defined using $\overline{M}(\mathbb{C}) = M(\mathbb{C})/C_M$.

Recall $H = \Gamma(H)H^0$, so $\widetilde{H} = \widetilde{\Gamma(H)}\widetilde{H}^0$ and $Z(\widetilde{H}) = Z(\widetilde{\Gamma(H)})\widetilde{H}^0$. Then $Z_0(H) = Z_0(\Gamma(H))H^0$, and since $\Gamma(H)$ is a two-group there exists $C_M \subset Z_0(\Gamma(H))$ such that $C_M \cap H^0 = \{1\}$ and $Z_0(H) = C_M H^0$. We have to show C_M satisfies the conditions of Definition 3.14(3). Condition (a) is obvious, and since $M = HM^0 = \Gamma(H)H^0M^0$ it follows easily that $Z_0(\Gamma(H)) \subset Z_0(M)$, which is (b). Finally $C_M \cap M_d \subset C_M \cap H^0 = \{1\}$, giving (c).

4 Cartan Subgroups and Cayley Transforms

We need some structural facts about Cartan subgroups. We first define commutators on G with respect to an admissible cover $\widetilde{G}$.

Assume for the moment that $G_d(\mathbb{C})$ is simply connected and τ is an automorphism of G. Then τ stabilizes $G_d = G_d^0$ (cf. (2.8)(b)) and by Lemma 3.2 it lifts uniquely to an automorphism $\widetilde{\tau}$ of the (unique) admissible cover

$\widetilde{G_d}$ of G_d. Suppose $g \in G_d$ and $\tau(g) = g$. Choose an inverse image $\widetilde{g}$ of g and set

$$(4.1) \qquad \{\tau, g\} = \widetilde{\tau}(\widetilde{g})\widetilde{g}^{-1}.$$

Then $p(\{\tau, g\}) = 1$ so $\{\tau, g\} = \pm 1$. It is independent of the choice of $\widetilde{g}$.

We now drop the assumption that $G_d(\mathbb{C})$ is simply connected, but suppose that $\widetilde{G}$ is an admissible cover of G. Suppose g, h are commuting elements of G. Choose inverse images $\widetilde{g}, \widetilde{h}$ of g, h in $\widetilde{G}$ and define

$$(4.2) \qquad \{g, h\} = \widetilde{g}\widetilde{h}\widetilde{g}^{-1}\widetilde{h}^{-1}.$$

Since $p(\widetilde{g}\widetilde{h}\widetilde{g}^{-1}\widetilde{h}^{-1}) = ghg^{-1}h^{-1} = 1$, $\{g, h\} = \pm 1$, independent of the choices. We say $\{g, h\}$ is the commutator of g, h (with respect to the cover $\widetilde{G}$). It is continuous in each factor so

$$(4.3) \qquad \{Z(G), G^0\} = 1.$$

Note that this implies

$$(4.4) \qquad Z(\widetilde{G}) = \widetilde{Z(G)} \quad \text{if } G \text{ is connected.}$$

In general we only have $Z(\widetilde{G}) \subset \widetilde{Z(G)}$, for example if G is a torus and $\widetilde{G}$ is nonabelian.

Suppose $\alpha \in \Phi(G, H)$ is a long real root. Let M_α be as in Section 3. By Lemma 3.2 α is metaplectic, and by the discussion at the beginning of Section 3, $M_\alpha \simeq SL(2, \mathbb{R})$. Let τ be an automorphism of G_d satisfying $\tau(H \cap G_d) = H \cap G_d$ and $\tau(\alpha) = \alpha$. Then $\tau(M_\alpha) = M_\alpha$, so for $g \in M_\alpha$ define $\{\tau, g\}$ by (4.1) applied to M_α . Let $T_\alpha \simeq S^1$ be a τ-stable compact Cartan subgroup of M_α. Then for all $t \in T_\alpha$, $\tau(t) = t$ or t^{-1}. Recall $m_\alpha = \alpha^\vee(-1) \in Z(M_\alpha)$.

For the next result we do not need to assume G is oddly laced.

Proposition 4.5 *Suppose α is a long real root. Then*

$$(4.6)(a) \qquad \{\tau, m_\alpha\} = \begin{cases} 1 & \tau(t) = t \quad \textit{for all } t \in T_\alpha \\ -1 & \tau(t) = t^{-1} \quad \textit{for all } t \in T_\alpha \end{cases}$$

For all $h \in H$,

$$\text{(4.6)(b)} \qquad \{h, m_\alpha\} = sgn(\alpha(h)).$$

If $\beta \in \Phi_r(G,H)$ then

$$\text{(4.6)(c)} \qquad \{m_\alpha, m_\beta\} = (-1)^{\langle \alpha, \beta^\vee \rangle}.$$

Proof. We can compute $\tau(m_\alpha)$ by working in T_α. If τ acts trivially on T_α then the same holds for the action of $\widetilde{\tau}$ on $\widetilde{T_\alpha}$ and $\{\tau, m_\alpha\} = 1$. Suppose $\tau(t) = t^{-1}$ for all $t \in T_\alpha$. Then $\widetilde{\tau}(\widetilde{t}) = \widetilde{t}^{-1}$ for all $\widetilde{t} \in \widetilde{T}_\alpha$. Let $\widetilde{m}_\alpha$ be an inverse image of m_α in $\widetilde{T}_\alpha$. Then $\{\tau, m_\alpha\} = \widetilde{m}_\alpha^{-2}$. Since α is metaplectic $\widetilde{m}_\alpha$ has order 4 (see Section 3) and $\widetilde{m}_\alpha^{-2} = -1$. This gives (a).

It is a standard fact (essentially a calculation in $GL(2,\mathbb{R})$) that for all $h \in H, t \in T_\alpha$, $hth^{-1} = t^{\mathrm{sgn}(\alpha(h))}$. This proves (b), and (c) follows from this and the identity $\alpha(m_\beta) = (-1)^{\langle \alpha, \beta^\vee \rangle}$. □

The following result is of fundamental importance, and does not hold in the two root length case (for example for $Sp(2n,\mathbb{R})$).

Proposition 4.7 *Assume G is oddly laced. Then*

$$\text{(4.8)} \qquad Z(\widetilde{H}) = Z(\widetilde{G})\widetilde{H^0}$$

We first prove

Lemma 4.9 *In the setting of the Proposition we have*

$$\text{(4.10)} \qquad Z(\widetilde{H}) \subset \widetilde{Z(G)}\widetilde{H}^0.$$

Proof. It is enough to show if $h \in H$ and $\{h, m_\alpha\} = 1$ for all real roots α then $h \in Z(G)H^0$. (This is where the oddly laced condition appears: otherwise we only have this identity for the long real roots.) By (4.6)(b) it is enough to show

$$\text{(4.11)} \qquad \alpha(h) > 0 \text{ for all } \alpha \in \Phi_r(G,H) \text{ implies } h \in Z(G)H^0$$

This is a straightforward calculation using roots and weights. Choose a basis of the root lattice of the form

$$\alpha_1, \dots, \alpha_m, \alpha_{m+1}, \dots, \alpha_{m+n}, \alpha_{m+n+1}, \dots \alpha_{m+n+2r}$$

where α_i is real for $1 \le i \le m$, imaginary for $m+1 \le i \le m+n$, and $\theta\alpha_{m+n+2i-1} = -\alpha_{m+n+2i}$ for all $1 \le i \le r$. It is a basic fact about lattices that such a basis exists. Let $\lambda_1^\vee, \dots, \lambda_{m+n+2r}^\vee$ be the dual basis of coweights.

For each i choose $x_i \in \mathbb{C}$ so that $e^{x_i} = \alpha_i(h)$. If $i \le m$ we may assume $x_i \in \mathbb{R}$ since $\alpha_i(h) > 0$. We may also assume $x_{m+n+2i} = \overline{x}_{m+n+2i-1}$ for all $1 \le i \le r$. Let $X = \sum_i x_i\lambda_i^\vee$, $h_1 = \exp(X)$. It follows easily that $X \in \mathfrak{h}$, $h_1 \in H^0$, and $\alpha(h) = \alpha(h_1)$ for all roots α. Let $z = hh_1^{-1}$; this is contained in G, and also $Z(G(\mathbb{C}))$, since $\alpha(z) = 1$ for all roots α. Therefore $z \in Z(G)$, and $h = zh_1 \in Z(G)H^0$. □

Proof of the Proposition. The statement is equivalent to

$$Z_0(H) = Z_0(G)H^0. \tag{4.12}$$

It is enough to show $Z_0(H) \subset Z_0(G)H^0$ (the reverse inclusion is obvious). We first prove this assuming H is a maximally split Cartan subgroup. Suppose $h \in Z_0(H)$. By the Lemma $h = zy$ with $z \in Z(G), y \in H^0$. Then $y \in Z_0(H)$, so $z = hy^{-1} \in Z_0(H)$. It is enough to show $z \in Z_0(G)$. We have $\{z, g\} = 1$ for all $g \in H$, and also for $g \in G^0$ by (4.3). Since H is maximally split $G = HG^0$, so $z \in Z_0(G)$.

The general case will be proved after Lemma 4.25, using Cayley transforms. □

Suppose H is a Cartan subgroup and α is a real or noncompact imaginary root. We define the Cayley transform H_α of H with respect to α as in [22, §11.15]; also see [26, Proposition 8.3.4 and 8.3.8]. Then H_α has a noncompact imaginary or real root β, and the Cayley transform of H_α with respect to β is H.

In order to emphasize the symmetry of the situation we change notation and let $H_\alpha = T_\alpha A_\alpha$ be a Cartan subgroup with a real root α. Then we let $H_\beta = T_\beta A_\beta$ be its Cayley transform, with noncompact imaginary root β.

Define $Z_\alpha \in \mathfrak{p}$ and $Z_\beta \in \mathfrak{k}$ as in [26, Proposition 8.3.4 and 8.3.8], and let $B_\alpha = \exp(\mathbb{R}Z_\alpha) \simeq \mathbb{R}^+ \subset H_\alpha$, and $B_\beta = \exp(\mathbb{R}Z_\beta) \simeq S^1 \subset H_\beta$. It follows easily from [26, 8.3.4 and 8.3.13] that we have

(4.13)(a) $\qquad (H_\alpha \cap H_\beta)B_\alpha \subset H_\alpha \quad$ index 1 or 2

(4.13)(b) $\qquad (H_\alpha \cap H_\beta)B_\beta = H_\beta.$

The root α takes positive real values on $(H_\alpha \cap H_\beta)B_\alpha$. We say α is type I if inclusion (a) is an equality. Otherwise α is of type II, in which case H_α is generated by the left hand side and an element t satisfying $\alpha(t) = -1$.

We say β is type I if $s_\beta \notin W(G, H_\beta)$, and type II otherwise. Then α, β are both of type I or both of type II.

There is an element $g \in G_{ad}(\mathbb{C})$ so that if $c = \mathrm{Ad}(g)$, $c^* = \mathrm{Ad}^*(g)$ then

(4.14)(a) $\quad c$ centralizes $\mathfrak{h}_\alpha(\mathbb{C}) \cap \mathfrak{h}_\beta(\mathbb{C})$

(4.14)(b) $\quad c(\mathfrak{h}_\beta(\mathbb{C})) = \mathfrak{h}_\alpha(\mathbb{C})$

(4.14)(c) $\quad c^*(\alpha) = \beta.$

Note that any c satisfying (a) and (b) satisfies $c^*(\alpha) = \pm\beta$.

Definition 4.15 *Suppose χ_α is a character of H_α, and χ_β is a character of H_β. We say χ_α is a Cayley transform of χ_β, and vice-versa, if*

(4.16)(a) $\quad \chi_\alpha(h) = \chi_\beta(h) \quad (h \in H_\alpha \cap H_\beta)$

(4.16)(b) $\quad Ad^*(c)(d\chi_\alpha) = d\chi_\beta.$

where $c \in G_{ad}(\mathbb{C})$ satisfies (4.14)*(a-c).*

Here is a convenient alternative characterization of Cayley transforms, which does not refer to the element c. The proof is elementary.

Lemma 4.17 *In the setting of the Definition, χ_α is a Cayley transform of χ_β, and vice versa, if and only if* (4.16)(a) *holds, and*

$$\langle d\chi_\alpha, \alpha^\vee \rangle = \langle d\chi_\beta, \beta^\vee \rangle. \tag{4.18}$$

Remark 4.19 In the definition of Cayley transform we are allowed to replace α and β with their negatives. This does not affect the Cartan subgroups. Suppose χ_β is the Cayley transform of χ_α with respect to $\{\alpha, \beta\}$. Then the Cayley transform of χ_α with respect to $\{\alpha, -\beta\}$ is $s_\beta(\chi_\beta)$, as is clear from the preceding Lemma. Similar commments apply to α.

Lemma 4.20 *Fix α and β.*
(1) Fix a character χ_α of H_α. There is a Cayley transform χ_β of χ_α if and only if

$$\langle d\chi_\alpha, \alpha^\vee \rangle \in \mathbb{Z} \tag{4.21}$$

and

$$\chi_\alpha(m_\alpha) = (-1)^{\langle d\chi_\alpha, \alpha^\vee\rangle}. \tag{4.22}$$

Assume these hold. Then the Cayley transform of χ_α is given by (4.16)(a) *and*

$$\chi_\beta(\exp(xZ_\beta)) = e^{i\langle d\chi, \alpha^\vee\rangle x}. \tag{4.23}$$

(2) Fix a character χ_β of H_β. There are one or two choices of Cayley transform χ_α of χ_β, given as follows. Define χ_α restricted to $(H_\alpha \cap H_\beta)B_\alpha$ by (4.16)(a)*, and*

$$\chi_\alpha(\exp(xZ_\alpha)) = e^{\langle d\chi_\beta, \beta^\vee\rangle x}. \tag{4.24}$$

If β is of type I this defines the character χ_α of H_α. If β is of type II define $\chi_\alpha^\pm$ to be the two extensions of χ_α to H_α.

Sketch of proof. This is elementary from the identities [26, 8.3.4 and 8.3.13]. See [26, Lemma 8.3.7 and 8.3.15], where the setting is a regular character and the construction differs from this one by a ρ-shift. For this reason the proof of the Lemma is in fact much easier than those in [26]. □

We now consider Cayley transforms for $\widetilde{G}$ in the oddly laced case. The analogue of (4.13) is:

Lemma 4.25 *Assume G is oddly laced and $\widetilde{G}$ is an admissible cover of G. Then*

$$(Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta))\widetilde{B_\beta} = Z(\widetilde{H}_\beta) \tag{4.26)(a}$$

$$(Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta))\widetilde{B_\alpha} = Z(\widetilde{H}_\alpha) \tag{4.26)(b}$$

Proof. We first prove (a). We have to show

$$(Z_0(H_\alpha) \cap Z_0(H_\beta))B_\beta = Z_0(H_\beta). \tag{4.27}$$

Since B_β is connected, $B_\beta \subset H_\beta^0 \subset Z_0(H_\beta)$, and the inclusion $\subset$ is clear. For the opposite inclusion suppose $g \in Z_0(H_\beta)$. By (4.13) write $g = hb$ with $h \in H_\alpha \cap H_\beta, b \in B_\beta$. Then $h = gb^{-1} \in Z_0(H_\beta)$ since both g and b are in $Z_0(H_\beta)$. It is enough to show $h \in Z_0(H_\alpha)$.

Note that $\{h,x\} = 1$ for $x \in H_\alpha \cap H_\beta$ (since $h \in Z_0(H_\beta)$). Since B_α is connected $B_\alpha \subset H_\alpha^0 \subset Z_0(H_\alpha)$, and therefore $\{h,x\} = 1$ for $x \in B_\alpha$. Therefore $\{h,x\} = 1$ for $x \in (H_\alpha \cap H_\beta)B_\alpha$. If α is type I the right hand side is H_α and we are done. Otherwise choose t satisfying $\alpha(t) = -1$, so that H_α is generated by $(H_\alpha \cap H_\beta)B_\alpha$ and t.

It is enough to show $\{h,t\} = 1$. If this is not the case replace h with hm_α and b with bm_α (note that $m_\alpha \in H_\alpha \cap H_\beta$ and B_β). By (4.6)(b) $\{m_\alpha, t\} = -1$, so $\{hm_\alpha, t\} = 1$.

For (b) the inclusion $\subset$ is immediate since B_α is connected. On the other hand suppose $g \in H_\alpha$ but $g \notin (H_\alpha \cap H_\beta)B_\alpha$. Then $\alpha(g) < 0$, and by (4.6)(b) $\{m_\alpha, g\} = -1$, so $g \notin Z_0(H_\alpha)$. This proves the reverse inclusion. □

We may now complete the proof of Proposition 4.7.

Proof. We have already shown this for the most split Cartan subgroup. By repeated use of the Cayley transform it is enough to show that in the previous setting

$$Z(\widetilde{H}_\alpha) = Z(\widetilde{G})\widetilde{H}_\alpha^0 \Rightarrow Z(\widetilde{H}_\beta) = Z(\widetilde{G})\widetilde{H}_\beta^0 \tag{4.28}$$

By the Lemma we have

$$\begin{aligned} Z(\widetilde{H}_\beta) &= (Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta))\widetilde{B_\beta} \\ &= (Z(\widetilde{G})\widetilde{H}_\alpha^0 \cap Z(\widetilde{H}_\beta))\widetilde{B_\beta} \\ &= Z(\widetilde{G})(\widetilde{H}_\alpha^0 \cap Z(\widetilde{H}_\beta))\widetilde{B_\beta} \\ &= Z(\widetilde{G})\widetilde{H}_\beta^0 \end{aligned} \tag{4.29}$$

For the last equality we have used that $(H_\alpha^0 \cap H_\beta)B_\beta \subset H_\beta^0$, which implies $(\widetilde{H}_\alpha^0 \cap Z(\widetilde{H}_\beta))\widetilde{B_\beta} \subset \widetilde{H}_\beta^0$. □

We now define Cayley transforms for genuine characters as in Definition 4.15, using $Z(\widetilde{H}_\alpha), Z(\widetilde{H}_\beta)$, and Lemma 4.25 in place of (4.13).

Definition 4.30 *Assume G is oddly laced and $\widetilde{G}$ is an admissible cover of G. Fix $\alpha, \beta, \widetilde{H}_\alpha$ and $\widetilde{H}_\beta$. Suppose $\widetilde{\chi}_\alpha$ is a genuine character of $Z(\widetilde{H}_\alpha)$, and $\widetilde{\chi}_\beta$ is a genuine character of $Z(\widetilde{H}_\beta)$. We say $\widetilde{\chi}_\alpha$ is a Cayley transform of $\widetilde{\chi}_\beta$, and vice-versa, if*

(4.31)(a) $$\widetilde{\chi}_\alpha(h) = \widetilde{\chi}_\beta(h) \quad (h \in Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta))$$

(4.31)(b) $$Ad^*(c)(d\widetilde{\chi}_\alpha) = d\widetilde{\chi}_\beta.$$

where $c \in G_{ad}(\mathbb{C})$ satisfies (4.14)*(a-c).*

Unlike in the linear case this defines $\widetilde{\chi}_\beta$ uniquely, since (4.26)(a) is an equality, rather than the containment of 4.13(a)).

The analogue of Lemma 4.17 follows easily from Proposition 4.7.

Lemma 4.32 *In the setting of the Definition $\widetilde{\chi}_\alpha$ and $\widetilde{\chi}_\beta$ are each others Cayley transforms if and only if*

$$d\widetilde{\chi}_\alpha(X) = d\widetilde{\chi}_\beta(X) \quad (X \in \mathfrak{h}_\alpha \cap \mathfrak{h}_\beta), \tag{4.33)(a}$$

$$\widetilde{\chi}_\alpha(z) = \widetilde{\chi}_\beta(z) \quad (z \in Z(\widetilde{G})) \tag{4.33)(b}$$

$$\langle d\widetilde{\chi}_\alpha, \alpha^\vee\rangle = \langle d\widetilde{\chi}_\beta, \beta^\vee\rangle. \tag{4.33)(c}$$

The analogue of Lemma 4.20 is:

Lemma 4.34 *Suppose we are in the setting of Definition 4.30.*
(1) Suppose $\widetilde{\chi}_\alpha$ is a genuine character of $Z(\widetilde{H}_\alpha)$. If α is type II, then the Cayley transform $\widetilde{\chi}_\beta$ of $\widetilde{\chi}_\alpha$ exists if and only if

$$\langle d\widetilde{\chi}_\alpha, \alpha^\vee\rangle \in \mathbb{Z} + \frac{1}{2}. \tag{4.35)(a}$$

Assume this holds. Then $\widetilde{\chi}_\beta$ is given by (4.31)(a) *and*

$$\widetilde{\chi}_\beta(\widetilde{\exp}(xZ_\beta)) = e^{\langle d\widetilde{\chi}_\alpha, \alpha^\vee\rangle ix}. \tag{4.35)(b}$$

If α is type I let $\widetilde{m}_\beta = \widetilde{\exp}(\pi Z_\beta) \in Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta)$. Note that $\widetilde{\chi}_\beta(\widetilde{m}_\beta) = \pm i$. Then the Cayley transform $\widetilde{\chi}_\beta$ of $\widetilde{\chi}_\alpha$ exists if and only if (4.35)(a) and

$$e^{i\pi\langle d\widetilde{\chi}_\alpha, \alpha^\vee\rangle} = \widetilde{\chi}(\widetilde{m}_\beta). \tag{4.35)(c}$$

In this case it is given by (4.31)(a) *and* (4.35)(b).
(2) Suppose $\widetilde{\chi}_\beta$ is a genuine character of $Z(\widetilde{H}_\beta)$. Then there is a unique Cayley transform $\widetilde{\chi}_\alpha$ of $\widetilde{\chi}_\beta$ given by (4.31)(a) *and*

$$\widetilde{\chi}_\alpha(\widetilde{\exp}(xZ_\alpha)) = e^{\langle d\widetilde{\chi}_\beta, \beta^\vee\rangle x}. \tag{4.35)(d}$$

Proof. For (1) it is immediate from the definition that $\widetilde{\chi}_\beta$ is defined by (4.31)(a) and (4.35)(b). If α is type II then $Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta) \cap \widetilde{B_\beta} = 1$, and there is no further condition. If α is type I then $\widetilde{m}_\beta \in Z(\widetilde{H}_\alpha) \cap Z(\widetilde{H}_\beta) \cap \widetilde{B_\beta}$, which gives condition (4.35)(c). Case (2) is similar, and easier. $\square$

5 Transfer Factors: Special Case

Let $(\widetilde{G}, G, \overline{G})$ be an admissible triple as in Definition 3.3. The basic idea of transfer factors is simple when $G = \overline{G}$ is connected and semisimple (for example take $G(\mathbb{C}) = \overline{G}(\mathbb{C})$ semisimple and simply connected). We discuss this case, before turning to the general case in the next two sections. We start with some general notation before specializing to the case at hand.

Suppose $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, with corresponding Cartan subgroups $\widetilde{H}, H$ and $\overline{H}$ of $\widetilde{G}, G$ and $\overline{G}$, respectively. Let $\widetilde{H}', H', \overline{H}'$ be the regular elements of these groups.

Let Φ^+ be a set of positive roots of H in G. For $h \in H'$ define

(5.1)(a) $$\Delta^0(h, \Phi^+) = \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}(h))$$

(5.1)(b) $$\epsilon_r(h, \Phi^+) = sign \prod_{\alpha \in \Phi_r^+} (1 - e^{-\alpha}(h))$$

(5.1)(c) $$\Delta^1(h, \Phi^+) = \epsilon_r(h, \Phi^+)\Delta^0(h, \Phi^+)$$

Let $D(h)$ be the coefficient of $t^{\text{rank G}}$ in $\text{Det}(t + 1 - \text{Ad}(h))$, and let n be the number of positive roots. It is easy to see that

(5.1)(d) $$(-1)^n D(h) = \Delta^0(h)^2 e^{2\rho}(h)$$

(5.1)(e) $$|D(h)|^{\frac{1}{2}} = |\Delta^0(h)||e^{\rho}(h)|$$

where the final term is shorthand for $|e^{2\rho}(h)|^{\frac{1}{2}}$. It is evident these expressions are independent of the choice of Φ^+.

If $G(\mathbb{C})$ is acceptable define

(5.1)(f) $$\Delta(h, \Phi^+) = e^{\rho}(h)\Delta^0(\Phi^+, h),$$

in which case

(5.1)(g) $$|D(h)|^{\frac{1}{2}} = |\Delta(h, \Phi^+)|,$$

independent of the choice of Φ^+.

The same definitions apply to $\widetilde{G}$ (by pulling back from G) and to $\overline{G}$.

Definition 5.2 *Let*

(5.3)(a) $$X(\overline{G}, \widetilde{G}) = \{(h, \widetilde{g}) \in \overline{G} \times \widetilde{G} \mid \phi(h) = p(\widetilde{g})\}.$$

and

$$(5.3)(b) \qquad X(\overline{H},\widetilde{H}) = X(\overline{G},\widetilde{G}) \cap \overline{H}\times\widetilde{H}.$$

For $\widetilde{g}\in\widetilde{G}$ *let*

$$(5.3)(c) \qquad X(\overline{G},\widetilde{g}) = \{h\in\overline{G}\,|\,\phi(h)=p(\widetilde{g})\} = \{h\in\overline{G}\,|\,(h,\widetilde{g})\in X(\overline{G},\widetilde{G})\}.$$

and define $X(\overline{H},\widetilde{g})$ *similarly.*

Let $X'(\overline{G},\widetilde{G})$, $X'(\overline{H},\widetilde{H})$ *be the subsets consisting of regular semisimple elements. With* G *playing the role of* $\overline{G}$ *we have* $\phi(h)=h^2$ *and*

$$(5.4) \qquad X(H,\widetilde{H}) = \{(h,\widetilde{g})\in H\times\widetilde{H}\,|\,p(\widetilde{g})=h^2\}.$$

For $(h,\widetilde{g})\in X(H,\widetilde{H})$, *choose* $\widetilde{h}$ *satisfying* $p(\widetilde{h})=h$ *and define*

$$(5.5) \qquad \tau(h,\widetilde{g}) = \widetilde{h}^2\widetilde{g}^{-1}.$$

This is clearly independent of the choice of $\widetilde{h}$, *and* $\tau(h,\widetilde{g})=\pm1$ *since* $p(\widetilde{h}^2\widetilde{g}^{-1}) = h^2p(\widetilde{g}^{-1})=1$.

Now suppose $G=\overline{G}$ is semisimple and connected. Recall (Definition 3.14) we always assume $G_d(\mathbb{C})$ is acceptable. Since $G=G_d$, Δ is defined, and for $(h,\widetilde{g})\in X(H,\widetilde{H})$ it is natural to consider

$$(5.6) \qquad \frac{\Delta(h,\Phi^+)}{\Delta(\widetilde{g},\Phi^+)}\Gamma_0(h,\widetilde{g})$$

for some factor $\Gamma_0(h,\widetilde{g})$. See the Introduction. Note that the quotient is independent of the choice of Φ^+. We want Γ_0 to be a genuine function of $\widetilde{g}$, i.e. $\Gamma_0(h,\pm\widetilde{g})=\pm\Gamma_0(h,\widetilde{g})$. We also want $\Gamma_0(h,\widetilde{g})$ to have small finite order for all $h,\widetilde{g}$.

Considerations of harmonic analysis make it natural to include the ϵ_r terms (cf. 5.1(c)) and instead define

$$(5.7) \qquad \Delta(h,\widetilde{g}) = \frac{\Delta(h,\Phi^+)\epsilon_r(h,\Phi^+)}{\Delta(g,\Phi^+)\epsilon_r(g,\Phi^+)}\Gamma_0(h,\widetilde{g})$$

with $\Gamma_0(h,\widetilde{g})$ to be determined. Now the quotient depends on the choice of Φ^+_r.

Fix $(h,\widetilde{g})\in X(H,\widetilde{H})$. The simplest genuine function on $X(H,\widetilde{H})$ is τ (5.5). Motivated by the case of $GL(n)$ (see the Introduction) it is natural to use this on H^0:

$$\Gamma_0(h,\widetilde{g})=\tau(h,\widetilde{g})=\widetilde{h}^2\widetilde{g}^{-1}=\pm 1 \quad (h\in H^0). \tag{5.8}$$

Here is another way to think of this term. Let $\widetilde{\chi}$ be a genuine character of $\widetilde{H^0}$. Then $\widetilde{\chi}^2$ factors to a character of H^0: for $h\in H^0$ define $\widetilde{\chi}^2(h)=\widetilde{\chi}(\widetilde{h}^2)$ where $p(\widetilde{h})=h$. Then it is easy to see that

$$\Gamma_0(h,\widetilde{g})=\frac{\widetilde{\chi}(\widetilde{g})}{\widetilde{\chi}^2(h)} \quad (h\in H^0). \tag{5.9}$$

This suggests a way to define $\Gamma_0(h,\widetilde{g})$ for all h; choose a character χ_0 of H restricting to $\widetilde{\chi}^2$ on H^0, and define:

$$\Gamma_0(h,\widetilde{g})=\frac{\widetilde{\chi}(\widetilde{g})}{\chi_0(h)} \quad \text{for all } (h,\widetilde{g})\in X(H,\widetilde{H}). \tag{5.10}$$

Recall since G is connected by (2.7) $H=\Gamma_r(H)H^0$. A basic property of $\widetilde{\chi}$ is the following (cf. 6.21):

$$\widetilde{\chi}^2(t)=e^{\rho_r}(t) \quad t\in\Gamma_r(H)\cap H^0 \tag{5.11}$$

where $\rho_r=\rho(\Phi_r^+)$. Note that e^{ρ_r} only depends on the positive real roots, and its restriction to H^0 is independent of all choices, since any two differ by a sum of real roots, which is necessarily trivial on $\Gamma_r(H)\cap H^0$.

Therefore we can define χ_0 by

$$\chi_0(h)=\begin{cases}\widetilde{\chi}^2(h) & h\in H^0\\ e^{\rho_r}(h) & h\in\Gamma_r(H).\end{cases} \tag{5.12}$$

Note that χ_0 depends on the choice of Φ^+, although its restriction to $\Gamma_r(H)\cap H^0$ does not. Defining $\Gamma_0(h,\widetilde{g})$ by (5.10), we see:

Lemma 5.13 *Fix a set of positive roots Φ^+, and let $\rho_r=\rho(\Phi_r^+)$. There is a unique function Γ_0 on $X(H,\widetilde{H})$ satisfying*

$$\begin{aligned}\Gamma_0(h,\widetilde{g})&=\tau(h,\widetilde{g}) \quad (h\in H^0)\\ \Gamma_0(th,\widetilde{g})&=e^{-\rho_r}(t)\Gamma_0(h,\widetilde{g}) \quad (t\in\Gamma_r(H),h\in H).\end{aligned} \tag{5.14}$$

Example 5.15 Let $G = SL(2,\mathbb{R})$ and let H be the diagonal Cartan subgroup, which we identify with $\mathbb{R}^\times$. If $p(\widetilde{x}) = x$ and $\epsilon = \pm 1$ then $(x, \widetilde{x}^2\epsilon) \in X(H, \widetilde{H})$ and

$$\Gamma_0(x, \widetilde{x}^2\epsilon) = \text{sgn}(x)\epsilon, \tag{5.16}$$

independent of Φ^+.

While this is a natural extension of $\Gamma_0(h, \widetilde{g})$ from H^0 to H, the compelling evidence that it is the correct definition amounts to the matching conditions involving different Cartan subgroups. See Lemma 6.37.

Definition 5.17 *Assume $G = \overline{G}$ is connected and semisimple. Fix a set of positive roots Φ^+. For $(h, \widetilde{g}) \in X(H, \widetilde{H})$ define the transfer factor*

$$\Delta(h, \widetilde{g}) = \frac{\Delta(h, \Phi^+)\epsilon_r(h, \Phi^+)}{\Delta(g, \Phi^+)\epsilon_r(g, \Phi^+)}.\Gamma_0(h, \widetilde{g}) \tag{5.18}$$

While $\Gamma_0(h, \widetilde{g})$ depends on a choice of Φ^+ (actually only Φ_r^+), the transfer factor itself is independent of this choice:

Lemma 5.19 *$\Delta(h, \widetilde{g})$ is independent of the choice of Φ^+.*

Proof. Write $h = th_0$ with $t \in \Gamma_r(H), h_0 \in H^0$. By (5.14)

$$\Delta(h, \widetilde{g}) = \frac{\Delta(h, \Phi^+)}{\Delta(g, \Phi^+)} \frac{\epsilon_r(h, \Phi^+)}{\epsilon_r(\widetilde{g}, \Phi^+)} e^{\rho_r(\Phi^+)}(t)\tau(h_0, \widetilde{g}) \tag{5.20}$$

Let $\Phi_r^+(h_0) = \{\alpha \in \Phi_r \mid e^\alpha(h_0) > 1\}$. By (7.6)(d) this gives

$$\Delta(h, \widetilde{g}) = \frac{\Delta(h, \Phi^+)}{\Delta(\widetilde{g}, \Phi^+)} e^{\rho(\Phi_r^+(h_0))}(t)\tau(h_0, \widetilde{g}) \tag{5.21}$$

which is obviously independent of Φ^+. □

Lemma 5.22 *For all $(h, \widetilde{g}) \in X(G, \widetilde{G})$ we have*

$$\Delta(h, \widetilde{g}) = c(h, \widetilde{g}) \frac{|D(h)|^{\frac{1}{2}}}{|D(\widetilde{g})|^{\frac{1}{2}}} \tag{5.23}$$

and

$$\Delta(h,\widetilde{g}) = c'(h,\widetilde{g})\frac{\Delta(h,\Phi^+)}{\Delta(\widetilde{g},\Phi^+)} \tag{5.24}$$

where $c(h,\widetilde{g})^4 = c'(h,\widetilde{g})^2 = 1$.

Suppose $\widetilde{x} \in \widetilde{G}$ *and let* $x = p(\widetilde{x})$. *Then*

$$\Delta(xhx^{-1}, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Delta(h,\widetilde{g}). \tag{5.25}$$

This is immediate.

In the next section we will drop the assumption that G is acceptable, in which case $\Delta(h,\Phi^+)$ and $\Delta(\widetilde{g},\Phi^+)$ are not necessarily well defined. With this in mind we rewrite (5.18) as follows:

$$\begin{aligned}\Delta(h,\widetilde{g}) &= \frac{\Delta(h,\Phi^+)\epsilon_r(h,\Phi^+)}{\Delta(\widetilde{g},\Phi^+)\epsilon_r(\widetilde{g},\Phi^+)}\frac{\widetilde{\chi}(\widetilde{g})}{\chi_0(h)}\\ &= \frac{\Delta^0(\Phi^+,h)\epsilon_r(h,\Phi^+)}{\Delta^0(\Phi^+,\widetilde{g})\epsilon_r(\widetilde{g},\Phi^+)}\frac{\widetilde{\chi}(\widetilde{g})e^\rho(h)}{\chi_0(h)e^\rho(\widetilde{g})}\\ &= \frac{\Delta^1(\Phi^+,h)}{\Delta^1(\Phi^+,\widetilde{g})}\frac{\widetilde{\chi}(\widetilde{g})}{(\chi_0 e^\rho)(h)}\end{aligned} \tag{5.26}$$

since $p(\widetilde{g}) = h^2$. Let $\chi = \chi_0 e^\rho$ and assume Φ^+ is special (Definition 2.1). Then by (2.4)(b) and (5.12) for $h \in \Gamma_r(H)$ we have

$$\chi(h) = e^{\rho_r}(h)e^\rho(h) = \zeta_{\mathrm{cx}}(h). \tag{5.27}$$

Thus χ is defined by

$$\chi(h) = \begin{cases}(\widetilde{\chi}^2 e^\rho)(h) & h \in H^0\\ \zeta_{\mathrm{cx}}(h) & h \in \Gamma_r(H).\end{cases} \tag{5.28)(a}$$

Define

$$\Gamma(h,\widetilde{g}) = \widetilde{\chi}(\widetilde{g})/\chi(h), \tag{5.28)(b}$$

and then (still assuming $G = \overline{G}$ is connected and semisimple)

$$\Delta(h,\widetilde{g}) = \frac{\Delta^1(\Phi^+,h)}{\Delta^1(\Phi^+,\widetilde{g})}\Gamma(h,\widetilde{g}). \tag{5.28)(c}$$

This has the advantage that the quotient is defined in general and this is the starting point of the definition of transfer factors. There are several issues in extending it to the general case. First of all e^ρ is only defined on H_d^0 (recall G_d is acceptable), so (5.28) only defines χ, and hence Γ and Δ, on $\Gamma_r(H)H_d^0 = H \cap G_d^0$ (cf. (2.7)). This may be a proper subgroup of H. Furthermore if $\overline{G} \neq G$ then some additional modification is required. We take up these issues in the next two sections.

6 Characters of Cartan Subgroups.

Let $(\widetilde{G}, G, \overline{G})$ be an admissible triple. In the previous section we defined the factor $\Gamma(h, \widetilde{g})$ (5.28)(b) if $G = \overline{G}$ is connected and semisimple. In this section we consider the general case, which we use in the subsequent section to define transfer factors in general.

Suppose H is a Cartan subgroup of G and let Φ^+ be a set of positive roots. Suppose $\widetilde{\chi}$ is a genuine character of $Z(\widetilde{H})$. Since $G_d(\mathbb{C})$ is acceptable ρ exponentiates to a character of $(H \cap G_d)^0$, and $\widetilde{\chi}^2 e^\rho$ is defined on $(H \cap G_d)^0$. Let $\overline{H}$ be the corresponding Cartan subgroup of $\overline{G}$ and set $\overline{H}_d = \overline{H} \cap \overline{G}_d$. We now use property (3c) of Definition 3.14, which says that $\widetilde{\chi}^2 e^\rho$ factors to $\overline{H}_d^0$. The canonical character ζ_{cx} of $\Gamma(\overline{H})$ was defined in Section 2.

We use (5.28) as our starting point.

Definition 6.1 *Let H be a Cartan subgroup of G. Choose*

1. *a special set of positive roots Φ^+ (Definition 2.1)*
2. *a genuine character $\widetilde{\chi}$ of $Z(\widetilde{H})$,*
3. *a character χ of $\overline{H}$.*

Assume these satisfy:

(6.2)(a) $$\chi(h) = (\widetilde{\chi}^2 e^\rho)(h), \quad (h \in \overline{H}_d^0)$$

(6.2)(b) $$\chi(h) = \zeta_{cx}(h) \quad (h \in \Gamma_r(\overline{H})).$$

Fix $(\Phi^+, \widetilde{\chi}, \chi)$ satisfying these conditions. Suppose $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$ (cf. (5.3)(b)). Then we define

(6.3) $$\Gamma(\widetilde{\chi}, \chi)(h, \widetilde{g}) = \frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)}.$$

Let

$$\text{(6.4)(a)} \qquad \mathcal{S}(H,\Phi^+) = \{(\widetilde{\chi},\chi)\} \textit{ satisfying } (6.2)\textit{(a) and (b)}\},$$
$$\text{(6.4)(b)} \qquad \mathcal{T}(H,\Phi^+) = \{\Gamma(\widetilde{\chi},\chi) \,|\, (\widetilde{\chi},\chi) \in \mathcal{S}(H,\Phi^+)\}$$
$$\text{(6.4)(c)} \qquad \mathcal{S}(H) = \cup_{\Phi^+}\mathcal{S}(H,\Phi^+)$$
$$\text{(6.4)(d)} \qquad \mathcal{T}(H) = \cup_{\Phi^+}\mathcal{T}(H,\Phi^+).$$

(The unions are over special sets of positive roots).

Example 6.5 Let $G = \overline{G} = SL(2,\mathbb{R})$ and let H be the diagonal Cartan subgroup, which we identify with $\mathbb{R}^\times$. Let $\Phi^+ = \{\alpha\}$ where $\alpha(x) = x^2$. If $p(\widetilde{x}) = x$ and $\epsilon = \pm 1$ then a short calculation gives

$$\text{(6.6)} \qquad \Gamma(x, \widetilde{x}^2\epsilon) = x^{-1}\epsilon.$$

This only depends on Φ^+; the other choice of Φ^+ gives $x\epsilon$. Compare Example 5.15.

We begin with some elementary properties. Fix H, $(\widetilde{\chi},\chi)$ as above and let $\Gamma = \Gamma(\widetilde{\chi},\chi)$. First of all we obtain a character λ of $Z_0(H)$.

Lemma 6.7 *Let $\lambda = (\widetilde{\chi}^2/\chi)$, viewed as a character of $Z_0(H)$, i.e. $\lambda(h) = \widetilde{\chi}^2(\widetilde{h})/\chi(\overline{p}(h))$ where $p(\widetilde{h}) = h$. Then*

$$\text{(6.8)(a)} \qquad \lambda(h) = e^{-\rho}(h) \quad (h \in Z_0(H) \cap G_d).$$

Also let

$$\text{(6.8)(b)} \qquad \mu = -d\lambda - \rho = d\chi - 2d\widetilde{\chi} - \rho.$$

Then $\mu|_{\mathfrak{h}_d} = 0$ and we identify it with an element of $\mathfrak{z}(\mathbb{C})^*$; as such it is the negative of the differential of $\lambda|_{Z(G)^0}$.

Furthermore for all $(h,\widetilde{g}) \in X(\overline{H},\widetilde{H})$

$$\text{(6.8)(c)} \qquad \Gamma(h,\widetilde{g})^2 = \lambda(\phi(h)).$$

In particular if $h \in \overline{G}_d$ then $\Gamma(h,\widetilde{g})^2 = e^{-2\rho}(h)$.

Proof. If $h \in H_d^0$ then by 6.2(b) $\chi(\overline{p}(h)) = (\widetilde{\chi}^2 e^\rho)(h)$. The same identity holds for $h \in \Gamma_r(H) \cap Z_0(H)$ by (6.21). In both cases we conclude $\lambda(h) =$

$\widetilde{\chi}^2(h)/\chi(\overline{p}(h)) = e^{-\rho}(h)$. This holds on $\Gamma_r(H)H_d^0 \cap Z_0(H)$, which equals $Z_0(H) \cap G_d$ by (2.7). This gives (a).

For (b), if $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$ then $\Gamma(h, \widetilde{g})^2 = \widetilde{\chi}^2(\widetilde{g})/\chi(h^2)$. Since $p(\widetilde{g}) = \phi(h)$ and $\overline{p}(\phi(h)) = h^2$ this equals $\widetilde{\chi}^2(\phi(h))/\chi(\phi(h)) = \lambda(\phi(h))$. The final equality follows from (a) and (b) and the fact that $\overline{p}(\phi(h)) = h^2$. □

Recall (5.5) for $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$, $\tau(h, \widetilde{g}) = \widetilde{h}^2 \widetilde{g}^{-1} = \pm 1$.

Lemma 6.9 *Suppose* $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$.
(1) Assume $h \in \overline{p}(Z_0(H))$*, and choose* $y \in Z_0(H)$ *satisfying* $\overline{p}(y) = h$*. Then with* λ *as in* (6.36)(c),

(6.10) $$\Gamma(h, \widetilde{g}) = \lambda(y)\tau(y, \widetilde{g}).$$

(2) If $h \in \overline{G}_d$ *write* $h = th_0$ *with* $t \in \Gamma_r(\overline{H})$*,* $h_0 \in \overline{H}_d^0$*, and choose* $y_0 \in H^0$ *satisfying* $\overline{p}(y_0) = h_0$*. Then*

(6.11) $$\Gamma(h, \widetilde{g}) = \zeta_{cx}(t) e^{-\rho}(y_0)\tau(y_0, \widetilde{g}),$$

and this is independent of all choices except that of Φ^+*.*
(3) In general write $h = th_0$ *with* $t \in \Gamma(\overline{H})$ *and* $h_0 \in \overline{H}^0$ (2.7)*. Write* $\widetilde{g} = \widetilde{\gamma}\widetilde{g}_0$ *where* $p(\widetilde{\gamma}) = \phi(t)$ *and* $p(\widetilde{g}_0) = \phi(h_0)$*. Finally choose* $y_0 \in H^0$ *satisfying* $\overline{p}(y_0) = h_0$*. Then*

(6.12) $$\Gamma(h, \widetilde{g}) = \frac{\widetilde{\chi}(\widetilde{\gamma})}{\chi(t)}\lambda(y_0)\tau(y_0, \widetilde{g}_0).$$

Note that $\widetilde{\chi}(\widetilde{\gamma})^4 = 1$ *and* $\chi(t)^2 = 1$*.*

Proof. For (1) choose $\widetilde{y} \in p^{-1}(y)$ and write $\widetilde{g} = \widetilde{y}^2\tau(y, \widetilde{g})$ (cf. 5.5). Then $\Gamma(h, \widetilde{g}) = \widetilde{\chi}(\widetilde{y}^2)\tau(y, \widetilde{g})/\chi(h) = (\widetilde{\chi}^2(y)/\chi(\overline{p}(y)))\tau(y, \widetilde{g}) = \lambda(y)\tau(y, \widetilde{g})$. Part (2) follows from this, (6.2)(b) and (6.8)(a). Part (3) is similar, noting that $\Gamma(h, \widetilde{g}) = \Gamma(t, \widetilde{\gamma})\Gamma(h_0, \widetilde{g}_0)$. □

Example 6.13 Let $G = \overline{G} = GL(n, \mathbb{R})$ and suppose H is the diagonal (split) Cartan subgroup. For simplicity we assume n is even, in which case $Z_0(H) = H^0$. Note that $\Gamma(H) = \{\text{diag}(\epsilon_1, \dots, \epsilon_n)\}$ with $\epsilon_i = \pm 1$, and $\Gamma_r(H)$ is the subset of $\Gamma(H)$ of elements of determinant 1. If $h \in H$ write $h = th_0$ with $t \in \Gamma(H)$ and $h \in H^0$. Let Φ^+ be any (necessarily special) set of positive roots.

The cover $Z(\widetilde{H}) \to Z_0(H)$ splits; let $\widetilde{\chi}$ be the unique, genuine quadratic character of $Z(\widetilde{H})$. Note that ρ exponentiates to H^0, and we can define $\chi(th_0) = e^{\rho}(h_0)$. Then $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)$ and by (6.12) we have

$$\Gamma(h, \widetilde{g}) = e^{-\rho}(h_0)\tau(h_0, \widetilde{g}). \tag{6.14}$$

Given $\widetilde{\chi}$, the only other allowed choice of χ and Γ is obtained by multiplying by $\nu(\det(h))$ for ν a character of $\mathbb{R}^\times$ (see Lemma 6.19).

Lemma 6.15 *Fix a special set of positive roots Φ^+.*
(1) Suppose ν is a character of $Z_0(H)$ and $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)$. Then $(\widetilde{\chi}(\nu \circ p), \chi(\nu \circ \phi)) \in \mathcal{S}(H, \Phi^+)$ and

$$\Gamma(\widetilde{\chi}(\nu \circ p), \chi(\nu \circ \phi)) = \Gamma(\widetilde{\chi}, \chi). \tag{6.16}$$

(2) Suppose Φ_1^+ is another special set of positive roots. There exists a character τ of $\overline{H}$ such that for all $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)$, $(\widetilde{\chi}, \chi\tau) \in \mathcal{S}(H, \Phi_1^+)$, and

$$\Gamma(\widetilde{\chi}, \chi)(\widetilde{g}, h) = \Gamma(\widetilde{\chi}, \chi\tau)(\widetilde{g}, h) \tag{6.17}$$

for all $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$ satisfying $h \in \Gamma(\overline{H})Z(\overline{G})$.

Proof. The first part is straightforward. For the second define $\tau(h) = e^{\rho(\Phi_1^+)-\rho(\Phi^+)}(h)$ for $h \in \overline{H}_d^0$, and $\tau(h) = 1$ for $h \in \Gamma(\overline{H})Z(\overline{G})$. It is easy to see that the fact that both Φ^+ and Φ_1^+ are special implies $e^{\rho(\Phi_1^+)-\rho(\Phi^+)}(h) = 1$ for all $h \in \Gamma(\overline{H})Z(\overline{G}) \cap \overline{H}_d^0$, so τ is well defined. It follows easily that it has the desired properties. $\square$

There is a natural action of the characters of $\overline{G}$ on $\mathcal{T}(H, \Phi^+)$: if ψ is a one-dimensional representation of $\overline{G}$ define

$$\psi \cdot \Gamma(\widetilde{\chi}, \chi) = \Gamma(\widetilde{\chi}, \chi\psi|_{\overline{H}}). \tag{6.18}$$

Lemma 6.19 *The space $\mathcal{T}(H, \Phi^+)$ is non-empty, and the action of the group of characters of $\overline{G}$ on $\mathcal{T}(H, \Phi^+)$ is transitive. If H is maximally split it is simply transitive.*

Proof. Fix any genuine character $\widetilde{\chi}$ of $Z(\widetilde{H})$. For existence it is enough to show that for any special set of positive roots,

$$\zeta_{\mathrm{cx}}(h) = e^{\rho}\widetilde{\chi}^2(h) \quad h \in \Gamma_r(\overline{H}) \cap \overline{H}_d^0. \tag{6.20}$$

Note that $p^{-1}(\overline{H}_d^0) \subset p^{-1}(\overline{H}^0) = H^0C \subset H^0Z_0(G) = Z_0(H)$ by 3.14(3b)) and Proposition 4.7. Pulling back to G it is therefore enough to show

$$\zeta_{cx}(h) = e^{\rho}\widetilde{\chi}^2(h) \quad h \in \Gamma_r(H) \cap Z_0(H). \tag{6.21}$$

Lemma 6.22 *$\widetilde{\chi}^2(h) = e^{\rho_r}(h)$ for all $h \in Z_0(H) \cap \Gamma_r(H)$.*

Proof. Fix $h \in Z_0(H) \cap \Gamma_r(H)$. By (2.6) write $h = \prod_{i=1}^n m_i$ where $\{\alpha_1, \ldots, \alpha_n\}$ is a subset of the simple roots for Φ_r^+, and $m_i = m_{\alpha_i}$. Then $e^{\rho_r}(h) = (-1)^n$.

For each i choose an inverse image $\widetilde{m}_i$ of m_i in $\widetilde{H}$. By Lemma 3.2 and the discussion preceding it $\widetilde{m}_i^2 = -1$ for all i. Assume for the moment that $\widetilde{m}_i, \widetilde{m}_j$ commute for all i, j. Then

$$\begin{aligned}\widetilde{\chi}^2(h) &= \widetilde{\chi}([\widetilde{m}_1 \ldots \widetilde{m}_n]^2)\\ &= \widetilde{\chi}(\widetilde{m}_1^2 \ldots \widetilde{m}_n^2)\\ &= \widetilde{\chi}((-1)^n) = (-1)^n\end{aligned} \tag{6.23}$$

since $\widetilde{\chi}$ is genuine, proving the result.

Thus it is enough to show the $\widetilde{m}_i$ commute. Using the commutator notation of Section 4, the fact that $\{m_i, h\} = 1$ implies by (4.6)(c) that, for each i, α_i is not orthogonal to an even number of other α_j. Consider the subdiagram of the Dynkin diagram of Φ_r consisting of the α_i ($1 \leq i \leq n$): every node is adjacent to an even number of other nodes. This implies the nodes are all orthogonal, and the $\widetilde{m}_i$ commute. □

Therefore

$$\widetilde{\chi}^2 e^{\rho}(h) = e^{\rho_r}(h) e^{\rho_r + \rho_{cx} + \rho_i}(h) = e^{\rho_{cx}}(h) \tag{6.24}$$

since $e^{2\rho_r}(h) = e^{\rho_i}(h) = 1$. This equals $\zeta_{cx}(h)$ by (2.4)(b).

Now suppose $\Gamma_1, \Gamma_2 \in \mathcal{T}(H, \Phi^+)$. If $h \in \overline{H}$ then $\psi_0(h) = \Gamma_1(h, \widetilde{g})/\Gamma_2(h, \widetilde{g})$ is independent of the choice of $\widetilde{g}$ so that $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$, and is a character of $\overline{H}$. By (6.9) $\psi_0(h) = 1$ for $h \in \overline{G}_d$.

We have a surjection

$$\mathrm{Hom}(\overline{G}/\overline{G}_d, \mathbb{C}) \to \mathrm{Hom}(\overline{H}/\overline{H}_d, \mathbb{C}) \tag{6.25}$$

dual to the injection $\overline{H}/\overline{H}_d \hookrightarrow \overline{G}/\overline{G}_d$, which is an isomorphism if H is maximally split. If we choose any preimage ψ of ψ_0 in (6.25) then it is easy to see that $\Gamma_1 = \psi\Gamma_2$. It follows that characters of $\overline{G}$ act transitively on $\mathcal{T}(H,\Phi^+)$, and this action is simply transitive if H is maximally split. $\square$

We now need to choose elements of $\mathcal{T}(H,\Phi^+)$ consistently for all H. We do this by reducing to the most split Cartan subgroup. For this we need a Lemma.

Lemma 6.26 *Suppose* $(\widetilde{\chi},\chi) \in \mathcal{S}(H,\Phi^+)$. *Then*

$$\chi(h) = \chi(wh) \quad (w \in W(\overline{G},\overline{H}), h \in \Gamma(\overline{H})Z(\overline{G})). \tag{6.27}$$

Proof. We need to show $\chi(g(wg)^{-1}) = 1$ for $g \in \Gamma(\overline{H})$. This is obvious if $w \in W(\Phi_i)$. If $w \in W(\Phi_r)$ then $g(wg)^{-1} \in \Gamma_r(\overline{H})$. Then $\chi(g(wg)^{-1}) = \zeta_{\mathrm{cx}}(g(wg)^{-1}) = 1$ by (6.2)(b) and (2.4)(a).

By [27, Proposition 3.12] $W(\overline{G}(\mathbb{C}),\overline{H}(\mathbb{C}))^\theta$, which contains $W(\overline{G},\overline{H})$, is generated by $W(\Phi_i), W(\Phi_r)$ and elements of the form $s_\alpha s_{\theta\alpha}$ where $\langle\alpha,\theta\alpha^\vee\rangle = 0$ and $s_\alpha s_{\theta\alpha}\Phi_r^+ = \Phi_r^+$. So it is enough to show $\chi(g(wg)^{-1}) = 1$ for w of this form.

Write $g = \overline{\exp}\pi i X$ with $X \in \mathfrak{a}$. Since $g^2 = 1$, $\alpha(X) \in \mathbb{Z}$. Then

$$g(wg)^{-1} = \overline{\exp}\pi i\alpha(X)(\alpha^\vee + \theta\alpha^\vee) \in \Gamma(\overline{H}) \cap \overline{H}_d^0 \tag{6.28}$$

Let $\rho = \rho(\Phi^+)$. Then

$$\begin{aligned}\chi((wg)g^{-1}) &= (\widetilde{\chi}^2 e^\rho)(g(wg)^{-1}) \quad \text{(by (6.2)(a))}\\ &= (-1)^{\alpha(X)\langle 2d\widetilde{\chi},\alpha^\vee+\theta\alpha^\vee\rangle} e^{\rho - w^{-1}\rho}(g)\end{aligned} \tag{6.29}$$

By [3, Lemma 6.11] $\langle d\widetilde{\chi},\alpha^\vee + \theta\alpha^\vee\rangle \in \mathbb{Z}$. The fact that Φ^+ is special and $w\Phi_r^+ = \Phi_r^+$ implies $\rho - w^{-1}\rho$ is a sum of imaginary roots and terms $\beta - \theta\beta$ with β complex. Therefore $e^{w^{-1}\rho-\rho}(g) = 1$. $\square$

Fix a maximally split Cartan subgroup $H_s = T_sA_s$ of G and $(\widetilde{\chi}_s,\chi_s) \in \mathcal{S}(H_s)$.

Suppose H is a Cartan subgroup, and write $\overline{H} = \overline{TA}$ as usual. There exists $x \in \overline{G}$ such that $x\overline{A}x^{-1} \subset \overline{A}_s$, and therefore $x\Gamma(\overline{H})x^{-1} \subset \Gamma(\overline{H}_s)$. Suppose $(h,\widetilde{g}) \in X(\overline{H},\widetilde{H})$ where $h \in \Gamma(\overline{H})Z(\overline{G})$. By Lemmas 3.11(5) and 3.12 $\phi(h) \in Z_0(G)$ so $\widetilde{g} \in Z(\widetilde{G}) \subset Z(\widetilde{H}_s)$. Therefore $(xhx^{-1},\widetilde{g}) \in X(\overline{H}_s,\widetilde{H}_s)$. Define

$$\Gamma_s(h,\widetilde{g}) = \Gamma(\widetilde{\chi}_s,\chi_s)(xhx^{-1},\widetilde{g}). \tag{6.30}$$

By the Lemma this is independent of the choice of x.

Proposition 6.31 *Fix a maximally split Cartan subgroup H_s and $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$. Suppose H is a Cartan subgroup, $\widetilde{\chi}$ is a genuine character of $Z(\widetilde{H})$, and Φ^+ is a special set of positive roots for H. Then there exists a unique character χ of $\overline{H}$ such that $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)$ and*

$$\Gamma(\widetilde{\chi}, \chi)(h, \widetilde{g}) = \Gamma_s(h, \widetilde{g}) \tag{6.32}$$

for all $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$ with $h \in \Gamma(\overline{H})Z(\overline{G})$.

Explicitly (6.32) is

$$\frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)} = \frac{\widetilde{\chi}_s(\widetilde{g})}{\chi_s(xhx^{-1})} \tag{6.33}$$

or alternatively

$$\chi(h) = (\widetilde{\chi}/\widetilde{\chi}_s)(\phi(h))\chi_s(xhx^{-1}). \tag{6.34}$$

Before giving the proof of the Proposition we give the main definition of this section.

Definition 6.35 *Fix a split Cartan subgroup H_s and $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$. Suppose H is a Cartan subgroup and Φ^+ is a special set of positive roots.*

(1) Let $\mathcal{S}(H, \Phi^+, \widetilde{\chi}_s, \chi_s)$ be the set of pairs $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+)$ satisfying (6.32)*.*

(2) Suppose $\widetilde{\chi}$ is any genuine character of $\widetilde{H}$. Choose χ so that $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+, \widetilde{\chi}_s, \chi_s)$ and let

$$\Gamma(H, \Phi^+) = \Gamma(\widetilde{\chi}, \chi) \in \mathcal{T}(H, \Phi^+). \tag{6.36)(a}$$

Also define

$$\lambda(H, \Phi^+) = \widetilde{\chi}^2/\chi \in \widehat{Z_0(H)} \tag{6.36)(b}$$

and

$$\mu = -d\lambda - \rho = d\chi - 2d\widetilde{\chi} - \rho \in \mathfrak{z}(\mathbb{C})^* \tag{6.36)(c}$$

(cf. Lemma 6.7). If it is necessary to indicate the dependence on $(\widetilde{\chi}_s, \chi_s)$ we write $\Gamma(\widetilde{\chi}_s, \chi_s, H, \Phi^+)$, $\lambda(\widetilde{\chi}_s, \chi_s, H, \Phi^+)$ and $\mu(\widetilde{\chi}_s, \chi_s)$.

It follows easily from Lemma 6.15 that $\Gamma(H,\Phi^+)$ and $\lambda(H,\Phi^+)$ are independent of the choice of $\widetilde{\chi}$. It is also easy to see that $\mu = d\chi_s - 2d\widetilde{\chi}_s$ restricted to $\mathfrak{z}(\mathbb{C})^*$, and is therefore independent of the choice of (H,Φ^+).

We now prove Proposition 6.31. Using Cayley transforms (Section 4) we will reduce to the following Lemma. Suppose H_α is a Cartan subgroup, with real root α, and H_β is the Cayley transform of H_α, with noncompact imaginary root β. See the discussion preceding (4.13). Fix $c \in \mathrm{Aut}(\mathfrak{g})$ satisfying (4.14)(a-c).

Lemma 6.37 *Suppose Φ^+_α is a special set of positive roots for H_α and $(\widetilde{\chi}_\alpha,\chi_\alpha) \in \mathcal{S}(H_\alpha,\Phi^+_\alpha)$. Suppose Φ^+_β is a special set of positive roots for H_β, and $\widetilde{\chi}_\beta$ is a genuine character of $Z(\widetilde{H}_\beta)$. Then there exists a unique character χ_β of H_β such that $(\widetilde{\chi}_\beta,\chi_\beta) \in \mathcal{S}(H_\beta,\Phi^+_\beta)$ and*

$$\Gamma(\widetilde{\chi}_\alpha,\chi_\alpha)(h,\widetilde{g}) = \Gamma(\widetilde{\chi}_\beta,\chi_\beta)(h,\widetilde{g}) \tag{6.38}$$

for all $(h,\widetilde{g}) \in X(\overline{H}_\beta,\widetilde{H}_\beta)$ satisfying $h \in \Gamma(\overline{H}_\beta)Z(\overline{G})$.

In addition assume $c^*(\Phi^+_\alpha) = \Phi^+_\beta$. Then (6.38) holds for all $h \in \overline{H}_\alpha \cap \overline{H}_\beta$.

Proof. To avoid runaway notation let $\overline{H} = \overline{H}_\beta$ for a moment. The character χ_β is determined on $\Gamma(\overline{H})Z(\overline{G})$ by (6.38) and on $\overline{H}^0_d$ by (6.2)(a). Since $\overline{H}^0 \subset Z(\overline{G})\overline{H}^0_d$, by (2.7) $\overline{H} = \Gamma(\overline{H})Z(\overline{G})\overline{H}^0_d$, and uniqueness is immediate.

For existence, by Lemma 6.15 it is enough to prove this for a single choice of Φ^+_α and Φ^+_β. It is not hard to see we can choose Φ^+_α special so that $\Phi^+_\beta = c^*(\Phi^+_\alpha)$ is also special. This implies α is simple for Φ^+_r (cf. Lemma 14.17).

Fix $\widetilde{\chi}_\beta$. By 6.15(1) we may choose $\widetilde{\chi}_\alpha$ arbitrarily, so by Lemma 4.34 choose $\widetilde{\chi}_\alpha$ to be the Cayley transform of $\widetilde{\chi}_\beta$. Since $\widetilde{\chi}_\beta$ is genuine and β is a noncompact imaginary root, $\langle d\widetilde{\chi}_\beta,\beta^\vee\rangle \in \mathbb{Z}+\frac{1}{2}$ by [3, Lemma 6.11]. Therefore $\langle d\widetilde{\chi}_\alpha,\alpha^\vee\rangle = \langle d\widetilde{\chi}_\beta,\beta^\vee\rangle \in \mathbb{Z}+\frac{1}{2}$. Let χ_α be any character of $\overline{H}_\alpha$ with $(\widetilde{\chi}_\alpha,\chi_\alpha) \in \mathcal{S}(H_\alpha,\Phi^+_\alpha)$. Then

$$\chi_\alpha(m_\alpha) = \zeta_{cx}(m_\alpha) = (-1)^{\langle\rho-\rho_r,\alpha^\vee\rangle} = -(-1)^{\langle\rho,\alpha^\vee\rangle} \tag{6.39}$$

since α is simple for Φ^+_r. Furthermore

$$\langle d\chi_\alpha,\alpha^\vee\rangle = \langle 2d\widetilde{\chi}_\alpha+\rho,\alpha^\vee\rangle \in 2\mathbb{Z}+1+\langle\rho,\alpha^\vee\rangle. \tag{6.40}$$

Thus χ_α satisfies the conditions of Lemma 4.20. Let χ_β be the (unique) Cayley transform of χ_α.

It is enough to show $(\widetilde{\chi}_\beta,\chi_\beta)\in\mathcal{S}(H_\beta,\Phi^+_\beta)$, for then it is obvious from the definition of Cayley transforms that (6.38) holds for all $h\in\overline{H}_\alpha\cap\overline{H}_\beta$.

We have

$$\begin{aligned} d\chi_\beta &= \mathrm{Ad}^*(c)(d\chi_\alpha)\\ &= \mathrm{Ad}^*(c)(2d\widetilde{\chi}_\alpha+\rho(\Phi^+_\alpha))\\ &= 2d\widetilde{\chi}_\beta+\rho(\Phi^+_\beta) \end{aligned} \tag{6.41}$$

by the choice of Φ^+_α and Φ^+_β. This verifies (6.2)(a).

We now verify (6.2)(b). Suppose γ is a real root of $\overline{H}_\beta$. Then $m_\gamma\in\overline{H}_\alpha\cap\overline{H}_\beta$, and $\chi_\beta(m_\gamma)=\chi_\alpha(m_\gamma)=\zeta_{cx}(\overline{G},\overline{H}_\alpha)(m_\gamma)$, by (6.2)(b) for $(\widetilde{\chi}_\alpha,\chi_\alpha)$. We want to show this equals $\zeta_{cx}(\overline{G},\overline{H}_\beta)(m_\gamma)$, i.e.

$$\zeta_{cx}(\overline{G},\overline{H}_\alpha)(m_\gamma)=\zeta_{cx}(\overline{G},\overline{H}_\beta)(m_\gamma). \tag{6.42}$$

If the simple factor containing α,β is of type G_2 an explicit calculation shows that both sides are 1, so assume G is simply laced.

The root $\widetilde{\gamma}=\mathrm{Ad}^*(c^{-1})(\gamma)$ of $\overline{H}_\alpha$ is also real, and $\langle\widetilde{\gamma},\alpha^\vee\rangle=0$. Suppose δ is a complex root of $\overline{H}_\alpha$ with $\langle\delta,\widetilde{\gamma}^\vee\rangle\neq 0$. Then the corresponding root of $\overline{H}_\beta$ is also complex. Conversely, suppose δ is a complex root of $\overline{H}_\beta$ with $\langle\delta,\gamma^\vee\rangle\neq 0$. Then the corresponding root $\widetilde{\delta}$ of $\overline{H}_\alpha$ is complex unless $s_\beta\delta=\sigma\delta$, in which case it is real. In this case, let $\delta'=s_\gamma\delta$. It is another complex root of $\overline{H}_\beta$ with $\widetilde{\delta}'$ real. Since $\langle\delta,\gamma^\vee\rangle\neq 0$, $\delta'\neq\delta$, and since γ is real, $\delta'\neq\sigma\delta$. Since $\delta\neq\pm\gamma,\delta'\neq-\delta$. Suppose $\delta'=-\sigma\delta=-s_\beta\delta$. Then

$$2\delta=\langle\delta,\gamma^\vee\rangle\gamma+\langle\delta,\beta^\vee\rangle\beta. \tag{6.43}$$

This cannot happen since in the simply laced case, two orthogonal roots have no other roots in their (real) span.

Therefore both δ and $s_\gamma(\delta)$ contribute to $\zeta_{cx}(\overline{G},\overline{H}_\beta)(m_\gamma)$, and their total contribution is $\delta(m_\gamma)(s_\gamma\delta)(m_\gamma)=\delta(m_\gamma)\delta(m_\gamma)=1$.

It follows that the terms in $\zeta_{cx}(\overline{G},\overline{H}_\alpha)$ and $\zeta_{cx}(\overline{G},\overline{H}_\beta)$ are the same, with the exception of those just discussed, which are 1. $\square$

Remark 6.44 Note that $\Gamma(\widetilde{\chi}_\alpha,\chi_\alpha)(m_\alpha,1)=\chi_\alpha(m_\alpha)$ and $\Gamma(\widetilde{\chi}_\beta,\chi_\beta)(m_\alpha,1)=\chi_\beta(m_\alpha)$. As in the proof of the Lemma $\chi_\beta(m_\alpha)=-(-1)^{\langle\rho,\alpha^\vee\rangle}$. The equality of the Lemma then implies $\chi_\alpha(m_\alpha)=\zeta_{cx}(m_\alpha)$. This motivates condition (6.2)(b).

Proof of the Proposition. This is now straightforward. If H is maximally split there is nothing to prove. If H is obtained from H_s by a series of Cayley transforms we conclude the result by a repeated application of Lemma 6.37. It is easy to check that if the conditions of the Proposition hold for a Cartan subgroup H, they hold for every G-conjugate of H. Up to conjugacy every Cartan subgroup is conjugate to one obtained by a series of Cayley transforms from H_s, and the result follows. $\square$

For later use we note a consequence of the last part of the proof of Lemma 6.37.

Lemma 6.45 *Suppose $H_1 = T_1A_1$ and $H_2 = T_2A_2$ are Cartan subgroups with $A_1 \subset A_2$. Suppose $\alpha \in \Phi_r(G, H_1)$.*
(1)

$$\zeta_{cx}(G, H_1)(m_\alpha) = \zeta_{cx}(G, H_2)(m_\alpha).$$

(2) Let $M_1 = Cent_G(A_1)$ and suppose $\alpha \in \Phi_r(M_1, H_2)$. Then

$$\zeta_{cx}(G, H_2)(m_\alpha) = \zeta_{cx}(M_1, H_2)(m_\alpha).$$

Proof. The first equality follows from a repeated application of (6.42). The second is similar. Choose a set S of complex roots such that

$$\{\beta \in \Phi_{cx}(G, H_2) \backslash \Phi_{cx}(M_1, H_2) \mid \langle \beta, \alpha^\vee \rangle \neq 0\} = \{\pm\beta, \pm\sigma\beta \mid \beta \in S\}. \tag{6.46}$$

If $\beta \in S$ then $s_\alpha\beta$ is also complex, is not contained in $\Phi(M_1, H_2)$, and is not equal to $\pm\beta, \pm\sigma\beta$ Then S is can be written as a union over pairs $\{\beta, s_\alpha\beta\}$ and the result follows as in the proof of (6.42). $\square$

The dependence of $\Gamma(H, \Phi^+)$ on Φ^+ follows easily from Lemma 6.15(2) and its proof.

Lemma 6.47 *Suppose Φ_1^+, Φ_2^+ are special sets of positive roots for H. Let $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$, and write $h = \gamma h_0$ with $\gamma \in \Gamma(\overline{H})$ and $h_0 \in \overline{H}^0$. Then*

$$\Gamma(H, \Phi_1^+)(h, \widetilde{g}) = \Gamma(H, \Phi_2^+)(h, \widetilde{g})e^{\rho(\Phi_2^+) - \rho(\Phi_1^+)}(h_0). \tag{6.48}$$

The factors $\Gamma(H, \Phi^+)$ are conjugation invariant in the following sense.

Lemma 6.49 *Suppose Φ^+ is a special set of positive roots for H, and $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$. Fix $\widetilde{x} \in \widetilde{G}$ and let $x = \overline{p}(p(\widetilde{x})) \in \overline{G}$. Let $\overline{H}_1 = x\overline{H}x^{-1}$ and $\Phi_1^+ = Ad(x)\Phi^+$. Then*

$$\Gamma(H_1, \Phi_1^+)(xhx^{-1}, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Gamma(H, \Phi^+)(h, \widetilde{g}) \tag{6.50}$$

Proof. Choose $(\widetilde{\chi},\chi)\in\mathcal{S}(H,\Phi^+,\widetilde{\chi}_s,\chi_s)$. Let $\widetilde{\chi}_1=\widetilde{x}\widetilde{\chi}\widetilde{x}^{-1}$ and $\chi_1=x\chi x^{-1}$. It is clearly enough to show $(\widetilde{\chi}_1,\chi_1)\in\mathcal{S}(H_1,\Phi_1^+,\widetilde{\chi}_s,\chi_s)$. Condition (6.2)(b) is immediate, and for all $h\in(\overline{H}_1\cap\overline{G}_d)^0$

$$\chi_1(h)=\chi(x^{-1}tx)=\widetilde{\chi}^2e^{\rho(\Phi^+)}(x^{-1}tx)=\widetilde{\chi}_1^2e^{\rho(\Phi_1^+)}(h)$$

so (6.2)(a) holds as well.

Assume $\gamma\in\Gamma(\overline{H}_1)Z(\overline{G})$ and choose y satisfying $y\Gamma(\overline{H})y^{-1}\subset\Gamma(\overline{H}_s)$. By (6.34)

$$\begin{aligned}\chi_1(\gamma)=\chi(x^{-1}\gamma x)&=(\widetilde{\chi}/\widetilde{\chi}_s)(\phi(x^{-1}\gamma x))\chi_s(yx^{-1}\gamma xy^{-1})\\&=(\widetilde{\chi}_1/\widetilde{\chi}_s)(\phi(\gamma))\chi_s((yx^{-1})\gamma(yx^{-1})^{-1}).\end{aligned}\tag{6.51}$$

This proves (6.34) holds for $(\widetilde{\chi}_1,\chi_1)$. □

We summarize the important properties of $\Gamma(H,\Phi^+)$, reformulating them slightly.

Proposition 6.52 *Fix a split Cartan subgroup, a special set Φ_s^+ of positive roots, and $(\widetilde{\chi}_s,\chi_s)\in\mathcal{S}(H_s,\Phi_s^+)$. Suppose H is any Cartan subgroup, Φ^+ is special, and $(h,\widetilde{g})\in X(\overline{H},\widetilde{H})$.*
(1)

$$\Gamma(H,\Phi^+)(h,\widetilde{g})=e^{-\rho}(y)\tau(y,\widetilde{g})\quad(h\in\overline{H}_d^0).\tag{6.53)(a}$$

(2) Suppose $\widetilde{x}\in\widetilde{G}$, $x=\overline{p}(p(\widetilde{x}))$. Then

$$\Gamma(xHx^{-1},Ad(x)\Phi^+)(xhx^{-1},\widetilde{x}\widetilde{g}\widetilde{x}^{-1})=\Gamma(H,\Phi^+)(h,\widetilde{g})\tag{6.53)(b}$$

(3) Suppose $(i=1,2)$ $(h_i,\widetilde{g}_i)\in X(\overline{H},\widetilde{H})$. Then

$$\Gamma(H,\Phi^+)(h_1h_2,\widetilde{g}_1\widetilde{g}_2)=\Gamma(H,\Phi^+)(h_1,\widetilde{g}_1)\Gamma(H,\Phi^+)(h_2,\widetilde{g}_2).\tag{6.53)(c}$$

(4) For $i=1,2$ let H_i be a Cartan subgroup and Φ_i^+ a special set of positive roots for H_i. Assume $(h,\widetilde{g})\in X(\overline{H}_i,\widetilde{H}_i)$ for $i=1,2$. Choose $c\in G_{ad}(\mathbb{C})$ satisfying $Ad(c)\mathfrak{h}_2=\mathfrak{h}_1$, $Ad(c)|_{\mathfrak{h}_1\cap\mathfrak{h}_2}=1$, and $Ad^*(c)(\Phi_1^+)=\Phi_2^+$. Then

$$\Gamma(H_1,\Phi_1^+)(h,\widetilde{g})=\Gamma(H_2,\Phi_2^+)(h,\widetilde{g}).\tag{6.53)(d}$$

Conditions (1,2,4) and Condition (3) for $h_1\in\Gamma(\overline{H})Z(\overline{G}),\widetilde{g}_1\in Z(\widetilde{G})$ uniquely determine the functions $\Gamma(H,\Phi^+)$, given the restriction of $\Gamma(H_s,\Phi_s^+)$ to $(\Gamma(\overline{H}_s)Z(\overline{G})\times Z(\widetilde{G}))\cap X(\overline{H}_s,\widetilde{H}_s)$.

7 Transfer Factors

We now define transfer factors for a general admissible triple $(\widetilde{G}, G, \overline{G})$, generalizing Definition 5.17.

Definition 7.1 *A set of lifting data for $(\widetilde{G}, G, \overline{G})$ is a pair $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$ for some maximally split Cartan subgroup H_s of G.*

The transfer factor $\Delta_{G}^{\widetilde{G}}$ associated to $(\widetilde{\chi}_s, \chi_s)$ is the function on $X'(\overline{G}, \widetilde{G})$ (Definition 5.2) defined as follows.

Suppose $(h, \widetilde{g}) \in X'(\overline{G}, \widetilde{G})$. Let $H = Cent_G(p(\widetilde{g}))$; this is a Cartan subgroup of G. Choose a special set of positive roots Φ^+ for H. Recall $\Gamma(H, \Phi^+)$ is given by Definition 6.35, and Δ^1 is given in (5.1)*(c). Define the transfer factors*

$$\Delta_{G}^{\widetilde{G}}(h, \widetilde{g}) = \frac{\Delta^1(\Phi^+, h)}{\Delta^1(\Phi^+, \widetilde{g})}\Gamma(H, \Phi^+)(h, \widetilde{g}). \tag{7.2}$$

Proposition 7.3 $\Delta_{G}^{\widetilde{G}}$ *is well defined, i.e. independent of the choice of Φ^+.*

Proof. We have to show that $\frac{\Delta^1(\Phi^+,h)}{\Delta^1(\Phi^+,\widetilde{g})}$ and $\Gamma(H, \Phi^+)$ satisfy inverse transformation properties with respect to Φ^+.

Suppose $(h, \widetilde{g}) \in X'(\overline{H}, \widetilde{H})$. By (2.7) write $h = \gamma h_0$ with $\gamma \in \Gamma(\overline{H})$ and $h_0 \in \overline{H}^0$. Suppose Φ_1^+, Φ_2^+ are two choices of special positive roots. By Lemma 6.47

$$\Gamma(H, \Phi_1^+)(h, \widetilde{g}) = \Gamma(H, \Phi_2^+)(h, \widetilde{g})e^{\rho(\Phi_2^+)-\rho(\Phi_1^+)}(h_0). \tag{7.4)(a}$$

By the definition of $\Delta_{G}^{\widetilde{G}}$ it is enough to prove

$$\frac{\Delta^1(\Phi_1^+, h)}{\Delta^1(\Phi_1^+, \widetilde{g})} = \frac{\Delta^1(\Phi_2^+, h)}{\Delta^1(\Phi_2^+, \widetilde{g})}e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h_0). \tag{7.4)(b}$$

Every term factors to $\overline{G}$, so we can replace $\widetilde{g}$ with $\overline{p}(p(\widetilde{g})) = \overline{p}(\phi(h)) = h^2$. This reduces us to showing

$$\frac{\Delta^1(\Phi_1^+, h)}{\Delta^1(\Phi_1^+, h^2)} = \frac{\Delta^1(\Phi_2^+, h)}{\Delta^1(\Phi_2^+, h^2)}e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h_0). \tag{7.5}$$

We first compute for any set of positive roots Φ^+:

$$(7.6)(a)\qquad \begin{aligned}\frac{\epsilon_r(h,\Phi^+)}{\epsilon_r(h^2,\Phi^+)} &= \operatorname{sgn}\frac{\prod_{\alpha\in\Phi_r^+}(1-e^{-\alpha}(h))}{\prod_{\alpha\in\Phi_r^+}(1-e^{-\alpha}(h^2))}\\ &= \operatorname{sgn}\prod_{\alpha\in\Phi_r^+}(1+e^{-\alpha}(\gamma)e^{-\alpha}(h_0))\end{aligned}$$

A given term is positive unless $e^\alpha(\gamma)=-1$ and $e^\alpha(h_0)<1$. Let

$$(7.6)(b)\qquad \Phi_r^+(h)=\{\alpha\in\Phi_r\,|\,e^\alpha(h_0)>1\}.$$

Then

$$(7.6)(c)\qquad \frac{\epsilon_r(h,\Phi^+)}{\epsilon_r(h^2,\Phi^+)}=\prod_{\alpha\in\Phi_r^+\cap(-\Phi_r^+(h))}e^\alpha(\gamma)=e^{\rho_r(\Phi^+)-\rho(\Phi_r^+(h))}(\gamma).$$

Letting $\Phi_{r,j}^+=\Phi_j^+\cap\Phi_r$ we have

$$(7.6)(d)\qquad \frac{\epsilon_r(h,\Phi_1^+)}{\epsilon_r(h^2,\Phi_1^+)}=e^{\rho(\Phi_{r,1}^+)-\rho(\Phi_{r,2}^+)}(\gamma)\frac{\epsilon_r(h,\Phi_2^+)}{\epsilon_r(h^2,\Phi_2^+)}.$$

Now consider the Δ^0 term of (5.1)(c). Write $\Phi_1^+=w\Phi_2^+$ for some $w\in W(\Phi)$, so that

$$(7.6)(e)\qquad \begin{aligned}\frac{\Delta^0(\Phi_1^+,h)}{\Delta^0(\Phi_1^+,h^2)} &= \frac{\epsilon(w)e^{\rho(\Phi_2^+)-\rho(\Phi_1^+)}(h)\Delta^0(\Phi_2^+,h)}{\epsilon(w)e^{\rho(\Phi_2^+)-\rho(\Phi_1^+)}(h^2)\Delta^0(\Phi_2^+,h^2)}\\ &= e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h)\frac{\Delta^0(\Phi_2^+,h)}{\Delta^0(\Phi_2^+,h^2)}\end{aligned}$$

since $\rho(\Phi_1^+)-\rho(\Phi_2^+)$ is a sum of roots.

Now

$$(7.6)(f)\qquad \begin{aligned}&e^{\rho(\Phi_{r,1}^+)-\rho(\Phi_{r,2}^+)}(\gamma)e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h)\\ &\qquad = e^{\rho(\Phi_{r,1}^+)-\rho(\Phi_{r,2}^+)-\rho(\Phi_2^+)+\rho(\Phi_1^+)}(\gamma)e^{-\rho(\Phi_2^+)+\rho(\Phi_1^+)}(h_0)\\ &\qquad = e^{\rho(\Phi_1^+)-\rho(\Phi_2^+)}(h_0)\end{aligned}$$

since γ has order 2 and (by (2.4)(b))

$$(7.6)(g)\qquad e^{\rho(\Phi_1^+)-\rho(\Phi_{r,1}^+)}(\gamma)=\zeta_{cx}(\overline{G},\overline{H})(\gamma)=e^{\rho(\Phi_2^+)-\rho(\Phi_{r,2}^+)}(\gamma).$$

Multiplying (c) and (e), and using (f) shows (7.5), and completes the proof. □

Here are some elementary properties of transfer factors.

Lemma 7.7 *Suppose* $(h,\widetilde{g})\in X(\overline{H},\widetilde{H})$. *Fix a special set of positive roots* Φ^+ *of* H, *and let* $\rho=\rho(\Phi^+)$, $\lambda=\lambda(H,\Phi^+)$ *(cf. 6.36)(b). Let* $g=p(\widetilde{g})$. *Then*

$$\Delta(h,\widetilde{g})^2=\frac{D(h)}{D(g)}e^{2\rho}(h)\lambda(\phi(h)). \tag{7.8)(a}$$

Write $h=th_0$ *with* $t\in\Gamma(\overline{H})Z(\overline{G})$ *and* $h_0\in\overline{H}^0_d$. *Then*

$$\Delta(h,\widetilde{g})^2=\frac{D(h)}{D(g)}e^{2\rho}(t)\lambda(\phi(t)). \tag{7.8)(b}$$

In particular

$$\Delta(h,\widetilde{g})=c(h,\widetilde{g})\lambda(\phi(t))^{\frac12}\frac{|D(h)|^{\frac12}}{|D(\widetilde{g})|^{\frac12}} \tag{7.8)(c}$$

where $c(h,\widetilde{g})^4=1$. *Finally if* λ *is unitary then*

$$|\Delta(h,\widetilde{g})|=\frac{|D(h)|^{\frac12}}{|D(\widetilde{g})|^{\frac12}}. \tag{7.8)(d}$$

Proof. The main point is that

$$\left[\frac{\Delta^0(\Phi^+,h)\epsilon_r(h,\Phi^+)}{\Delta^0(\Phi^+,\widetilde{g})\epsilon_r(\widetilde{g},\Phi^+)}\right]^2=\frac{\Delta^0(\Phi^+,h)^2e^{2\rho}(h)}{\Delta^0(\Phi^+,\widetilde{g})^2e^{2\rho}(\widetilde{g})}\frac{e^{2\rho}(\widetilde{g})}{e^{2\rho}(h)}=\frac{D(h)}{D(\widetilde{g})}e^{2\rho}(h), \tag{7.9}$$

the last equality follows from (5.1)(d) and the fact $\overline{p}(p(\widetilde{g}))=h^2$. Then (7.8)(a) and (7.8)(b) follow from (6.36)(c), and (c) and (d) are elementary consequences of this. □

Remark 7.10 Recall $\mu\in\mathfrak{z}(\mathbb{C})^*$ is given in Definition 6.35. If $\mu=0$ (i.e. $\lambda|_{Z(G)^0}=1$) then in (7.8)(c) $\lambda(\phi(t))^2=1$, and $\Delta(h,\widetilde{g})$ differs from $|D(h)/D(g)|^{\frac12}$ by a fourth root of unity. By varying our choice of lifting data $(\widetilde{\chi}_s,\chi_s)$ we are free to choose the restriction of μ to $Z(G)^0$, up to the constraints (6.8)(a). For simplicity assume $G=\overline{G}$. It is easy to see we can take this restriction to be trivial if and only if $e^\rho(z)=1$ for all $z\in Z(G)^0\cap H^0_d=Z(G)^0\cap Z(G_d)$. This is equivalent to: ρ exponentiates to a character of H^0. By the right hand side of the equality this is independent of the Cartan, and we say G is *real-admissible* if it holds.

This condition is empty unless $Z(G)$ contains a compact torus, and it is weaker than admissibility. For example $GL(2,\mathbb{C})$ is not admissible; $GL(2,\mathbb{R})$ is real-admissible, and $U(1,1)$ is not.

In any event we can always choose the restriction of λ to $Z_0(G)$ to be unitary, so that (7.8)(d) holds.

Example 7.11 Let $G = U(1,1)$ (see Example 3.4). Then G is not real-admissible, since $-I \in Z(G)^0$ and $e^\rho(-I) = -1$. Let T be the compact diagonal Cartan subgroup, and write $X^*(T) = \mathbb{Z}^2$ in the usual coordinates. Let $\widetilde{G}$ be an admissible cover of G. Recall (cf. Example 3.4) $X^*(\widetilde{T}) = \frac{1}{2}\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$.

Suppose $\widetilde{\chi} = (x, y) \in X^*(\widetilde{T})$ and $\chi = (a, b) \in X^*(T)$. Then (6.2)(a) holds if and only if $a - 2x - \frac{1}{2} = b - 2y + \frac{1}{2}$, in which case

(7.12) $$\mu = (a - 2x - \frac{1}{2}, a - 2x - \frac{1}{2}) \in \mathfrak{z}(\mathbb{C})^*.$$

Since $a, 2x \in \mathbb{Z}$ we cannot choose μ to be 0.

The transfer factors satisfy the following invariance property with respect to conjugation by $\widetilde{G}$.

Lemma 7.13 *Suppose $(h, \widetilde{g}) \in X'(\overline{H}, \widetilde{H})$. Fix $\widetilde{x} \in \widetilde{G}$ and let $x = \overline{p}(p(\widetilde{x})) \in \overline{G}$. Then*

(7.14) $$\Delta^{\widetilde{G}}_{\overline{G}}(xhx^{-1}, \widetilde{x}\widetilde{g}\widetilde{x}^{-1}) = \Delta^{\widetilde{G}}_{\overline{G}}(h, \widetilde{g}).$$

Follows easily from Lemma 6.49.

From Lemma 6.19 we see:

Lemma 7.15 *The group of characters of $\overline{G}$ acts simply transitively on the set of transfer factors.*

More precisely suppose for $i = 1, 2$ $(\widetilde{\chi}^i_s, \chi^i_s) \in \mathcal{S}(H_s)$, and let $\Delta^{\widetilde{G}}_{\overline{G}}(\chi^i_s, \widetilde{\chi}^i_s)$ denote the corresponding transfer factors. Then there is a character $\psi : \overline{G} \to \mathbb{C}^\times$ so that for all $(h, \widetilde{g}) \in X'(\overline{G}, \widetilde{G})$

(7.16) $$\Delta^{\widetilde{G}}_{\overline{G}}(\chi^1_s, \widetilde{\chi}^1_s)(h, \widetilde{g}) = \psi(h)\Delta^{\widetilde{G}}_{\overline{G}}(\chi^2_s, \widetilde{\chi}^2_s)(h, \widetilde{g})$$

Conversely, suppose that $\psi : \overline{G} \to \mathbb{C}^\times$ is a character and $(\widetilde{\chi}^1_s, \chi^1_s) \in \mathcal{S}(H_s)$. Then there exists $(\widetilde{\chi}^2_s, \chi^2_s) \in \mathcal{S}(H_s)$ such that (7.16) is satisfied.

Suppose $(h,\widetilde{g})\in X(\overline{H},\widetilde{H})$ and $e^\rho(h)$ is defined. This happens, for example, if $\overline{G}$ is admissible, or if $\overline{G}_d$ is admissible and $h\in\overline{G}_d$. Then

$$\frac{\Delta^1(\Phi^+,h)\epsilon_r(h,\Phi^+)}{\Delta^1(\Phi^+,\widetilde{g})\epsilon_r(\widetilde{g},\Phi^+)}=\frac{\Delta(h,\Phi^+)\epsilon_r(h,\Phi^+)}{\Delta(\widetilde{g},\Phi^+)\epsilon_r(\widetilde{g},\Phi^+)}e^\rho(h). \tag{7.17}$$

Together with (6.9) this gives simpler formulas for the transfer factor in some situations. For example (5.20) holds for $(h,\widetilde{g})$ if $\overline{G}_d$ is acceptable and $h\in\overline{G}_d$.

Example 7.18 Let $G=\overline{G}=GL(n,\mathbb{R})$ and let H be the diagonal split Cartan subgroup of G. Choose $(\widetilde{\chi},\chi)$ as in Example 6.13, and use other notation as in that Example. Suppose $(h,\widetilde{g})\in X'(H,\widetilde{H})$ and write $h=th_0\in\Gamma(H)H^0$. An easy calculation gives

$$\Delta^1(\Phi^+,h)e^\rho(h_0)=|D(h)|^{\frac{1}{2}}. \tag{7.19}$$

From this and (6.14) we compute

$$\Delta(h,\widetilde{g})=\frac{|D(h)|^{\frac{1}{2}}}{|D(\widetilde{g})|^{\frac{1}{2}}}\tau(h_0,\widetilde{g}). \tag{7.20}$$

This agrees with the transfer factors of [19].

Lemma 7.21 *Let H be a Cartan subgroup of G. Suppose $(h,\widetilde{g})\in X'(\overline{H},\widetilde{H})$. Also suppose $z\in Z(\overline{G}),\widetilde{z}\in Z(\widetilde{G})$ and $(z,\widetilde{z})\in X(\overline{H},\widetilde{H})$. Then $(zh,\widetilde{z}\widetilde{g})\in X'(\overline{H},\widetilde{H})$ and*

$$\Delta_{\overline{G}}^{\widetilde{G}}(zh,\widetilde{z}\widetilde{g})=\widetilde{\chi}_s(\widetilde{z})/\chi_s(z)\Delta_{\overline{G}}^{\widetilde{G}}(\widetilde{g},h). \tag{7.22}$$

Proof. Let Φ^+ be a special set of positive roots and choose $(\widetilde{\chi},\chi)\in\mathcal{S}(H,\Phi^+,\widetilde{\chi}_s,\chi_s)$. Since z and $\widetilde{z}$ are central

$$\Delta^1(\Phi^+,zh)=\Delta^1(\Phi^+,h),\ \Delta^1(\widetilde{\widetilde{g}},\Phi^+)=\Delta^1(\Phi^+,\widetilde{g}).$$

Furthermore

$$\Gamma(H,\Phi^+)(zh,\widetilde{z}\widetilde{g})=\Gamma(H,\Phi^+)(z,\widetilde{z})\Gamma(H,\Phi^+)(h,\widetilde{g}) \tag{7.23}$$

and $\Gamma(H,\Phi^+)(z,\widetilde{z})=\widetilde{\chi}_s(\widetilde{z})/\chi_s(z)$ by (6.33). Inserting this into the definition of $\Delta_{\overline{G}}^{\widetilde{G}}$ gives the result. □

8 Some Constants

We need to take care of some constants. Fix an admissible triple $(\widetilde{G}, G, \overline{G})$. Suppose H is a Cartan subgroup of G, with corresponding Cartan subgroups $\widetilde{H}$ of $\widetilde{G}$ and $\overline{H}$ of $\overline{G}$. Let $\phi_{\overline{H}}$ be the restriction of ϕ to $\overline{H}$. Define

Definition 8.1

$$
\begin{aligned}
c(H) &= |Ker(\phi_{\overline{H}})||\widetilde{H}/Z(\widetilde{H})|^{-\frac{1}{2}}|Z(\widetilde{H})/p^{-1}(\phi(\overline{H}))|^{-1} \\
&= |Ker(\phi_{\overline{H}})||H/Z_0(H)|^{-\frac{1}{2}}|Z_0(H)/\phi(\overline{H})|^{-1}
\end{aligned} \tag{8.2}
$$

Fix a maximally split Cartan subgroup H_s of G, and define

$$c = c(H_s), \quad C(H) = c(H)/c(H_s). \tag{8.3}$$

If it is necessary to specify $\overline{G}$ we write $c_{\overline{G}}(H)$, $c_{\overline{G}} = c_{\overline{G}}(H_s)$ and $C_{\overline{G}}(H) = c_{\overline{G}}(H)/c_{\overline{G}}(H_s)$.

Let $H_2 = \{h \in H \,|\, h^2 = 1\}$ and $H_2^0 = H^0 \cap H_2$. Note that $|H_2^0| = 2^{\dim(H\cap K)}$. Let H_s, H_f be maximally split and fundamental Cartan subgroups, respectively.

Proposition 8.4
(1) $c_G(H) = |H_2^0||H/Z_0(H)|^{\frac{1}{2}}$.

(2) $c_{\overline{G}}(H) = c_G(H)\frac{|\Gamma(H)\cap C|}{|C|}$.

(3) $c(H)$ and $C(H)$ are integers, and are powers of 2.

(4) $c = c(H_s) \leq c(H) \leq c(H_f)$.

(5) $1 \leq C(H) \leq C(H_f)$

Proof.
By definition we have

$$
\begin{aligned}
c_{\overline{G}}(H) &= |\mathrm{Ker}(\phi_{\overline{H}})||H/Z_0(H)|^{-\frac{1}{2}}|Z_0(H)/\phi(\overline{H})|^{-1} \\
&= |\mathrm{Ker}(\phi_{\overline{H}})||H/Z_0(H)|^{-\frac{1}{2}}|H/\phi(\overline{H})|^{-1}|H/Z_0(H)| \\
&= |\mathrm{Ker}(\phi_{\overline{H}})||H/Z_0(H)|^{\frac{1}{2}}|H/\phi(\overline{H})|^{-1}.
\end{aligned} \tag{8.5}
$$

If $\overline{G} = G$ then $\mathrm{Ker}(\phi_{\overline{H}}) = H_2$, $\phi(\overline{H}) = H^0$, and

$$c_G(H) = |H_2||H/Z_0(H)|^{\frac{1}{2}}|H/H^0|^{-1}. \tag{8.6}$$

It is easy to see $H \simeq H^0 \times H/H^0$, and H/H^0 is a two-group, which implies $|H_2| = |H_2^0||H/H^0|$. Therefore $c_G(H) = |H_2^0||H/Z_0(H)|^{\frac{1}{2}}$, which is (1).

By (1) and (8.5)

(8.7)(a)
$$c_{\overline{G}}(H)/c_G(H) = \frac{|\mathrm{Ker}(\phi_{\overline{H}})||Z_0(H)/\phi(\overline{H})|^{-1}}{|H_2||Z_0(H)/H^0|^{-1}} = \frac{|\mathrm{Ker}(\phi_{\overline{H}})||\phi(\overline{H})/H^0|}{|H_2|}$$

Recall (Lemma 3.11) $\phi(\overline{H}) = (\Gamma(H) \cap C)H^0$. The composition of maps $\overline{H} \overset{\phi_{\overline{H}}}{\to} \phi_{\overline{H}}(\overline{H}) \to \phi_{\overline{H}}(\overline{H})/H^0$ gives an exact sequence

(8.7)(b)
$$1 \to \frac{\mathrm{Ker}(\phi_{\overline{H}})}{\mathrm{Ker}(\phi_{\overline{H}}) \cap \overline{H}^0} \to \frac{\overline{H}}{\overline{H}^0} \to \frac{\phi_{\overline{H}}(\overline{H})}{H^0} \to 1$$

Solving the resulting identity for $|\phi_{\overline{H}}(\overline{H})/H^0|$, inserting in (8.7)(a) and cancelling terms gives

(8.7)(c)
$$c_{\overline{G}}(H)/c_G(H) = |\overline{H}/\overline{H}^0||\mathrm{Ker}(\phi_{\overline{H}}) \cap \overline{H}^0|/|H_2|.$$

It is easy to see that $\mathrm{Ker}(\phi_{\overline{H}}) \cap \overline{H}^0 = H_2^0/C \cap H^0$, so the right hand side equals

(8.7)(d)
$$\frac{|\overline{H}/\overline{H}^0|}{|H_2/H_2^0||C \cap H^0|}$$

On the other hand there is an exact sequence

(8.7)(e)
$$1 \to \frac{C}{C \cap H^0} \to \frac{H}{H^0} \to \frac{\overline{H}}{\overline{H}^0} \to \Gamma(H) \cap C \to 1.$$

The map to $\Gamma(H) \cap C$ is given by $h \to y^{-1}\sigma(y)$ where $y \in H(\mathbb{C})$ satisfies $\overline{p}(y) = h$. This gives

(8.7)(f)
$$\frac{|\overline{H}/\overline{H}^0|}{|H/H^0|} = \frac{|\Gamma(H) \cap C||C \cap H^0|}{|C|}.$$

Using the fact that $H/H^0 \simeq H_2/H_2^0$ and plugging this into (8.7)(d) gives (2).

We will see in Lemma 10.8 that $|H/Z_0(H)|^{\frac{1}{2}}$ is an integer. Further, $H/Z_0(H)$ is a quotient of H/H^0, which is a two group, so by (1) $c_G(H)$

is an integer and a power of 2. The general case of (3) follows from (1) and (2) if we can show

$$|H_2^0||\Gamma(H)\cap C|/|C|\in\mathbb{Z}. \tag{8.8}$$

To see this note that $C\subset\Gamma(H)H_2^0$. For $g\in C$ write $g=\gamma h$ accordingly, and define $\psi(\gamma h)=h(H_2^0\cap C)$. This is a well defined homomorphism from C to $H_2^0/H_2^0\cap C$. It factors to an inclusion $C/C\cap\Gamma(H)\hookrightarrow H_2^0/H_2^0\cap C$, and (8.8) follows from this. This proves (3) for $c_{\overline{G}}(H)$.

It is enough to prove (4); assertion (3) for $C_{\overline{G}}(H)$ and (5) follow easily. For this we use Cayley transforms. Suppose H_α, H_β are as in Section 4. It is enough to show

$$c_{\overline{G}}(H_\alpha)\le c_{\overline{G}}(H_\beta). \tag{8.9}$$

First assume $\overline{G}=G$. Write $H_\alpha=T_\alpha A_\alpha$, $H_\beta=T_\beta A_\beta$. Then $\dim(T_\beta)=\dim(T_\alpha)+1$, and by part (1) of the Lemma it is enough to show

$$|H_\alpha/Z_0(H_\alpha)|^{\frac12}\le 2|H_\beta/Z_0(H_\beta)|^{\frac12}. \tag{8.10}$$

If α is of type II choose $t\in H_\alpha$ satisfying $\alpha(t)=-1$; otherwise let $t=1$. By (4.13) write $H_\alpha=\langle(H_\alpha\cap H_\beta)B_\alpha,t\rangle$. By (4.26)(a)

$$\begin{aligned}\frac{H_\alpha}{Z_0(H_\alpha)}&\simeq\frac{\langle(H_\alpha\cap H_\beta)B_\alpha,t\rangle}{[Z_0(H_\alpha)\cap Z_0(H_\beta)]B_\alpha}\\&\simeq\frac{\langle H_\alpha\cap H_\beta,t\rangle}{[Z_0(H_\alpha)\cap Z_0(H_\beta)][\langle H_\alpha\cap H_\beta,t\rangle\cap B_\alpha]}\\&\simeq\frac{\langle H_\alpha\cap H_\beta,t\rangle}{Z_0(H_\alpha)\cap Z_0(H_\beta)}\end{aligned}\tag{8.11}$$

since $\langle H_\alpha\cap H_\beta,t\rangle\cap B_\alpha\subset\langle H_\beta,t\rangle\cap B_\alpha=1$. Similarly

$$\begin{aligned}\frac{H_\beta}{Z_0(H_\beta)}&\simeq\frac{(H_\alpha\cap H_\beta)B_\beta}{[Z_0(H_\alpha)\cap Z_0(H_\beta)]B_\beta}\\&\simeq\frac{H_\alpha\cap H_\beta}{[Z_0(H_\alpha)\cap Z_0(H_\beta)][H_\alpha\cap H_\beta\cap B_\beta]}\\&\simeq\frac{H_\alpha\cap H_\beta}{\langle Z_0(H_\alpha)\cap Z_0(H_\beta),m_\alpha\rangle}\end{aligned}\tag{8.12}$$

We have $m_\alpha \in H_\beta^0 \subset Z_0(H_\beta)$. By (4.6)(b) $m_\alpha \in Z_0(H_\alpha)$ if and only if α is of type I. By (8.11) and (8.12) we see $|H_\alpha/Z_0(H_\alpha)| = \tau|H_\beta/Z_0(H_\beta)|$ where $\tau = 1$ (resp. 4) if α is of type I (resp. II). This proves (8.10).

Now supppose $\overline{G}$ is not equal to G. By (2) of the Lemma

(8.13)
$$\frac{c_{\overline{G}}(H_\alpha)}{c_{\overline{G}}(H_\beta)} = \frac{c_G(H_\alpha)}{c_G(H_\beta)} \frac{|\Gamma(H_\alpha) \cap C|}{|\Gamma(H_\beta) \cap C|}$$

We need to show the right hand side is ≤ 1.

By the preceding argument the first quotient on the right hand side is equal to 1 if α is of type II for H_α, or $\frac{1}{2}$ otherwise. It is clear that $\Gamma(H_\beta) \subset \Gamma(H_\alpha)$ of index 1 or 2, so $\Gamma(H_\beta) \cap C$ is of index 1 or 2 in $\Gamma(H_\alpha) \cap C$. We need to show that if α is of type II for H_α then $\Gamma(H_\alpha) \cap C = \Gamma(H_\beta) \cap C$.

By the assumption on α we can choose $t \in \Gamma(H_\alpha)$ such that $\alpha(t) = -1$. Since $\alpha(g) = 1$ for all $g \in \Gamma(H_\beta)$, $t \notin \Gamma(H_\beta)$, so $\Gamma(H_\alpha) = \langle \Gamma(H_\beta), t \rangle$. Therefore $\alpha(g) = -1$ for all $g \in \Gamma(H_\beta)t$, so $\Gamma(H_\alpha) \cap C = \Gamma(H_\beta) \cap C$.

□

In fact we have shown that

(8.14)
$$\frac{c_{\overline{G}}(H_\alpha)}{c_{\overline{G}}(H_\beta)} = \begin{cases} 1 & \alpha \text{ type II for } H_\alpha \\ 1 \text{ or } \frac{1}{2} & \alpha \text{ type I for } H_\alpha \end{cases}$$

(Only 1/2 occurs in the second case if $C = 1$). This implies that for any Cartan subgroup H

(8.15)
$$C_{\overline{G}}(H) \leq 2^{r(H_s) - r(H)}$$

where $r(*)$ denotes split rank.

Example 8.16 If $G = GL(n, \mathbb{R})$ every real root α is of type II, so $c_G(H)$ is independent of H and $C_G(H) = 1$ for all H. See [19].

On the other hand if $G = \overline{G} = U(p, q)$ then every real root is of type I, and equality holds in (8.15).

9 Lifting: Definition and Basic Properties

Assume that $(\widetilde{G}, G, \overline{G})$ is an admissible triple. In this section we define lifting from $\overline{G}$ to $\widetilde{G}$ and derive some of its basic properties. Fix lifting data $(\widetilde{\chi}_s, \chi_s)$ with corresponding transfer factors $\Delta_{\overline{G}}^{\widetilde{G}}$ (Definition 7.1).

Recall (Section 3) $\mathcal{O}(G,g)$ is the conjugacy class of $g\in G$, and

$$\mathcal{O}^{\mathrm{st}}(G,g)=\{xgx^{-1}\,|\,x\in G(\mathbb{C})\}\cap G \tag{9.1}$$

is the stable orbit. We also have $\mathcal{O}(\overline{G},g)$ and $\mathcal{O}^{\mathrm{st}}(\overline{G},g)$, and the map (Lemma 3.6(3)) $\phi:\mathrm{Orb}^{st}(\overline{G})\rightarrow\mathrm{Orb}^{st}(G)$.

Definition 9.2 *Suppose $\widetilde{g}$ is a strongly regular semisimple element of $\widetilde{G}$. Let $\widetilde{H}=p^{-1}(Cent_G(p(\widetilde{g})))$. We say $\widetilde{g}$ and $\mathcal{O}(\widetilde{G},\widetilde{g})$ are relevant if $\widetilde{g}\in Z(\widetilde{H})$.*

The reason for this terminology is:

Lemma 9.3 ([2], Proposition 2.7) *Let $\widetilde{\pi}$ be a genuine admissible representation of $\widetilde{G}$. Suppose $\widetilde{g}$ is a strongly regular semisimple element which is not relevant. Then $\Theta_{\widetilde{\pi}}(\widetilde{g})=0$.*

This is elementary, and goes back to $GL(2)$ [9].

Here are a few basic properties about orbits.

Lemma 9.4 *Suppose $\widetilde{g}$ is a semisimple element of $\widetilde{G}$. Then*

1. $p(\mathcal{O}(\widetilde{G},\widetilde{g}))=\mathcal{O}(G,p(\widetilde{g}))$
2. $p^{-1}(\mathcal{O}(G,p(\widetilde{g})))=\mathcal{O}(\widetilde{G},\widetilde{g})\cup\mathcal{O}(\widetilde{G},-\widetilde{g})$
3. *Suppose $\widetilde{g}$ is relevant and strongly regular. Then $\mathcal{O}(\widetilde{G},\widetilde{g})\neq\mathcal{O}(\widetilde{G},-\widetilde{g})$.*

Proof. Part (1) and (2) are routine. For (3) suppose not. Then $\widetilde{x}\widetilde{g}\widetilde{x}^{-1}=-\widetilde{g}$ for some $\widetilde{x}\in\widetilde{G}$. Let $H=\mathrm{Cent}_G(p(\widetilde{g}))$. Then $p(\widetilde{x})p(\widetilde{g})p(\widetilde{x}^{-1})=p(\widetilde{g})$, so $p(\widetilde{x})\in H$. But then $\widetilde{x}\widetilde{g}\widetilde{x}^{-1}=\widetilde{g}$ since $\widetilde{g}\in Z(\widetilde{H})$ and $\widetilde{x}\in\widetilde{H}$. □

Lemma 9.5 *Suppose $\overline{\mathcal{O}}^{st}\in Orb^{st}(\overline{G})$ and write $\phi(\overline{\mathcal{O}}^{st})=\mathcal{O}^{st}(G,g)$ for some $g\in G$. Assume g is strongly regular. Then there is a unique $h\in\overline{\mathcal{O}}^{st}$ such that $\phi(h)=g$. Furthermore $h\in\overline{H}$ and is also strongly regular.*

Proof. By definition there are $h'\in\overline{\mathcal{O}}^{\mathrm{st}}$ and $g'\in\mathcal{O}^{\mathrm{st}}(G,g)$ such that $\phi(h')=g'$. Let $x\in G(\mathbb{C})$ such that $g'=xgx^{-1}$. Since $g,g'\in G$ we have $\sigma(x)^{-1}x\in\mathrm{Cent}_{G(\mathbb{C})}(g)=H(\mathbb{C})$. Let $y=\overline{p}(x)\in\overline{G}(\mathbb{C})$ and $h=y^{-1}h'y$. Since $g=\phi(h)=s(h)^2$, $s(h)\in\mathrm{Cent}_{G(\mathbb{C})}(g)=H(\mathbb{C})$. But $\sigma(y)^{-1}y\in\overline{H}(\mathbb{C})$ so $\sigma(h)=h$ and $h\in\overline{H}\cap\overline{\mathcal{O}}^{\mathrm{st}}$ with $\phi(h)=g$.

Suppose $r \in \mathrm{Cent}_{\overline{G}(\mathbb{C})}(h)$, and choose a preimage s of r in $G(\mathbb{C})$. Then $g = \phi(h) = \phi(rhr^{-1}) = s\phi(h)s^{-1} = sgs^{-1}$ so that $s \in H(\mathbb{C})$. Thus $r \in \overline{H}(\mathbb{C})$, so that h is also strongly regular.

Suppose that $h_1 \in \overline{\mathcal{O}}^{\mathrm{st}}$ with $\phi(h_1) = g$. Then there is $u \in \overline{G}(\mathbb{C})$ with $uhu^{-1} = h_1$. Choose a preimage v of u in $G(\mathbb{C})$. Then $g = \phi(h_1) = \phi(uhu^{-1}) = v\phi(h)v^{-1} = vgv^{-1}$. Thus $v \in H(\mathbb{C})$. Now $u \in \overline{H}(\mathbb{C})$ so $h_1 = h$. □

Remark 9.6 Note that Lemma 9.5 implies the following. Let $\mathcal{O}^{st} \in \mathrm{Orb}^{st}(G)$ and $\overline{\mathcal{O}}^{\mathrm{st}} \in \mathrm{Orb}^{st}(\overline{G})$ with $\phi(\overline{\mathcal{O}}^{\mathrm{st}}) = \mathcal{O}^{st}$. If $\mathcal{O}^{st}$ is strongly regular and semisimple, then so is $\overline{\mathcal{O}}^{\mathrm{st}}$, and the restriction of ϕ to $\overline{\mathcal{O}}^{\mathrm{st}}$ gives a bijection between $\overline{\mathcal{O}}^{\mathrm{st}}$ and $\mathcal{O}^{\mathrm{st}}$. Further, for any $g \in \mathcal{O}^{\mathrm{st}} \cap H$,

$$\{\overline{\mathcal{O}}^{\mathrm{st}} \mid \phi(\overline{\mathcal{O}}^{\mathrm{st}}) = \mathcal{O}^{\mathrm{st}}\} = \{\mathrm{Orb}^{st}(\overline{G}, h) \mid h \in \overline{H}, \phi(h) = g\}. \tag{9.7}$$

Definition 9.8 *Suppose $\widetilde{\mathcal{O}} \in Orb(\widetilde{G})$ and $\overline{\mathcal{O}}^{st} \in Orb^{st}(\overline{G})$ are strongly regular and semisimple. Let $p(\widetilde{\mathcal{O}})^{st}$ be the stable orbit for G containing $p(\widetilde{\mathcal{O}})$.*

Define $\Delta_{\overline{G}}^{\widetilde{G}}(\overline{\mathcal{O}}^{st}, \widetilde{\mathcal{O}}) = 0$ unless $\phi(\overline{\mathcal{O}}^{st}) = p(\widetilde{\mathcal{O}})^{st}$.

Suppose $\phi(\overline{\mathcal{O}}^{st}) = p(\widetilde{\mathcal{O}})^{st}$. Choose $\widetilde{g} \in \widetilde{\mathcal{O}}$. By Lemma 9.5 there is a unique $h \in \overline{\mathcal{O}}^{st}$ with $\phi(h) = p(\widetilde{g})$. Define

$$\Delta_{\overline{G}}^{\widetilde{G}}(\overline{\mathcal{O}}^{st}, \widetilde{\mathcal{O}}) = \Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g}). \tag{9.9}$$

By Lemma 7.13 this is independent of the choice of $\widetilde{g}$.

Recall a virtual character π of $\overline{G}$ is said to be *stable* if the function θ_π representing its character is constant on the stable orbit $\mathcal{O}^{st}(G, g)$ for all strongly regular semisimple elements g.

Definition 9.10 *Suppose $\overline{\Theta}$ is a stable virtual character of $\overline{G}$. Then $\overline{\Theta}(\overline{\mathcal{O}}^{st})$ is defined for any strongly regular semisimple orbit $\overline{\mathcal{O}}^{st} \in Orb^{st}(\overline{G})$. Suppose $\widetilde{\mathcal{O}} \in Orb(\widetilde{G})$ is strongly regular and semisimple. Recall $c = c_{\overline{G}}$ is given by Definition 8.1.*

Define

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{\mathcal{O}}) = c^{-1} \sum_{\{\overline{\mathcal{O}}^{st} \in Orb^{st}(\overline{G})\}} \Delta_{\overline{G}}^{\widetilde{G}}(\overline{\mathcal{O}}^{st}, \widetilde{\mathcal{O}}) \overline{\Theta}(\overline{\mathcal{O}}^{st}). \tag{9.11(a)}$$

This is a finite sum, over

(9.11)(b) $$\{\overline{\mathcal{O}}^{st}\,|\,\phi(\overline{\mathcal{O}}^{st})=p(\widetilde{\mathcal{O}})^{st}\}.$$

Thus $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}\overline{\Theta}$ is a genuine class function defined on the set of strongly regular semisimple orbits in $\widetilde{G}$. The following lemma follows easily from (9.7) and the fact that ϕ restricted to $\overline{H}$ is a homomorphism. Recall $X(\overline{H},\widetilde{g})$ is given by Definition 5.2.

Lemma 9.12 *Suppose $\widetilde{g}$ is a strongly regular semisimple element. Let $H = Cent_G(p(\widetilde{g}))$ and $\overline{H}=\overline{p}(H)$. Then*

(9.13) $$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{g}) = c^{-1}\sum_{\{h\in X(\overline{H},\widetilde{g})\}}\Delta_{\overline{G}}^{\widetilde{G}}(h,\widetilde{g})\overline{\Theta}(h).$$

In particular $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{g})=0$ *unless $p(\widetilde{g})$ is in the image of ϕ. Assume this holds, and choose h satisfying $\phi(h)=p(\widetilde{g})$. Then*

(9.14) $$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{g}) = c^{-1}\sum_{\{t\in\overline{H}\,|\,\phi(t)=1\}}\Delta_{\overline{G}}^{\widetilde{G}}(th,\widetilde{g})\overline{\Theta}(th).$$

Remark 9.15 Formula (9.13) can be used to extend $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta)$ to a genuine class function on all regular (not just strongly regular) semisimple elements.

We derive some elementary properties of lifting. Recall the definitions of (5.1).

Lemma 9.16 $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})$ *is supported on $Z(\widetilde{G})\widetilde{G}_d^0$.*

Proof. Let $\overline{H}$ be a Cartan subgroup of $\overline{G}$. Then $\overline{H}=Z(\overline{G})\Gamma(\overline{H})\overline{H}_d^0$, so that $\phi(\overline{H})\subset Z_0(G)H_d^0$. Thus $p^{-1}\phi(\overline{H})\subset Z(\widetilde{G})\widetilde{G}_d^0$. □

Define $Z_1(\overline{G})=\{z\in Z(\overline{G}):\phi(z)=1\}$.

Lemma 9.17 *Let $\overline{\Theta}$ be a stable character of $\overline{G}$ with central character ζ. Then* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})\equiv 0$ *unless $\zeta(z)=\chi_s(z)$ for all $z\in Z_1(\overline{G})$.*

Proof. Fix a Cartan subgroup H, $\widetilde{g}\in\widetilde{H}'$, and $z\in Z_1(\overline{G})$. Then

$$\{h\in\overline{H}\,|\,\phi(h)=p(\widetilde{g})\}=\{zh\,|\,h\in\overline{H},\phi(h)=p(\widetilde{g})\}.$$

Further, for all $h \in \overline{H}$ such that $\phi(h) = \widetilde{p}(\widetilde{g})$,

$$\Delta^{\widetilde{G}}_{\overline{G}}(\widetilde{g}, zh) = \chi_s(z^{-1})\Delta^{\widetilde{G}}_{\overline{G}}(\widetilde{g}, h)$$

by Lemma 7.21 applied to $h, \widetilde{g}, z$, and $\widetilde{z} = 1$. We have

$$Lift^{\widetilde{G}}_{\overline{G}}(\overline{\Theta})(\widetilde{g}) = c^{-1} \sum_{X(\overline{H},\widetilde{g})} \Delta^{\widetilde{G}}_{\overline{G}}(\widetilde{g}, zh)\overline{\Theta}(zh) = c^{-1}\zeta(z)\chi_s(z^{-1})Lift^{\widetilde{G}}_{\overline{G}}(\overline{\Theta})(\widetilde{g}).$$

□

Fix a character ζ of $Z(\overline{G})$ satisfying $\zeta(z) = \chi_s(z)$ for all $z \in Z_1(\overline{G})$. For $\widetilde{z} \in \widetilde{S} = p^{-1}\phi(Z(\overline{G}))$ define

(9.18) $$\widetilde{\zeta}(\widetilde{z}) = \widetilde{\chi}_s(\widetilde{z})(\zeta\chi_s^{-1})(z)$$

where $z \in Z(\overline{G})$, $\phi(z) = p(\widetilde{z})$. This is independent of the choice of z, and is a genuine character of $\widetilde{S}$.

Lemma 9.19 *Let $\overline{\Theta}$ be a stable character of $\overline{G}$ with central character ζ such that $\zeta(z) = \chi_s(z)$ for all $z \in Z_1(\overline{G})$. Then for all $\widetilde{g} \in \widetilde{G}, \widetilde{z} \in \widetilde{S}$,*

$$\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\overline{\Theta})(\widetilde{z}\widetilde{g}) = \widetilde{\zeta}(\widetilde{z})Lift^{\widetilde{G}}_{\overline{G}}(\overline{\Theta})(\widetilde{g}).$$

Proof. Fix a Cartan subgroup H of G, $\widetilde{g} \in \widetilde{H}'$ and $\widetilde{z} \in \widetilde{S}$. Choose $z \in Z(\overline{G})$ such that $\phi(z) = p(\widetilde{z})$. Then

$$\{h \in \overline{H} \,|\, (h, \widetilde{z}\widetilde{g}) \in X(\overline{H}, \widetilde{H})\} = \{zh \,|\, h \in \overline{H}, (h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})\}.$$

Thus by Lemma 7.21,

$$\begin{aligned} Lift^{\widetilde{G}}_{\overline{G}}(\overline{\Theta})(\widetilde{z}\widetilde{g}) &= c^{-1} \sum_{X(\overline{H},\widetilde{g})} \Delta^{\widetilde{G}}_{\overline{G}}(\widetilde{z}\widetilde{g}, zh)(\overline{\Theta}(zh) \\ &= (\widetilde{\chi}_s(\widetilde{z})/\chi_s(z))\zeta(z)Lift^{\widetilde{G}}_{\overline{G}}(\overline{\Theta})(\widetilde{g}) = \widetilde{\zeta}(\widetilde{z})Lift^{\widetilde{G}}_{\overline{G}}\overline{\Theta})(\widetilde{g}). \end{aligned}$$

□

10 Lifting for Tori

We give some details of lifting for tori. This illustrates some of the basic principles, and plays an important role in lifting for general reductive groups.

Let $G(\mathbb{C})$ be an algebraic torus, with real points G. Let $p : \widetilde{G} \to G$ be any two-fold cover; any such cover is necessarily admissible (Definition 3.3). Let C be a subgroup of $p(Z(\widetilde{G}))$ consisting of elements of order 2, and let $\overline{G} = G(\mathbb{C})/C$, with real points $\overline{G}$. Then $(\widetilde{G}, G, \overline{G})$ is an admissible triple.

Since G is a torus, $G = H = H_s$ is a maximally split Cartan subgroup. Let $X_g(Z(\widetilde{G}))$ denote the genuine characters of $Z(\widetilde{G})$ and let $X(\overline{G})$ denote the characters of $\overline{G}$. Since $\mathfrak{g}_d = 0$ condition (6.2) is empty, so $(\widetilde{\chi}, \chi) \subset \mathcal{S}(G)$ for all $\widetilde{\chi} \in X_g(Z(\widetilde{G})), \chi \in X(G)$.

The transfer factor $\Delta^{\widetilde{G}}_{\overline{G}}$ of Definition 7.1 is then given by

$$\Delta^{\widetilde{G}}_{\overline{G}}(h, \widetilde{g}) = \widetilde{\chi}(\widetilde{g})\chi(h)^{-1} \quad ((h, \widetilde{g}) \in X(\overline{G}, \widetilde{G})).$$

Let

$$\widetilde{S} = \{\widetilde{g} \in \widetilde{G} \,|\, p(\widetilde{g}) \in \phi(\overline{G})\} = \{\widetilde{g} \,|\, X(\overline{G}, \widetilde{g}) \neq \emptyset\}. \tag{10.1}$$

If $\psi \in X(\overline{G})$ then

$$Lift^{\widetilde{G}}_{\overline{G}}(\psi)(\widetilde{g}) = c^{-1}\widetilde{\chi}(\widetilde{g}) \sum_{h \in X(\overline{G}, \widetilde{g})} \chi^{-1}(h)\psi(h), \tag{10.2}$$

with $c \in \mathbb{Z}$ given by Definition 8.1. In particular $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\psi)(\widetilde{g}) = 0$ for $\widetilde{g} \notin \widetilde{S}$.

Suppose $\widetilde{g} \in \widetilde{S}$ and fix $h \in X(\overline{G}, \widetilde{g})$. Then

$$Lift^{\widetilde{G}}_{\overline{G}}\psi(\widetilde{g}) = c^{-1}\widetilde{\chi}(\widetilde{g})\chi^{-1}(h)\psi(h) \sum_{\{t \in \overline{G} \,|\, \phi(t)=1\}} \chi^{-1}(t)\psi(t). \tag{10.3}$$

Thus $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}\psi = 0$ unless $\chi(t) = \psi(t)$ for all $t \in \mathrm{Ker}(\phi)$. Assume this holds. For $\widetilde{g} \in \widetilde{S}$ choose $h \in X(\overline{G}, \widetilde{g})$ and define

$$\widetilde{\psi}_0(\widetilde{g}) = \widetilde{\chi}(\widetilde{g})\psi(h)\chi(h)^{-1}. \tag{10.4}$$

This is independent of the choice of $h \in X(\overline{G}, \widetilde{g})$, so $\widetilde{\psi}_0$ is a well defined genuine character of $\widetilde{S}$, and it follows immediately that

$$(10.5)\qquad Lift_G^{\widetilde{G}}(\psi)(\widetilde{g}) = \begin{cases} 0 & \widetilde{g} \notin \widetilde{S} \\ c^{-1}|\mathrm{Ker}(\phi)|\widetilde{\psi}_0(\widetilde{g}) & \widetilde{g} \in \widetilde{S}. \end{cases}$$

It follows easily from the induced character formula that

$$(10.6)\qquad \mathrm{Lift}_G^{\widetilde{G}}(\psi) = c^{-1}|\mathrm{Ker}(\phi)||\widetilde{G}/\widetilde{S}|^{-1}\,\mathrm{Ind}_{\widetilde{S}}^{\widetilde{G}}(\widetilde{\psi}_0).$$

By induction by stages we have

$$(10.7)\qquad \mathrm{Ind}_{\widetilde{S}}^{\widetilde{G}}(\widetilde{\psi}_0) = \mathrm{Ind}_{Z(\widetilde{G})}^{\widetilde{G}}\mathrm{Ind}_{\widetilde{S}}^{Z(\widetilde{G})}(\widetilde{\psi}_0)$$

In order to identify this as a sum of characters of $\widetilde{G}$, we use the following elementary result.

Lemma 10.8 ([1], Proposition 2.2) *Let $\widetilde{\psi} \in X_g(Z(\widetilde{G}))$. Then there is a unique irreducible genuine representation $\widetilde{\tau} = \widetilde{\tau}(\widetilde{\psi})$ of $\widetilde{G}$ such that $\widetilde{\tau}|_{Z(\widetilde{G})}$ is a multiple of $\widetilde{\psi}$. The map $\widetilde{\psi} \to \widetilde{\tau}(\widetilde{\psi})$ is a bijection between $X_g(Z(\widetilde{G}))$ and the set of equivalence classes of irreducible genuine representations of $\widetilde{G}$, and*

$$Ind_{Z(\widetilde{G})}^{\widetilde{G}}\widetilde{\psi} = |\widetilde{G}/Z(\widetilde{G})|^{\frac{1}{2}}\widetilde{\tau}(\widetilde{\psi}).$$

The dimension of $\widetilde{\tau}(\widetilde{\psi})$ is $|\widetilde{G}/Z(\widetilde{G})|^{\frac{1}{2}}$; in particular this is an integer.

Let

$$(10.9)\qquad X_g(Z(\widetilde{G}),\widetilde{\psi}_0) = \{\widetilde{\psi} \in X_g(Z(\widetilde{G})) \,|\, \widetilde{\psi}|_{\widetilde{S}} = \widetilde{\psi}_0\}.$$

By Frobenius reciprocity we have

$$(10.10)\qquad \mathrm{Lift}_G^{\widetilde{G}}(\psi) = c^{-1}|\mathrm{Ker}(\phi)||\widetilde{G}/\widetilde{S}|^{-1}|\widetilde{G}/Z(\widetilde{G})|^{\frac{1}{2}} \sum_{\widetilde{\psi}\in X_g(Z(\widetilde{G}),\widetilde{\psi}_0)} \Theta_{\widetilde{\tau}(\widetilde{\psi})}.$$

By (8.2) the term in front of the sum is equal to 1 (of course we defined c precisely to make this hold).

We summarize these results. As defined, $\mathrm{Lift}_G^{\widetilde{G}}(\psi)$ is a class function on $\widetilde{G}$, which by (10.10)) the character of a representation. We identify $\mathrm{Lift}_G^{\widetilde{G}}(\psi)$ with this representation.

Proposition 10.11 *Fix* $\widetilde{\chi} \in X_g(Z(\widetilde{G}))$ *and* $\chi \in X(\overline{G})$*, and use them to define* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}$*. Fix* $\psi \in X(\overline{G})$*. Then* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\psi) \neq 0$ *if and only if* $\chi(t) = \psi(t)$ *for all* $t \in Ker(\phi)$*. Assume this holds. Define* $\widetilde{\psi}_0$ *as in* (10.4) *and* $X_g(Z(\widetilde{G}), \widetilde{\psi}_0)$ *as in* (10.9)*. Then*

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\psi) = \sum_{\widetilde{\psi} \in X_g(Z(\widetilde{G}), \widetilde{\psi}_0)} \widetilde{\tau}(\widetilde{\psi}). \tag{10.12}$$

This is an identity of genuine representations of $\widetilde{G}$*; the right hand side is the direct sum of* $|p(Z(\widetilde{G}))/\phi(\overline{G})|$ *irreducible representations. The differentials satisfy*

$$d\widetilde{\psi} = \frac{1}{2}(d\psi - d\mu) \tag{10.13}$$

where $d\mu = d\chi - 2d\widetilde{\chi} \in \mathfrak{g}(\mathbb{C})^*$ *as in* (6.36)(c)*.*

In particular assume C *is chosen so that* $\phi(\overline{G}) = p(Z(\widetilde{G}))$*. Then* $X_g(Z(\widetilde{G}), \widetilde{\psi}_0)$ *consists of the single character* $\widetilde{\psi}_0$*, and*

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\psi) = \widetilde{\tau}(\widetilde{\psi}_0). \tag{10.14}$$

If it is necessary to indicate the dependence of $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}$ on $\widetilde{\chi}$ and χ we will write

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\widetilde{\chi}, \chi, \psi) = \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\psi). \tag{10.15}$$

Corollary 10.16 *Fix* $(\widetilde{\chi}, \chi)$ *as above, and let* $\widetilde{\psi}$ *be a genuine character of* $Z(\widetilde{G})$*. Then there is a unique character* ψ *of* G *such that* $\widetilde{\tau}(\widetilde{\psi})$ *occurs in* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\widetilde{\chi}, \chi, \psi)$*. It is given by*

$$\psi(h) = \chi(h)(\widetilde{\psi}\widetilde{\chi}^{-1})(\phi(h)) \quad (h \in \overline{G}). \tag{10.17}$$

Example 10.18 Let $G(\mathbb{C})$ be an algebraic torus with connected real points G and let $\widetilde{G}$ be a two-fold cover. Then $\widetilde{G} = Z(\widetilde{G})$ is also abelian. We may as well take $\overline{G} = G$. Let $\widetilde{\chi}$ be any genuine character of $\widetilde{G}$, and let $\chi = \widetilde{\chi}^2$. Suppose $\psi \in X(G)$. Using Proposition 10.11 $\mathrm{Lift}_G^{\widetilde{G}}(\psi) \neq 0$ if and only if (10.4) gives a well-defined genuine character $\widetilde{\psi}$ of $\widetilde{G}$. It is easy to see that in this case $\widetilde{\psi}$ is the unique genuine character with $\psi = \widetilde{\psi}^2$.

Therefore, with this choice of $(\widetilde{\chi}, \chi)$, $Lift_G^{\widetilde{G}}(\psi) \neq 0$ if and only if ψ has a genuine square root $\widetilde{\psi}$ on $\widetilde{G}$, in which case $Lift_G^{\widetilde{G}}(\psi) = \widetilde{\psi}$.

11 Example: Minimal Principal Series of Split Groups

Now assume $(\widetilde{G}, G, \overline{G})$ is an admissible triple where G is split. Let H_s be a split Cartan subgroup of G, let Φ denote the roots of H_s, and let $W = W(\Phi)$. Fix a set of positive roots Φ^+ and choose $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(G, H_s, \Phi^+)$ (cf. 6.4). Let $\rho = \rho(\Phi^+)$ and define $\Delta^{\widetilde{G}}_{\overline{G}}$ as in Definition 7.1 accordingly.

Corresponding to each character ψ of $\overline{H}_s$ is a principal series representation $\pi(\psi)$ of $\overline{G}$. Its character $\Theta_{\pi(\psi)}$ is supported on the conjugates of $\overline{H}_s$ and satisfies

$$\Theta_{\pi(\psi)}(h) = |D_{\overline{G}}(h)|^{-\frac{1}{2}} \sum_{w \in W} \psi(wh) \quad (h \in \overline{H}'_s). \tag{11.1}$$

We wish to compute $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\pi(\psi))$. Using (9.13) it is clear that $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\Theta_{\pi(\psi)})(\widetilde{g}) = 0$ unless $\widetilde{g}$ is conjugate to an element of $\widetilde{H}_s$.

Suppose $\widetilde{g} \in \widetilde{H}'_s$. Then by (9.13)

$$\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\Theta_{\pi(\psi)})(\widetilde{g}) = c^{-1} \sum_{h \in X(\overline{H}_s, \widetilde{g})} \Delta^{\widetilde{G}}_{\overline{G}}(h, \widetilde{g})||D_{\overline{G}}(h)|^{-\frac{1}{2}} \sum_{w \in W} \psi(wh). \tag{11.2}$$

Suppose $h \in X(\overline{H}_s, \widetilde{g})$. From the definitions we have

$$\Delta^{\widetilde{G}}_{\overline{G}}(h, \widetilde{g})|D_{\overline{G}}(h)|^{-\frac{1}{2}} = |D_{\widetilde{G}}(\widetilde{g})|^{-\frac{1}{2}} |e^{\rho}(h)| \widetilde{\chi}_s(\widetilde{g}) \chi_s^{-1}(h). \tag{11.3}$$

If $w \in W$ then $wh \in X(\overline{H}_s, w\widetilde{g})$ and by Lemma 7.13 we have

$$\Delta^{\widetilde{G}}_{\overline{G}}(h, \widetilde{g})|D_{\overline{G}}(h)|^{-\frac{1}{2}} = \Delta^{\widetilde{G}}_{\overline{G}}(wh, w\widetilde{g})|D_{\overline{G}}(wh)|^{-\frac{1}{2}}. \tag{11.4}$$

Therefore $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\Theta_{\pi(\psi)})(\widetilde{g}) =$

$$c^{-1} \sum_{w \in W} |D_{\widetilde{G}}(w\widetilde{g})|^{-\frac{1}{2}} \widetilde{\chi}_s(w\widetilde{g}) \sum_{h \in X(\overline{H}_s, \widetilde{g})} |e^{\rho}(wh)| \chi_s^{-1}(wh) \psi(wh). \tag{11.5}$$

Since $\widetilde{\chi}_s \in X_g(Z(\widetilde{H}_s))$ and $\chi_s|e^{-\rho}| \in X(\overline{H}_s)$, we can use the pair $(\widetilde{\chi}_s, \chi_s|e^{-\rho}|)$ to define $\mathrm{Lift}^{\widetilde{H}_s}_{\overline{H}_s}$ as in §10. By Proposition 10.11 and a short calculation, we can write

$$\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\Theta_{\pi(\psi)}(\widetilde{g}) = |D_{\widetilde{G}}(\widetilde{g})|^{-\frac{1}{2}} \sum_{w \in W} \mathrm{Lift}^{\widetilde{H}_s}_{\overline{H}_s}(\psi))(w\widetilde{g}). \tag{11.6}$$

By Proposition 10.11 the lift is non-zero if and only if

$$\psi(h) = |e^{-\rho}(h)|\chi_s(h) \quad (h \in \overline{H}_s, \phi(h) = 1). \tag{11.7}$$

If $h = m_\alpha$ then by (6.2)(b) the right hand side is trivial ($\zeta_{\mathrm{cx}} = 1$ since there are no complex roots). Therefore $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}} = 0$ independent of the choice of transfer factors unless $\psi(m_\alpha) = 1$ for all $\alpha \in \Phi$.

Assume (11.7) holds. We apply Proposition 10.11 to the summands of (11.6). Let $\widetilde{S} = p^{-1}(\phi(\overline{H}_s)) \subset Z(\widetilde{H}_s)$ and let $n = |Z(\widetilde{H}_s)/\widetilde{S}|^{\frac{1}{2}}$. Define a character of $\widetilde{S}$:

$$\widetilde{\psi}_0(\widetilde{g}) = \widetilde{\chi}_s(\widetilde{g})\psi(h)\chi_s(h)^{-1}|e^{\rho}(h)| \tag{11.8}$$

where $\phi(h) = p(\widetilde{g})$. Write

$$\{\widetilde{\psi} \in X_g(Z(\widetilde{H}_s)) \,|\, \widetilde{\psi}|_{\widetilde{S}} = \widetilde{\psi}_0\} = \{\widetilde{\psi}_1, \dots, \widetilde{\psi}_n\}. \tag{11.9}$$

By Proposition 10.11

$$\mathrm{Lift}_{\overline{H}_s}^{\widetilde{H}_s}(\psi) = \sum_{i=1}^{n} \pi(\widetilde{\tau}(\widetilde{\psi}_i)) \tag{11.10}$$

and plugging this into (11.6) gives

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\pi(\psi)})(\widetilde{g}) = |D_{\widetilde{G}}(\widetilde{g})|^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{w \in W} \Theta_{\widetilde{\tau}(\widetilde{\psi}_i)}(w\widetilde{g}). \tag{11.11}$$

It is straightforward to identify the right hand side of (11.11) with a sum of principal series characters.

Suppose $\widetilde{\tau}$ is an irreducible genuine representation of $\widetilde{H}_s$. Associated to $\widetilde{\tau}$ is a genuine principal series representation $\pi(\widetilde{\tau})$ of $\widetilde{G}$; see Section 16 for details. Then $\Theta_{\pi(\widetilde{\tau})}$ is supported on the conjugates of $Z(\widetilde{H}_s)$, and for $\widetilde{g} \in \widetilde{H}_s'$ we have

$$\Theta_{\pi(\widetilde{\tau})}(\widetilde{g}) = |D_{\widetilde{G}}(\widetilde{g})|^{-\frac{1}{2}} \sum_{w \in W} \mathrm{Tr}(\widetilde{\tau}(w\widetilde{g})). \tag{11.12}$$

Write $\widetilde{\tau} = \widetilde{\tau}(\widetilde{\psi})$ for $\widetilde{\psi}$ a genuine character of $Z(\widetilde{H}_s)$ as in Lemma 10.8, and let $\pi(\widetilde{\psi}) = \pi(\widetilde{\tau}(\widetilde{\psi}))$. Comparing (11.11) and (11.12) we obtain the following result. Let $\mu = \mu(\widetilde{\chi}_s, \chi_s) = d\chi_s - 2d\widetilde{\chi}_s - \rho \in \mathfrak{z}(\mathbb{C})^*$ (Definition 6.35).

Proposition 11.13 *Fix* $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$ *for* $(\widetilde{G}, G, \overline{G})$*, and use this to define* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}$*. Define* $\mathrm{Lift}_{\overline{H}_s}^{\widetilde{H}_s}$ *using* $(\widetilde{\chi}_s, \chi_s|e^{-\rho}|) \in \mathcal{S}(H_s)$ *for* $(\widetilde{H}_s, H_s, \overline{H}_s)$*. Suppose* $\psi \in X(\overline{H}_s)$*. Then* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\pi(\psi)) = 0$ *unless* (11.7) *holds, so assume this is the case. Define* $\mu, \widetilde{S}, n = |Z(\widetilde{H}_s)/\widetilde{S}|$ *and* $\{\widetilde{\psi}_1, \dots, \widetilde{\psi}_n\}$ *as above. Then*

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\pi(\psi)) = \sum_{i=1}^{n} \pi(\widetilde{\psi}_i). \tag{11.14}$$

Each term in the sum has infinitesimal character $\frac{1}{2}(d\psi - \mu)$*.*

Conversely, for $\widetilde{\psi} \in X_g(Z(\widetilde{H}_s))$ *define*

$$\psi(h) = |e^{-\rho}(h)|\chi_s(h)(\widetilde{\psi}\widetilde{\chi}_s^{-1})(\phi(h)) \quad (h \in \overline{H}). \tag{11.15}$$

Then $\Theta_{\pi(\widetilde{\psi})}$ *occurs in* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\pi(\psi))$*.*

Remark 11.16 Note that $\widetilde{H_s^0} \subset \widetilde{S} \subset Z(\widetilde{H}_s) = Z(\widetilde{G})\widetilde{H_s^0}$. This implies $d\widetilde{\psi}_i = \frac{1}{2}(d\psi - \mu)$ (independent of i) and that the restrictions of the $\widetilde{\psi}_i$ to $Z(\widetilde{G})$ are distinct. Therefore the principal series representations $\pi(\widetilde{\psi}_i)$ occuring in (11.14) have distinct central characters, and are *a fortiori* not isomorphic.

Example 11.17 Suppose $G(\mathbb{C})$ is simple and simply connected. In this case lifting is canonical (cf. Section 5) and $\mu = 0$. Also $Z(\widetilde{G}) = \widetilde{Z(G)}$ and $|Z(\widetilde{H}_s)/\widetilde{H_s^0}| = |Z(G)|$. In fact $|Z(G)| = 4$ (in type D_{2n}), 2 (in type A_{2n+1}, D_{2n+1} or E_7) or 1 otherwise. See [1, Table I].

Suppose $\overline{G} = G$. Then $\widetilde{S} = \widetilde{H}_s^0$ and $n = |Z(G)|$. By (11.7) $\mathrm{Lift}_G^{\widetilde{G}}(\pi(\psi)) \neq 0$ if and only if $\pi(\psi)$ is spherical. Thus the lift of the spherical principal series representation with infinitesimal character λ is the sum of the n genuine principal series representations of $\widetilde{G}$ with infinitesimal character $\lambda/2$.

On the other hand suppose $C = Z(G)$; this is allowed by Lemma 3.15. Then $\widetilde{S} = Z(\widetilde{H})$, and the lift consists of a single principal series.

Example 11.18 We specialize the preceding example. Suppose $G = \overline{G} = SL(2, \mathbb{R})$. Write $H_s \simeq \mathbb{R}^\times$ and suppose $\psi_\nu(x) = |x|^\nu$. Then $\widetilde{H}_s \simeq \mathbb{R}^\times \cup i\mathbb{R}^\times$. Suppose $\widetilde{\psi}_\nu^\pm(x) = |x|^\nu$ and $\widetilde{\psi}_\nu^\pm(i) = \pm i$. Then

$$\mathrm{Lift}_G^{\widetilde{G}}(\pi(\psi_\nu)) = \pi(\widetilde{\psi}_{\nu/2}^+) + \pi(\widetilde{\psi}_{\nu/2}^-). \tag{11.19}$$

Note that $Z(\widetilde{G}) = \{\pm 1, \pm i\}$ and $\psi_\nu^\pm$ have different central characters. Also note that the character of $\pi(\widetilde{\psi}_\nu^+) + \pi(\widetilde{\psi}_\nu^-)$ vanishes on $i\mathbb{R}^\times$. In this case $\phi(H_s) = H_s^0$.

Now take $\overline{G} = PSL(2,\mathbb{R}) \simeq SO(2,1)$. Then (see Example 1.11) $\phi : \overline{H}_s \to H_s$ is an isomorphism. By (11.7) the lift of any principal series of $\overline{G}$ is non-zero. With the obvious notation we have

$$\text{Lift}_{\overline{G}}^{\widetilde{G}}(\pi(\psi_\nu^\pm)) = \pi(\widetilde{\psi}_{\nu/2}^\pm). \tag{11.20}$$

12 Example: Discrete Series on the Compact Cartan

Suppose $G = \overline{G}$ is connected, semisimple, acceptable, and equal rank. We assume also that $\widetilde{G}$ is connected. Let H be a compact Cartan subgroup of G. We compute the character formula for the lift of discrete series representation restricted to $\widetilde{H}$.

The group H is connected. Suppose $\lambda \in \mathfrak{h}(\mathbb{C})^*$ satisfies $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\neq 0}$ for all roots α. Associated to λ is a stable sum of discrete series, $\pi^{\text{st}}(\lambda)$ of G, whose character $\Theta_{\pi^{\text{st}}(\lambda)}$ satisfies

$$\Theta_{\pi^{\text{st}}(\lambda)}(h) = (-1)^q \Delta(\Phi^+, h)^{-1} \sum_{w \in W} \epsilon(w) e^{w\lambda}(h) \quad (h \in H'). \tag{12.1}$$

Here $\Phi^+ = \Phi^+(\lambda) = \{\alpha \,|\, \langle \lambda, \alpha^\vee \rangle > 0\}$ and $q = \frac{1}{2}\dim(G/K)$.

Suppose $(h, \widetilde{g}) \in X(H, \widetilde{H})$. By Definition 5.17 (see (5.20)) the transfer factors are canonical, and given by

$$\Delta(h, \widetilde{g}) = \frac{\Delta(h, \Phi^+)}{\Delta(\widetilde{g}, \Phi^+)} \tau(h, \widetilde{g}). \tag{12.2}$$

By (12.1) and (9.13) we have:
(12.3)
$$\text{Lift}_G^{\widetilde{G}}(\Theta_{\pi^{\text{st}}(\lambda)})(\widetilde{g}) = c^{-1}(-1)^q \Delta(\widetilde{g}, \Phi^+)^{-1} \sum_W \epsilon(w) \sum_{h \in X(H, \widetilde{g})} e^{w\lambda}(h) \tau(h, \widetilde{g}).$$

Define $\text{Lift}_H^{\widetilde{H}}$ using lifting data $(\widetilde{\chi}, \widetilde{\chi}^2)$ as in Example 10.18. Then the inner sum is equal to $c(H)\text{Lift}_H^{\widetilde{H}}(e^{w\lambda})(\widetilde{g})$. By Definition 8.1 $C(H) = c(H)/c$, so

$$\text{(12.4)} \qquad \mathrm{Lift}_G^{\widetilde{G}}(\Theta_{\pi^{\mathrm{st}}(\lambda)})(\widetilde{g}) = C(H)(-1)^q\Delta(\widetilde{g},\Phi^+)^{-1}\sum_W \epsilon(w)\mathrm{Lift}_H^{\widetilde{H}}(e^{w\lambda})(\widetilde{g}).$$

By Example 10.18 $\mathrm{Lift}_H^{\widetilde{H}}(e^{w\lambda}) \neq 0$ if and only if $e^{w\lambda/2}$ is a genuine character of $\widetilde{H}$. Let $W_\#$ be the set of $w \in W$ for which this holds. Assume $W_\# \neq \emptyset$. By replacing λ with $w\lambda$ we may assume $1 \in W_\#$. Then

$$\text{(12.5)} \qquad W_\# = \{w \in W \mid w\frac{\lambda}{2} - \frac{\lambda}{2} \in X^*(H)\}$$

where $X^*(H)$ is the character lattice of H. From Lemma 3.2 it is clear that

$$\text{(12.6)} \qquad W(G,H) = \{w \in W \mid w\frac{\lambda}{2} - \frac{\lambda}{2} \in R\} \subset W_\#.$$

where R is the root lattice. A short calculation gives

$$\text{(12.7)} \quad \mathrm{Lift}_G^{\widetilde{G}}(\Theta_{\pi^{\mathrm{st}}(\lambda)})(\widetilde{g}) = C(H)\sum_{w\in W(G,H)\backslash W_\#}(-1)^q\Delta(\widetilde{g},\Phi^+(w\lambda))^{-1}\sum_{v\in W(G,H)}\epsilon(v)e^{vw\lambda/2}(\widetilde{g}).$$

For each $w \in W_\#$ the summand is the formula for the character of $\pi(w\lambda/2)$, the genuine discrete series representation of $\widetilde{G}$ with Harish-Chandra parameter $w\lambda/2$. Therefore

$$\text{(12.8)} \qquad \mathrm{Lift}_G^{\widetilde{G}}(\Theta_{\pi^{\mathrm{st}}(\lambda)})(\widetilde{g}) = C(H)\sum_{w\in W(G,H)\backslash W_\#}\Theta_{\pi(w\lambda/2)}(\widetilde{g}) \quad (\widetilde{g} \in \widetilde{H}').$$

Since formulas for discrete series characters are more complicated on noncompact Cartan subgroups, we will not directly compute $\mathrm{Lift}_G^{\widetilde{G}}(\Theta_{\pi^{\mathrm{st}}(\lambda)})(\widetilde{g})$ for $\widetilde{g}$ in an arbitrary Cartan subgroup. In Section 14 we will use results of Hirai and Harish-Chandra to show that the lift of a stable, supertempered eigendistribution on G is a supertempered eigendistribution on $\widetilde{G}$. Thus both sides of (12.8) are supertempered eigendistributions. Now a theorem of Harish-Chandra that says if two supertempered eigendistributions on $\widetilde{G}$ agree on the compact Cartan subgroup, they are equal. This allows us to conclude that (12.8) is valid for all $\widetilde{g} \in \widetilde{G}'$. See Section 18.

Remark 12.9 If $w \in W(G,H)\backslash W_{\#}$ then $w\frac{\lambda}{2} - \frac{\lambda}{2} \in X^*(H)/R \simeq \widehat{Z(G(\mathbb{C}))}$. Since G is of equal rank $Z(G(\mathbb{C})) = Z(G)$. Therefore $\pi(w\lambda/2)$ and $\pi(\lambda/2)$ have distinct central characters, and the terms in (12.8) are *a fortiori* not isomorphic. Compare Remark 11.16.

Example 12.10 Let $G = Spin(2p,2q)$, and write $\lambda = (a_1,\dots,a_p;b_1,\dots,b_q)$ in the usual coordinates. Then the lift is non-zero if and only if all a_i are even, and all b_i are odd, or vice-versa. If $p \neq q$ then the sum is a singleton, and $\mathrm{Lift}_G^{\widetilde{G}}(\pi(\lambda)) = C(H)\pi(\lambda/2)$. If $p = q$ then the sum consists of two elements, $\lambda/2$ and $\lambda'/2 = \frac{1}{2}(b_1,\dots,b_p;a_1,\dots,a_p)$.

Example 12.11 Suppose G is compact, so $\widetilde{G} \simeq G \times \mathbb{Z}/2\mathbb{Z}$ (which is admissible, but not connected). Then $\pi(\lambda)$ is the finite dimensional representation with infinitesimal character λ. The calculation of $\mathrm{Lift}_G^{\widetilde{G}}(\pi(\lambda))$ is the same as above, except that $\mathrm{Lift}_H^{\widetilde{H}}(e^{w\lambda}) \neq 0$ if and only if $e^{w\lambda/2}$ is a character of H, in which the genuine square root of $e^{w\lambda}$ is $e^{w\lambda/2} \otimes \mathrm{sgn}$. In this case $W = W(G,H) = W_{\#}$ so the sum on the right hand side of (12.8) has one element. Thus

$$\mathrm{Lift}_G^{\widetilde{G}}(\pi(\lambda)) = \pi(\lambda/2) \otimes \mathrm{sgn} \tag{12.12}$$

if $\lambda/2 \in X^*(H)$, or is 0 otherwise.

Also see Example 18.26.

13 Invariant Eigendistributions

In this section we review results of Hirai and Harish-Chandra on invariant eigendistributions. In the next section we apply these results to study lifting from G to $\widetilde{G}$.

Througout this section we assume G is a group in Harish-Chandra's class (see the Appendix). This holds for the groups $(\widetilde{G}, G, \overline{G})$ in an admissible triple (see the beginning of Section (14)). Let H be a Cartan subgroup of G. Recall H' is the set of regular elements of H and suppose $F : H' \to \mathbb{C}$ is differentiable. Corresponding to $X \in \mathfrak{h}$ is the differential operator

$$D_X F(h) = \frac{d}{dt}\bigg|_{t=0} F(h\exp(tX)) \quad (h \in H'). \tag{13.1}$$

Fix a set of positive roots Φ^+, with corresponding $\rho = \rho(\Phi^+)$. Then we also define

$$(13.2) \qquad D^{\rho}_X F(h) = \frac{d}{dt}\bigg|_{t=0} e^{<\rho, tX>} F(h\exp(tX)) \quad (h \in H').$$

The map $D_X \to D^{\rho}_X = < \rho, X > + D_X$ can be extended to an automorphism $D \to D^{\rho}$ of $S(\mathfrak{h}(\mathbb{C}))$.

Let ν be a character of the center $\mathfrak{Z}$ of the universal enveloping algebra of $\mathfrak{g}(\mathbb{C})$. Using the Harish-Chandra homomorphism we identify ν with a character of $I(\mathfrak{h}(\mathbb{C})) = S(\mathfrak{h}(\mathbb{C}))^W$.

We say $F : H' \to \mathbb{C}$ satisfies condition (C1, Φ^+, ν) if F is real analytic on H' and

$$(\text{C1},\Phi^+,\nu) \qquad D^{\rho} F(h) = \nu(D) F(h) \quad (h \in H', D \in I(\mathfrak{h}(\mathbb{C}))).$$

Let $H'(R) = \{h \in H \,|\, e^{\alpha}(h) \neq 1, \alpha \in \Phi_r\}$. We say F satisfies condition (C2) if

(C2) $\qquad$ F extends to a real analytic function on $H'(R)$.

Let G' be the set of regular semisimple elements of G, and suppose Θ is a class function on G'. With $\Delta^1(\Phi^+, h)$ defined as in (5.1) let

$$(13.3) \qquad \Psi(H, \Phi^+, h) = \Delta^1(\Phi^+, h)\Theta(h) \quad (h \in H').$$

Assume $\Psi(H, \Phi^+)$ satisfies (C1, Φ^+, ν) and (C2).

Let α be a simple root of Φ^+_r. Let J be the corresponding Cayley transform of H as in Section 4. Choose $c \in \mathrm{Aut}(\mathfrak{g}(\mathbb{C}))$ satisfing (4.14)(a) and (b), and let $\beta = c^*(\alpha)$ be the imaginary root of J corresponding to α. Let $\Phi^+_J = c^*\Phi^+$ and $\rho_J = \rho(\Phi^+_J)$. Let $H(\alpha)$ be the set of $h \in H$ such that $e^{\alpha}(h) = 1$, but $e^{\gamma}(h) \neq 1$ for all $\gamma \neq \pm\alpha$.

Fix $h \in H(\alpha)$. Then $h\exp(t\alpha^\vee) \in H'$ for $0 \neq t$ sufficiently small, and there are well-defined one-sided limits

$$(13.4) \qquad D^{\rho;\pm}_{\alpha^\vee}\Psi(H, \Phi^+, h) = \lim_{t \to 0^{\pm}} D^{\rho}_{\alpha^\vee}\Psi(H, \Phi^+, h\exp(t\alpha^\vee)).$$

Note that $i\beta^\vee \in \mathfrak{j}$ and $h \in J'(R)$. Thus $D^{\rho_J}_{\beta^\vee} = -iD^{\rho_J}_{i\beta^\vee}$ is defined as in (13.2), and $D^{\rho_J}_{\beta^\vee}\Psi(J, \Phi^+_J, h)$ is defined. We say $\Psi(H, \Phi^+)$ satisfies condition (C3) if

(C3) $$[D^{\rho,+}_{\alpha^\vee} - D^{\rho,-}_{\alpha^\vee}]\Psi(H,\Phi^+,h) = 2D^{\rho_J}_{\beta^\vee}\Psi(J,\Phi^+_J,h) \quad (h \in H(\alpha)).$$

The following theorem was proved by Hirai [15], [17] assuming two extra conditions (see the Appendix) which may fail to hold in our situation. In the Appendix we extend Hirai's theorem to a class of reductive groups that includes Harish-Chandra's class.

Theorem 13.5 (cf. Theorem 20.23) *Let Θ be a class function on G' and let ν be a character of $\mathfrak{Z}$. Then Θ is an invariant eigendistribution with infinitesimal character ν if and only for every Cartan subgroup H of G and every real root α of H, there is a choice of positive roots Φ^+ such that α is simple for Φ^+_r and the function $\Psi(H,\Phi^+)$ satisfies conditions (C1, Φ^+,ν), (C2) , and (C3).*

Moreover, suppose Θ is an invariant eigendistribution. Then $\Psi(H,\Phi^+)$ satisfies (C1, Φ^+,ν) and (C2) for every choice of Φ^+, and satisfies (C3) for every choice of Φ^+ such that α is simple for Φ^+_r.

We now review some results of Harish-Chandra. Write $G = K\exp(\mathfrak{p})$ where K is a maximal compact subgroup and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition. Let $\|\cdot\|_G$ be a Euclidean norm on $\mathfrak{p}$ and define

(13.6) $$\tau_G(k\exp X) = \|X\|,\ k \in K, X \in \mathfrak{p}.$$

Let Θ be an invariant eigendistribution on G. Then by §12 of [11], Θ is tempered if and only if for each Cartan subgroup H of G there are numbers $C, r \geq 0$, so that

(13.7) $$|D(h)|^{\frac{1}{2}}|\Theta(h)| \leq C(1+\tau_G(h))^r,\ h \in H',$$

(see 5.1 for the definition of $|D(h)|^{\frac{1}{2}}$).

Let $\mathfrak{z}$ be the center of $\mathfrak{g}$. As in [11] we can decompose $G = {}^0GZ_{\mathfrak{p}}$ where ${}^0G$ has Lie algebra $[\mathfrak{g},\mathfrak{g}] \oplus (\mathfrak{z} \cap \mathfrak{k})$ and $Z_{\mathfrak{p}} = \exp(\mathfrak{z} \cap \mathfrak{p})$. Let Θ be an invariant eigendistribution on G. We say Θ is relatively tempered (relatively supertempered) on G if its restriction to ${}^0G$ is tempered (supertempered) on ${}^0G$. By §4 of [12], Θ is supertempered on ${}^0G$ if and only if for every Cartan subgroup ${}^0H$ of ${}^0G$ and every $r \geq 0$,

(13.8) $$\sup_{g \in {}^0H'} |D(h)|^{\frac{1}{2}}|\Theta(h)|(1+\tau_G(h))^r < \infty.$$

We will also use the following result from §4 of [12].

Theorem 13.9 *[12] Assume that G has a relatively compact Cartan subgroup B and let Θ be a relatively supertempered invariant eigendistribution such that $\Theta(b) = 0, b \in B'$. Then $\Theta = 0$.*

14 Lifting of Invariant Eigendistributions

We now return to the context of lifting. Suppose $(\widetilde{G}, G, \overline{G})$ is an admissible triple. We note that each of the groups $\widetilde{G}, G$, and $\overline{G}$ are in Harish-Chandra's class (see the Appendix). For G and $\overline{G}$ this is immediate. For $\widetilde{G}$ note that $\mathrm{Cent}_{\widetilde{G}}(\widetilde{G}^0) = \widetilde{Z(G)}$ by (4.3). Therefore $\mathrm{Ad}(\widetilde{G}) \simeq \mathrm{Ad}(G)$, so $\widetilde{G}$ also satisfies Condition A of the Appendix, and the remaining conditions are easy. Therefore the results of Section 13 apply.

Fix a maximally split Cartan subgroup H_s of G and lifting data $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$ as in Definition 7.1. Recall $\mu \in \mathfrak{z}(\mathbb{C})^*$ is given by (6.36)(c). Let Θ be a stable invariant eigendistribution on $\overline{G}$. Define $\widetilde{\Theta} = \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}\Theta$ as in Definition 9.10; $\widetilde{\Theta}$ is a genuine class function on $\widetilde{G}'$.

Theorem 14.1 *Let Θ be a stable invariant eigendistribution on $\overline{G}$ with infinitesimal character ν. Then $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}\Theta$ is an invariant eigendistribution on $\widetilde{G}$ with infinitesimal character $(\nu - \mu)/2$.*

We apply Theorem 13.5. We need to show that conditions (C1-3) hold.

Let H be a Cartan subgroup of G, and let Φ^+ be a choice of positive roots. Define

(14.2)(a) $$\Psi(\overline{H}, \Phi^+, h) = \Delta^1(\Phi^+, h)\Theta(h) \quad (h \in \overline{H}'),$$

(14.2)(b) $$\Psi(\widetilde{H}, \Phi^+, \widetilde{g}) = \Delta^1(\Phi^+, \widetilde{g})\widetilde{\Theta}(\widetilde{g}) \quad (\widetilde{g} \in \widetilde{H}').$$

The second part of Theorem 13.5 says:

Lemma 14.3 $\Psi(\overline{H}, \Phi^+)$ *satisfies conditions (C1, Φ^+, ν) and (C2) for every choice of Φ^+. Let $\alpha \in \Phi_r^+$. Then $\Psi(\overline{H}, \Phi^+)$ satisfies condition (C3) for any choice of Φ^+ such that α is a simple root for Φ_r^+.*

We now verify conditions (C1)-(C3) for $\Psi(\widetilde{H}, \Phi^+)$. We begin with (C1). Note that μ defines a character of $I(\mathfrak{h}(\mathbb{C})) = S(\mathfrak{h}(\mathbb{C}))^W$ and also by the Harish-Chandra homomorphism a character of $\mathfrak{Z}$. Assume Φ^+ is a special set of positive roots.

Proposition 14.4 *For all $D \in I(\mathfrak{h}(\mathbb{C}))$ and $\widetilde{g} \in \widetilde{H}'$,*

$$D^\rho \Psi(\widetilde{H}, \Phi^+, \widetilde{g}) = ((\nu - \mu)/2)(D)\Psi(\widetilde{H}, \Phi^+, \widetilde{g}).$$

In other words $\Psi(\widetilde{H}, \Phi^+)$ satisfies condition (C1,Φ^+,$(\nu-\mu)/2$).

Since $\widetilde{H}, \overline{H}$ and Φ^+ are fixed we write $\widetilde{\Psi}(\widetilde{g}) = \Psi(\widetilde{H}, \Phi^+, \widetilde{g})$ and $\overline{\Psi}(h) = \Psi(\overline{H}, \Phi^+, h)$. Let $\Gamma = \Gamma(H, \Phi^+)$ (Definition 6.35). Thus for $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$, $\Gamma(h, \widetilde{g}) = \widetilde{\chi}(\widetilde{g})\chi^{-1}(h)$ where $(\widetilde{\chi}, \chi) \in S(H, \Phi^+, \widetilde{\chi}_s, \chi_s)$.

Lemma 14.5 *For all $\widetilde{g} \in \widetilde{H}, X \in \mathfrak{h}$ such that $\widetilde{g}\widetilde{\exp}X \in \widetilde{H}'$,*

$$\widetilde{\Psi}(\widetilde{g}\widetilde{\exp}X) = c^{-1} \sum_{h \in X(\overline{H}, \widetilde{g})} \Gamma(h, \widetilde{g}) e^{-<\rho+\mu, X/2>} \overline{\Psi}(h\overline{\exp}(X/2)).$$

Proof. Let $\widetilde{g} \in \widetilde{H}'$. Recall the definition of $X(\overline{H}, \widetilde{g})$ (5.3)(c). From the definitions it is clear that

$$\widetilde{\Psi}(\widetilde{g}) = c^{-1} \sum_{h \in X(\overline{H}, \widetilde{g})} \Gamma(h, \widetilde{g}) \overline{\Psi}(h).$$

Fix $\widetilde{g} \in \widetilde{H}, X \in \mathfrak{h}$ such that $\widetilde{g}\widetilde{\exp}X \in \widetilde{H}'$. Then $\phi(\overline{\exp}(X/2)) = \widetilde{\exp}X$ so

$$X(\overline{H}, \widetilde{g}\widetilde{\exp}X) = \{h\overline{\exp}(X/2) : h \in X(\overline{H}, \widetilde{g})\}.$$

Thus

$$\widetilde{\Psi}(\widetilde{g}\widetilde{\exp}X) = \sum_{h \in X(\overline{H}, \widetilde{g})} \Gamma(h, \widetilde{g})\Gamma(\overline{\exp}(X/2), \widetilde{\exp}X)\overline{\Psi}(h\overline{\exp}(X/2).$$

But

$$\Gamma(\overline{\exp}(X/2), \widetilde{\exp}X) = e^{<2d\widetilde{\chi} - d\chi, X/2>} = e^{-<\rho+\mu, X/2>}$$

by (6.36)(c). □

Proof of Proposition 14.4.

It follows from Lemmas 14.3 and 14.5 that $\widetilde{\Psi}$ is real analytic on $\widetilde{H}'$. Suppose $X \in \mathfrak{h}$ and $t \in \mathbb{R}$. Using Lemma 14.5 we have

$$e^{<\rho,tX>}\widetilde{\Psi}(\widetilde{g}\widetilde{\exp}tX) = c^{-1} \sum_{h \in X(\overline{H},\widetilde{g})} \Gamma(h,\widetilde{g}) e^{<\rho-\mu,tX/2>}\overline{\Psi}(h\overline{\exp}(tX/2)).$$

Thus

$$D_X^\rho \widetilde{\Psi}(\widetilde{g}) = c^{-1} \sum_{h \in X(\overline{H},\widetilde{g})} \Gamma(h,\widetilde{g})(D^\rho_{X/2} - <\mu, X/2>)\overline{\Psi}(h).$$

First assume $X \in \mathfrak{z}$. Then $D_{X/2} \in I(\mathfrak{h}(\mathbb{C}))$, so $D^\rho_{X/2}\overline{\Psi}(h) = \nu(X/2)\overline{\Psi}(h)$ by Lemma 14.3. Thus

$$D_X^\rho \widetilde{\Psi}(\widetilde{g}) = c^{-1} \sum_{h \in X(\overline{H},\widetilde{g})} \Gamma(h,\widetilde{g}) < \nu - \mu, X/2 > \overline{\Psi}(h) = ((\nu-\mu)/2)(D_X)\widetilde{\Psi}(\widetilde{g}).$$

By induction we see that

$$D^\rho \widetilde{\Psi}(\widetilde{g}) = ((\nu-\mu)/2)(D)\widetilde{\Psi}(\widetilde{g}) \quad \text{for all } D \in S(\mathfrak{z}(\mathbb{C})). \tag{14.6}$$

Now suppose $X \in \mathfrak{h}_d$. Then $<\mu, X>= 0$, so

$$D_X^\rho \widetilde{\Psi}(\widetilde{g}) = \frac{1}{2c} \sum_{h \in X(\overline{H},\widetilde{g})} \Gamma(h,\widetilde{g}) D_X^\rho \overline{\Psi}(h). \tag{14.7}$$

Now suppose $Y \in \mathfrak{h}_d$, replace $\widetilde{g}$ with $\widetilde{g}\widetilde{\exp}(tY)$, and multiply both sides by $e^{\langle\rho,tY\rangle}$. The right hand side is

$$e^{\langle\rho,tY\rangle}\frac{1}{2c} \sum_{h \in X(\overline{H},\widetilde{g})} \Gamma(h\overline{\exp}(tY/2), \widetilde{g}\widetilde{\exp}(tY)) D_X^\rho \overline{\Psi}(h\overline{\exp}(tY/2)). \tag{14.8}$$

As in the proof of Lemma 14.5 we can write

$$\Gamma(h\overline{\exp}(tY/2), \widetilde{g}\widetilde{\exp}(tY)) = \Gamma(h,\widetilde{g})e^{\langle -\rho, tY/2\rangle}$$

and we see

(14.9)

$$e^{\langle\rho,tY\rangle} D_X^\rho \widetilde{\Psi}(\widetilde{g}\widetilde{\exp}(tY)) = \frac{1}{2c} \sum_{h \in X(\overline{H},\widetilde{g})} \Gamma(h,\widetilde{g}) e^{\langle\rho,tY/2\rangle} D_X^\rho \overline{\Psi}(h\overline{\exp}(tY/2)).$$

Taking the derivative and evaluating at $t = 0$ we see

$$D^\rho_Y D^\rho_X \widetilde{\Psi}(\widetilde{g}) = \frac{1}{4c} \sum_{h \in X(\overline{H}, \widetilde{g})} \Gamma(h, \widetilde{g}) D^\rho_Y D^\rho_X \overline{\Psi}(h). \tag{14.10}$$

By induction we conclude that for $D \in I(\mathfrak{h}_d(\mathbb{C}))$, homogeneous of degree k, we have

$$\begin{aligned} D^\rho \widetilde{\Psi}(\widetilde{g}) &= \frac{1}{2^k c} \sum_{h \in X(\overline{H}, \widetilde{g})} \Gamma(h, \widetilde{g}) D^\rho \overline{\Psi}(h) \\ &= \frac{1}{2^k c} \nu(D) \sum_{h \in X(\overline{H}, \widetilde{g})} \Gamma(h, \widetilde{g}) \overline{\Psi}(h) \\ &= (\nu/2)(D) \widetilde{\Psi}(\widetilde{g}) \\ &= ((\nu - \mu)/2)(D) \widetilde{\Psi}(\widetilde{g}). \end{aligned} \tag{14.11}$$

The last equality comes from the fact that $\mu(D) = 0$.

We have $I(\mathfrak{h}(\mathbb{C})) = I(\mathfrak{h}_d(\mathbb{C})) S(\mathfrak{z}(\mathbb{C}))$, and the result follows from (14.11) and (14.6). □

We now consider condition (C2). We continue to write $\widetilde{\Psi}(\widetilde{g}) = \Psi(\widetilde{H}, \Phi^+, \widetilde{g})$ and $\overline{\Psi}(h) = \Psi(\overline{H}, \Phi^+, h)$. Recall

$$\widetilde{H}'(R) = \{\widetilde{g} \in \widetilde{H} \mid e^\alpha(\widetilde{g}) \neq 1,\ \alpha \in \Phi_r\}. \tag{14.12}$$

Proposition 14.13 *$\widetilde{\Psi}$ extends to a real analytic function on $\widetilde{H}'(R)$, i.e. $\widetilde{\Psi}$ satisfies condition (C2).*

Proof. Since the restriction of $\widetilde{\Theta}$ to $\widetilde{H}$ is supported on $Z(\widetilde{G})\widetilde{H}^0$, $\widetilde{\Psi}$ is also supported on $Z(\widetilde{G})\widetilde{H}^0$. Fix $\widetilde{t} \in Z(\widetilde{G})\widetilde{T}^0$. If Φ_r^+ is a set of positive roots of Φ_r let

$$C(\Phi_r^+) = \{X \in \mathfrak{a} \mid \alpha(X) > 0 \text{ for all } \alpha \in \Phi_r^+\}. \tag{14.14}$$

Then the connected components of $\widetilde{t}\widetilde{H^0} \cap \widetilde{H}'(R)$ are of the form $\widetilde{t}\widetilde{T^0}\widetilde{\exp}(C(\Phi_r^+))$.

Fix $h \in X(\overline{H}, \widetilde{t})$. Note that for all $\alpha \in \Phi_r$, $e^{2\alpha}(h) = e^\alpha(\widetilde{t}) = 1$, so that $e^\alpha(h) = \pm 1$. Let $\Phi_r(h) = \{\alpha \in \Phi_r \mid e^\alpha(h) = 1\}$. Then the connected components of $h\overline{H}^0 \cap \overline{H}'(R)$ have the form $h\overline{T}^0\overline{\exp}(C(\Phi_r^+(h)))$ with $C(\Phi_r^+(h))$ defined as in (14.14).

Given Φ_r^+, let $\Phi_r^+(h) = \Phi_r^+ \cap \Phi_r(h)$. If $X \in \mathfrak{t} + C(\Phi_r^+)$ then $X/2 \in \mathfrak{t} + C(\Phi_r^+(h))$. Thus $X \to e^{<-\rho-\mu, X/2>}\overline{\Psi}(h\overline{\exp}(X/2))$ extends to a real analytic function on $\mathfrak{t} + C(\Phi_r^+)$. Therefore

$$X \to \sum_{h \in X(\overline{H}, \widetilde{t})} \Gamma(h, \widetilde{t}) e^{<-\rho-\mu, X/2>}\overline{\Psi}(h\overline{\exp}(X/2)) \tag{14.15}$$

extends to a real analytic function on $\mathfrak{t} + C(\Phi_r^+)$.

By Lemma 14.5, for all $X \in \mathfrak{h}$ such that $\widetilde{t}\widetilde{\exp}X \in \widetilde{H}'$,

$$\widetilde{\Psi}(\widetilde{t}\widetilde{\exp}X) = c^{-1} \sum_{h \in X(\overline{H}, \widetilde{t})} \Gamma(h, \widetilde{t}) e^{<-\rho-\mu, X/2>}\overline{\Psi}(h\overline{\exp}(X/2)). \tag{14.16}$$

By (14.15) there is an open neighborhood U of $\widetilde{t}$ in $\widetilde{t}\widetilde{T}^0$ such that $\widetilde{\Psi}$ extends to be real analytic on $U\widetilde{\exp}(C(\Phi_r^+))$. □

We now turn to condition (C3). Fix $\alpha \in \Phi_r$ and let J be the Cayley transform of H as in Section 4. Choose $c \in \mathrm{Aut}(\mathfrak{g}(\mathbb{C}))$ satisfying (4.14)(a) and (b), and let $\beta = c^*(\alpha)$.

Lemma 14.17 *Assume that Φ^+ is a special set of positive roots, $\alpha \in \Phi^+$, and that $\Phi_J^+ = c^*\Phi^+$ is also special. Then α is a simple root for Φ_r^+.*

Proof. A basic property of Cayley transforms is that for any root γ of H, $\sigma(c^*\gamma) = c^*(s_\alpha\sigma\gamma)$. Suppose $\gamma \in \Phi_r^+$, $\gamma \neq \alpha$. Then

$$\sigma(c^*\gamma) = c^*(s_\alpha\gamma). \tag{14.18}$$

If $\langle\gamma, \alpha^\vee\rangle = 0$ then $s_\alpha(\gamma) = \gamma \in \Phi_r^+$. If $\langle\gamma, \alpha^\vee\rangle \neq 0$ then $c^*\gamma \in \Phi_J^+$, and by (14.18) $c^*\gamma$ is complex. Since Φ_J^+ is special $\sigma(c^*\gamma) \in \Phi_J^+$, and so by (14.18) again $s_\alpha\gamma \in \Phi_r^+$. Therefore $s_\alpha\gamma \in \Phi_r^+$ for all $\alpha \neq \gamma \in \Phi_r^+$, so α is simple for Φ_r^+. □

Proposition 14.19 *Suppose Φ^+ is a special set of positive roots for H, such that Φ_J^+ is a special set of positive roots for J. Let $\widetilde{g} \in \widetilde{H}(\alpha)$. Then*

$$[D_{\alpha^\vee}^{\rho,+} - D_{\alpha^\vee}^{\rho,-}]\Psi(\widetilde{H}, \Phi^+, \widetilde{g}) = 2D_{\beta^\vee}^{\rho_J}\Psi(\widetilde{J}, \Phi_J^+, \widetilde{g}).$$

That is $\Psi(\widetilde{H}, \Phi^+)$ satisfies condition (C3).

Proof. Define $\Gamma(H,\Phi^+)$ and $\Gamma(J,\Phi^+_J)$ as in Definition 6.35.

By (14.7) we have

$$[D^{\rho,+}_{\alpha^\vee}-D^{\rho,-}_{\alpha^\vee}]\Psi(\widetilde{H},\Phi^+,\widetilde{g})=\frac{1}{2c}\sum_{h\in X(\overline{H},\widetilde{g})}\Gamma(H,\Phi^+,h,\widetilde{g})[D^{\rho,+}_{\alpha^\vee}-D^{\rho,-}_{\alpha^\vee}]\Psi(\overline{H},\Phi^+,h).$$

If $h\in X(\overline{H},\widetilde{g})$ then for all $\gamma\in\Phi$ by (3.10) we have $e^{2\gamma}(h)=e^\gamma(\widetilde{g})$. Thus $e^\alpha(h)=\pm1$ and $e^\gamma(h)\neq\pm1$ for all $\gamma\neq\pm\alpha$. Thus $X(\overline{H},\widetilde{g})=X(\overline{H},\widetilde{g},\alpha)_+\cup X(\overline{H},\widetilde{g},\alpha)_-$ where

$$X(\overline{H},\widetilde{g},\alpha)_\pm=\{h\in X(\overline{H},\widetilde{g}):e^\alpha(h)=\pm1\}. \tag{14.20}$$

Suppose that $h\in X(\overline{H},\widetilde{g},\alpha)_-$. Then h is regular so that

$$D^{\rho,+}_{\alpha^\vee}\Psi(\overline{H},\Phi^+,h)=D^{\rho,-}_{\alpha^\vee}\Psi(\overline{H},\Phi^+,h)\quad(h\in X(\overline{H},\widetilde{g},\alpha)_-). \tag{14.21}$$

On the other hand if $h\in X(\overline{H},\widetilde{g},\alpha)_+$, since $\Psi(\overline{H},\Phi^+)$ satisfies (C3),

$$[D^{\rho,+}_{\alpha^\vee}-D^{\rho,-}_{\alpha^\vee}]\Psi(\overline{H},\Phi^+,h)=2D^{\rho_J}_{\beta^\vee}\Psi(\overline{J},\Phi^+_J,h) \tag{14.22}$$

where $\beta=c^*\alpha$. Therefore

(14.23)(a)

$$[D^{\rho,+}_{\alpha^\vee}-D^{\rho,-}_{\alpha^\vee}]\Psi(\widetilde{H},\Phi^+,\widetilde{g})=c^{-1}\sum_{h\in X(\overline{H},\widetilde{g},\alpha)_+}\Gamma(H,\Phi^+,h,\widetilde{g})D^{\rho_J}_{\beta^\vee}\Psi(\overline{J},\Phi^+_J,h).$$

On the other hand, by (14.7) applied to $\widetilde{J}$

$$2D^{\rho_J}_{\beta^\vee}\Psi(\widetilde{J},\Phi^+_J,\widetilde{g})=c^{-1}\sum_{b\in X(\overline{J},\widetilde{g})}\Gamma(J,\Phi^+_J,b,\widetilde{g})D^{\rho_J}_{\beta^\vee}\Psi(\overline{J},\Phi^+_J,b). \tag{14.23)(b}$$

Write $X(\overline{J},\widetilde{g})=X(\overline{J},\widetilde{g},\beta)_+\cup X(\overline{J},\widetilde{g},\beta)_-$ as above, so we have

$$\begin{aligned}2D^{\rho_J}_{\beta^\vee}\Psi(\widetilde{J},\Phi^+_J,\widetilde{g})=c^{-1}&\sum_{b\in X(\overline{J},\widetilde{g},\beta)_+}\Gamma(J,\Phi^+_J,b,\widetilde{g})D^{\rho_J}_{\beta^\vee}\Psi(\overline{J},\Phi^+_J,b)\\+c^{-1}&\sum_{b\in X(\overline{J},\widetilde{g},\beta)_-}\Gamma(J,\Phi^+_J,b,\widetilde{g})D^{\rho_J}_{\beta^\vee}\Psi(\overline{J},\Phi^+_J,b)\end{aligned} \tag{14.23)(c}$$

Suppose $h\in X(\overline{J},\widetilde{g},\beta)_+$. Then $h\in\overline{H}$, so $h\in X(\overline{H},\widetilde{g},\alpha)_+$, and $X(\overline{J},\widetilde{g},\beta)_+=X(\overline{H},\widetilde{g},\alpha)_+$. Also by Lemma 6.37 $\Gamma(H,\Phi^+,h,\widetilde{g})=\Gamma(J,\Phi^+_J,h,\widetilde{g})$. Therefore

$$(14.23)(d)\quad \begin{aligned} 2D^{\rho_J}_{\beta^\vee}\Psi(\widetilde{J},\Phi^+_J,\widetilde{g}) &= c^{-1}\sum_{b\in X(\overline{H},\widetilde{g},\alpha)_+}\Gamma(H,\Phi^+,b,\widetilde{g})D^{\rho_J}_{\beta^\vee}\Psi(\overline{J},\Phi^+_J,b)\\ &+c^{-1}\sum_{b\in X(\overline{J},\widetilde{g},\beta)_-}\Gamma(J,\Phi^+_J,b,\widetilde{g})D^{\rho_J}_{\beta^\vee}\Psi(\overline{J},\Phi^+_J,b)\end{aligned}$$

It is enough to show the second sum on the right hand side is 0, for then the result follows from (14.23)(a) and (14.23)(d). We restate this as:

Lemma 14.24 *Suppose α is an imaginary noncompact root and $\widetilde{g}\in\widetilde{H}(\alpha)$. Then*

$$(14.25)\qquad \sum_{h\in X(\overline{H},\widetilde{g},\alpha)_-}\Gamma(h,\widetilde{g})D^{\rho}_{\alpha^\vee}\Psi(\overline{H},\Phi^+,h)=0.$$

Proof. Since $\overline{H}$ and Φ^+ are fixed let $\overline{\Psi}(h)=\Psi(\overline{H},\Phi^+,h)$. Let w be the reflection in α and suppose $h\in\overline{H}'$. Since α is imaginary $\epsilon_R(\Phi^+,wh)=\epsilon_R(\Phi^+,h)$, and since Θ is stable $\Theta(wh)=\Theta(h)$. Furthermore since $\epsilon(w)=-1$,

$$\Delta^0(\Phi^+,wh)=-e^{\rho-w\rho}(h)\Delta^0(\Phi^+,h).$$

Thus

$$\overline{\Psi}(wh)=-e^{\rho-w\rho}(h)\overline{\Psi}(h).$$

Now assume $h\in X(\widetilde{g},\alpha)_-$, so $e^\alpha(h)=-1$. Then

$$(14.26)\qquad wh=h\overline{\exp}(\pi i\alpha^\vee).$$

Therefore $\phi(wh)=p(\widetilde{g})$ and $e^\alpha(wh)=-1$, so $wh\in X(\widetilde{g},\alpha)_-$. Also, for all $t\in\mathbb{R}$,

$$(14.27)\quad e^{<\rho,it\alpha^\vee>}\overline{\Psi}((wh)\overline{\exp}(it\alpha^\vee))=-e^{\rho-w\rho}(h)e^{<\rho,-it\alpha^\vee>}\overline{\Psi}(h\overline{\exp}(-it\alpha^\vee)).$$

It follows from this and the definitions that

$$(14.28)\qquad D^\rho_{\alpha^\vee}\overline{\Psi}(wh)=e^{\rho-w\rho}(h)D^\rho_{\alpha^\vee}\overline{\Psi}(h).$$

From (14.26) we have $e^{\rho-w\rho}(h)=(-1)^{\langle\rho,\alpha^\vee\rangle}$ and

$$(14.29)\qquad \Gamma(wh,\widetilde{g})=\frac{\widetilde{\chi}(\widetilde{g})}{\chi(wh)}=\frac{\widetilde{\chi}(\widetilde{g})}{\chi(h)\chi(\overline{\exp}(\pi i\alpha^\vee))}=\Gamma(h,\widetilde{g})(-1)^{\langle d\chi,\alpha^\vee\rangle},$$

where $(\widetilde{\chi}, \chi) \in S(H, \Phi^+, \widetilde{\chi}_s, \chi_s)$. Thus

$$\Gamma(wh, \widetilde{g}) D^{\rho}_{\alpha^\vee} \Psi(\overline{H}, wh) = (-1)^{<d\chi+\rho,\alpha^\vee>} \Gamma(h, \widetilde{g}) D^{\rho}_{\alpha^\vee} \Psi(\overline{H}, h). \tag{14.30}$$

By (6.2)

$$\langle d\chi + \rho, \alpha^\vee \rangle = \langle 2d\widetilde{\chi} + 2\rho, \alpha^\vee \rangle \equiv \langle 2d\widetilde{\chi}, \alpha^\vee \rangle \pmod 2.$$

By [3, Section 7] $\langle 2d\widetilde{\chi}, \alpha^\vee \rangle = 1 \pmod 2$. Therefore

$$\Gamma(wh, \widetilde{g}) D^{\rho}_{\alpha^\vee} \Psi(\overline{H}, wh) = -\Gamma(h, \widetilde{g}) D^{\rho}_{\alpha^\vee} \Psi(\overline{H}, h). \tag{14.31}$$

Therefore if $wh \neq h$ then the two terms cancel, and if $wh = h$, $D^{\rho}_{\alpha^\vee}\Psi(h) = 0$.
□

This completes the proof of Proposition 14.19. □

Theorem 14.1 follows from Theorem 13.5, Propositions 14.4, 14.13 and 14.19.

Remark 14.32 The only place that stability is used in the proof of Theorem 14.1 is in the proof of Lemma 14.24.

It is easy to see that lifting preserves the property of being relatively tempered and supertempered, and (provided $\mu \in i\mathfrak{z}^*$) tempered.

Proposition 14.33 *Let Θ be a stable invariant eigendistribution on $\overline{G}$.*

1. *If Θ is relatively tempered then $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}\Theta$ is relatively tempered.*

2. *If Θ is relatively supertempered then $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}\Theta$ is relatively supertempered.*

3. *Assume $\mu \in i\mathfrak{z}^*$. If Θ is tempered then $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}\Theta$ is tempered.*

Proof. Since $\overline{G}$ and $\widetilde{G}$ have the same Lie algebra, we can use the same Euclidean norm $\|\cdot\|$ on $\mathfrak{p}$ to define both τ_H and $\tau_{\widetilde{G}}$.

Assume Θ is tempered and $\mu \in i\mathfrak{z}^*$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Fix $\widetilde{g} \in \widetilde{H}'$ and write $\widetilde{g} = \widetilde{t}\widetilde{\exp} X$ where $\widetilde{t} \in \widetilde{T}', X \in \mathfrak{a}$. Then $\tau_{\widetilde{G}}(\widetilde{g}) = \|X\|$. Further, for any $h \in \overline{H}$ such that $\phi(h) = p(\widetilde{g})$, we have $h = t\overline{\exp} X/2$ where $t \in \overline{T}$ with $\phi(t) = p(\widetilde{t})$. Thus $\tau_H(h) = \|X/2\| = \tau_{\widetilde{G}}(\widetilde{g})/2$.

Now, using (7.8)(d) and (13.7) applied to $\overline{G}$, there are $C, r \geq 0$ such that

$$|D(\widetilde{g})|^{\frac{1}{2}}|\widetilde{\Theta}(\widetilde{g})| \leq c^{-1} \sum_{h \in X(\overline{H}\widetilde{g})} |D(\widetilde{g})|^{\frac{1}{2}}|\Delta_{\overline{G}}^{\widetilde{G}}(h,\widetilde{g})||\Theta(h)|$$

$$= c^{-1} \sum_{h \in X(\overline{H}\widetilde{g})} |D(h)|^{\frac{1}{2}}|\Theta(h)| \leq c^{-1}[\overline{H}_1]C(1+\tau_{\widetilde{G}}(\widetilde{g})/2)^r,$$

where $\overline{H}_1 = \{h \in \overline{H} : \phi(h) = 1\}$. Thus there is a constant C' such that

$$|D(\widetilde{g})|^{\frac{1}{2}}|\widetilde{\Theta}(\widetilde{g})| \leq C'(1+\tau_{\widetilde{G}}(\widetilde{g}))^r$$

for all $\widetilde{g} \in \widetilde{H}'$. Now $\widetilde{\Theta}$ is tempered using (13.7) applied to $\widetilde{G}$. This proves (3).

Part (1) is similar, using the fact that by (7.8)(c) $|D(\widetilde{g})|^{\frac{1}{2}}|\Delta_{\overline{G}}^{\widetilde{G}}(h,\widetilde{g})| = |D(h)|^{\frac{1}{2}}$ for any choice of $(\widetilde{\chi}_s, \chi_s)$ provided $\widetilde{g} \in {}^0\widetilde{G}$. Part (2) is also similar, using (13.8) in place of (13.7). □

15 Cuspidal Levi Subgroups

Let $(\widetilde{G}, G, \overline{G})$ be an admissible triple, and fix a cuspidal Levi subgroup M of G. Let $\widetilde{M} = p^{-1}(M)$, and $\overline{M} = \overline{p}(M)$. These are cuspidal Levi subgroups of $\widetilde{G}$ and $\overline{G}$ respectively.

Proposition 15.1 $(\widetilde{M}, M, \overline{M})$ *is an admissible triple (Definition 3.14).*

Proof. It is well known that M is the set of real points of the connected reductive complex group $M(\mathbb{C})$ and that $M_d(\mathbb{C})$ is acceptable, and it is clear that every simple factor of $M(\mathbb{C})$ is oddly laced. Let Φ_M be the set of roots of $H(\mathbb{C})$ in $M(\mathbb{C})$ where $H = TA$ is a relatively compact Cartan subgroup of M. Since $\widetilde{G}$ is an admissible cover of G, every noncompact root in Φ_M is metaplectic. Thus $p : \widetilde{M} \to M$ is also an admissible cover. Further, $\overline{M}$ is the set of real points of $\overline{M}(\mathbb{C}) = M(\mathbb{C})/C$, where $C \subset Z_0(G) \subset Z_0(M)$ with $c^2 = 1$ for all $c \in C$. This leaves only condition 3(c) of Definition 3.14.

Let Φ^+ be a special set of positive roots for H in G and let $\Phi_M^+ = \Phi^+ \cap \Phi_M$. Let $\rho = \rho(\Phi^+)$ and $\rho_M = \rho(\Phi_M^+)$. Then $\rho - \rho_M$ is zero on $\mathfrak{t}$. But $C \cap M_d^0 \subset T^0$, so that $e^\rho(c) = e^{\rho_M}(c)$ for all $c \in C \cap M_d^0$. Thus if $\widetilde{\chi}$ is a genuine character of $Z(\widetilde{M})$, then $\widetilde{\chi}^2(c)e^{\rho_M}(c) = \widetilde{\chi}^2(c)e^\rho(c) = 1$ for all $c \in C \cap M_d^0$. □

Suppose H is an arbitrary Cartan subgroup of M. Let Φ and Φ_M be the sets of roots of $H(\mathbb{C})$ in $G(\mathbb{C})$ and $M(\mathbb{C})$, respectively. Let Φ^+ be a special set of positive roots of Φ, and let $\Phi_M^+ = \Phi^+ \cap \Phi_M$. This is a special set of positive roots of Φ_M. Suppose χ is a character of H. Define

$$\chi_M(h) = |e^{\rho_M - \rho}(h)|\chi(h) \quad (h \in \overline{H}). \tag{15.2}$$

To be precise we are writing $|e^{\rho_M-\rho}(h)|$ for $|e^{2\rho_M-2\rho}(h)|^{\frac{1}{2}}$. Note that $\Phi_i \subset \Phi_M$, so $2\rho_M - 2\rho$ is a sum of real and complex roots; since Φ^+ is special this takes real values on $\mathfrak{h}$. Therefore $\rho_M - \rho$ exponentiates to $\overline{H}^0$, and

$$|e^{\rho_M-\rho}(h)| = e^{\rho_M-\rho}(h) \quad (h \in \overline{H}^0). \tag{15.3}$$

In particular let H_s be a maximally split Cartan subgroup of M. This is also a maximally split Cartan subgroup of G. Let Φ_s^+ be a special set of positive roots for $H_s(\mathbb{C})$ in $G(\mathbb{C})$. Let $\Phi_{M,s}$ be the roots of $H_s(\mathbb{C})$ in $M(\mathbb{C})$, and let $\Phi_{M,s}^+ = \Phi_s^+ \cap \Phi_{M,s}$.

Lemma 15.4 *Fix $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s, \Phi_s^+)$.*

(1) Let $\chi_{M,s} = (\chi_s)_M$ (cf. 15.2). Then $(\widetilde{\chi}_s, \chi_{M,s}) \in \mathcal{S}(H_s, \Phi_{M,s}^+)$.

(2) Let H be any Cartan subgroup of M and suppose $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+, \widetilde{\chi}_s, \chi_s)$. Then $(\widetilde{\chi}, \chi_M) \in \mathcal{S}(H, \Phi_M^+, \widetilde{\chi}_s, \chi_{M,s})$.

(3) Let $\mu = \mu(\widetilde{\chi}_s, \chi_s)$ and $\mu_M = \mu(\widetilde{\chi}_s, \chi_{M,s})$ (defined with respect to M). Then $\mu = \mu_M$.

Proof. Fix $h \in (\overline{H}_s \cap M_d)^0$. By (15.3) $\chi_{M,s}(h) = \chi_s(h)e^{\rho_M-\rho}(h)$. On the other hand since $(\overline{H}_s \cap M_d)^0 \subset (\overline{H}_s \cap G_d)^0$, by (6.2)(a) we have $\chi_s(h) = (\widetilde{\chi}^2 e^{\rho})(h)$. Therefore

$$\chi_{M,s}(h) = \chi_s(h)e^{\rho_M-\rho}(h) = (\widetilde{\chi}^2 e^{\rho})(h)e^{\rho_M-\rho}(h) = (\widetilde{\chi}_s^2 e^{\rho_M})(h). \tag{15.5}$$

Therefore (6.2)(a) holds.

Next let $\Gamma_r(\overline{G}, \overline{H})$ (resp. $\Gamma_r(\overline{M}, \overline{H})$) be the group $\Gamma_r(\overline{H})$ with respect to the group $\overline{G}$ (resp. $\overline{M}$) (cf. (2.6)). Then $\Gamma_r(\overline{M}, \overline{H}) \subset \Gamma_r(\overline{G}, \overline{H})$. Fix $h \in \Gamma_r(\overline{M}, \overline{H})$. Then $|e^{\rho_M-\rho}(h)| = 1$, and by Lemma 6.45

$$\chi_{M,s}(h) = \chi_s(h) = \zeta_{cx}(\overline{G}, \overline{H})(h) = \zeta_{cx}(\overline{M}, \overline{H})(h). \tag{15.6}$$

This verifies condition (6.2)(b), and proves (1).

Now consider (2). Let χ_0 be the character of $\overline{H}$ such that $(\widetilde{\chi}, \chi_0) \in \mathcal{S}(H, \Phi_M^+, \widetilde{\chi}_s, \chi_{M,s})$ (cf. Proposition 6.31 and Definition 6.35). We need to show $\chi_0 = \chi_M$. We may assume $A \subset A_s$. Recall $\overline{H} = \Gamma(\overline{H})Z(\overline{G})\overline{H}_d^0$. First assume $h \in \Gamma(\overline{H})Z(\overline{G}) \subset \Gamma(\overline{H}_s)Z(\overline{G})$. Then $|e^{\rho_M-\rho}(h)| = |e^{\rho_{M,s}-\rho_s}(h)| = 1$, so $\chi_M(h) = \chi(h)$ and $\chi_{M,s}(h) = \chi_s(h)$. Therefore

$$\begin{aligned}\chi_0(h) &= (\widetilde{\chi}/\widetilde{\chi}_s)(\phi(h))\chi_{M,s}(h) \quad \text{(by (6.34))}\\ &= (\widetilde{\chi}/\widetilde{\chi}_s)(\phi(h))\chi_s(h)\\ &= \chi(h) \quad \text{(by (6.34))}\\ &= \chi_M(h).\end{aligned} \tag{15.7}$$

Now suppose $h = (\overline{H} \cap M_d)^0 \subset \overline{H}_d^0$. Then $\chi(h) = (\widetilde{\chi}^2 e^\rho)(h)$, and

$$\chi_0(h) = (\widetilde{\chi}^2 e^{\rho_M})(h) = (\chi e^{-\rho} e^{\rho_M})(h) = \chi_M(h). \tag{15.8}$$

Next let $X \in \mathfrak{h}_d \cap \mathfrak{z}_M$ where $\mathfrak{z}_M$ is the center of $\mathfrak{m}$. Write $h = \overline{\exp}X$. Then:

$$\begin{aligned}\chi_0(h) &= (\widetilde{\chi}/\widetilde{\chi}_s)(\phi(h))\chi_{M,s}(h)\\ &= (\widetilde{\chi}/\widetilde{\chi}_s)(\phi(h))\chi_s(h)e^{\rho_{M,s}-\rho_s}(h)\\ &= (\widetilde{\chi}/\widetilde{\chi}_s)(\phi(h))(\widetilde{\chi}_s^2 e^{\rho_s})(h)e^{\rho_{M,s}-\rho_s}(h)\\ &= (\widetilde{\chi}/\widetilde{\chi}_s)(\exp 2X)(\widetilde{\chi}_s^2)(\exp X)e^{<\rho_s,X>}e^{<\rho_{M,s}-\rho_s,X>}\end{aligned} \tag{15.9}$$

But $(\widetilde{\chi}/\widetilde{\chi}_s)(\exp 2X) = (\widetilde{\chi}/\widetilde{\chi}_s)^2(\exp X)$. Also $< \rho_{M,s}, X >=< \rho_M, X >= 0$ since $X \in \mathfrak{z}_M$. Therefore

$$\begin{aligned}\chi_0(h) &= \widetilde{\chi}^2(\exp X)\\ &= \chi(h)e^{-<\rho,X>} \quad \text{(by (6.2)(a))}\\ &= \chi(h)e^{<\rho_M-\rho,X>} = \chi_M(h).\end{aligned} \tag{15.10}$$

The result follows from the fact that $\overline{H}_d^0 = \overline{\exp}(\mathfrak{h}_d \cap \mathfrak{z}_M)(\overline{H} \cap M_d)^0$.

For the final assertion $\mu = d\chi_s - 2d\widetilde{\chi}_s - \rho$ and $\mu_M = d\chi_{M,s} - 2d\widetilde{\chi}_s - \rho_M$, and the result follows immediately from (15.2). □

Now fix lifting data $(\widetilde{\chi}_s, \chi_s)$ for G (Definition 7.1.) By Part (1) of the Lemma $(\widetilde{\chi}_s, \chi_{M,s})$ is lifting data for $(\widetilde{M}, M, \overline{M})$, and we use it to define transfer factors $\Delta_{\overline{M}}^{\widetilde{M}}$ and $\text{Lift}_{\overline{M}}^{\widetilde{M}}$. The remainder of this section is devoted to proving that with these choices, lifting commutes with parabolic induction.

Suppose $(h, \widetilde{g}) \in X(\overline{H}, \widetilde{H})$. We need to compare the transfer factors defined for G and for M. Recall

$$\text{(15.11)(a)} \qquad \Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g}) = \frac{\epsilon_r(h, \Phi^+)\Delta^0(\Phi^+, h)\widetilde{\chi}(\widetilde{g})}{\epsilon_r(\widetilde{g}, \Phi^+)\Delta^0(\Phi^+, \widetilde{g})\chi(h)}$$

$$\text{(15.11)(b)} \qquad \Delta_{\overline{M}}^{\widetilde{M}}(h, \widetilde{g}) = \frac{\epsilon_r(h, \Phi_M^+)\Delta^0(\Phi_M^+, h)\widetilde{\chi}(\widetilde{g})}{\epsilon_r(\widetilde{g}, \Phi_M^+)\Delta^0(\Phi_M^+, \widetilde{g})\chi_M(h)}.$$

Lemma 15.12 *Suppose* $(h, \widetilde{g}) \in X'(\overline{H}, \widetilde{H})$*. Then*

$$\text{(15.13)} \qquad \Delta_{\overline{G}}^{\widetilde{G}}(h, \widetilde{g})\frac{|D_{\widetilde{G}}(\widetilde{g})|^{\frac{1}{2}}}{|D_{\overline{G}}(h)|^{\frac{1}{2}}} = \Delta_{\overline{M}}^{\widetilde{M}}(h, \widetilde{g})\frac{|D_{\widetilde{M}}(\widetilde{g})|^{\frac{1}{2}}}{|D_{\overline{M}}(h)|^{\frac{1}{2}}}.$$

Proof. We first expand the left hand side, using (15.11)(a) and (5.1)(d) for the $|D|^{\frac{1}{2}}$ terms. Let Φ^+ be a special set of positive roots. The result is

$$\text{(15.14)} \qquad \frac{\epsilon_r(h, \Phi^+)\Delta^0(\Phi^+, h)|\Delta^0(\Phi^+, \widetilde{g})||e^\rho(\widetilde{g})|\widetilde{\chi}(\widetilde{g})}{\epsilon_r(\widetilde{g}, \Phi^+)\Delta^0(\Phi^+, \widetilde{g})|\Delta^0(\Phi^+, h)||e^\rho(h)|\chi(h)}.$$

Write Φ_r^+, Φ_i^+ and Φ_{cx}^+ for the real, complex and imaginary roots in Φ^+, respectively. Then with the obvious notation

$$\text{(15.15)} \qquad \Delta^0(h, \Phi^+) = \Delta^0(h, \Phi_r^+)\Delta^0(h, \Phi_{cx}^+)\Delta^0(h, \Phi_i^+).$$

Then

$$\text{(15.16)} \qquad \begin{aligned} \epsilon_r(h, \Phi^+)\Delta^0(\Phi_r^+, h) &= |\Delta^0(\Phi_r^+, h)| \\ \Delta^0(\Phi_{cx}^+, h) &= |\Delta^0(\Phi_{cx}^+, h)| \end{aligned}$$

Therefore

$$\text{(15.17)} \qquad \frac{\epsilon_r(h, \Phi^+)\Delta^0(\Phi^+, h)|\Delta^0(\Phi^+, \widetilde{g})|}{\epsilon_r(\widetilde{g}, \Phi^+)\Delta^0(\Phi^+, \widetilde{g})|\Delta^0(\Phi^+, h)|} = \frac{\Delta^0(\Phi_i^+, h)|\Delta^0(\Phi_i^+, \widetilde{g})|}{\Delta^0(\Phi_i^+, \widetilde{g})|\Delta^0(\widetilde{h}, \Phi_i^+)|}.$$

A similar argument applies to the right hand side of (15.13). Since $\Phi_i = \Phi_{M,i}$ and the right hand side of (15.17) only depends on the imaginary roots, we conclude

$$\text{(15.18)} \quad \frac{\epsilon_r(h, \Phi^+)\Delta^0(\Phi^+, h)|\Delta^0(\Phi^+, \widetilde{g})|}{\epsilon_r(\widetilde{g}, \Phi^+)\Delta^0(\Phi^+, \widetilde{g})|\Delta^0(\Phi^+, h)|} = \frac{\epsilon_r(h, \Phi_M^+)\Delta^0(\Phi_M^+, h)|\Delta^0(\Phi_M^+, \widetilde{g})|}{\epsilon_r(\widetilde{g}, \Phi_M^+)\Delta^0(\Phi_M^+, \widetilde{g})|\Delta^0(\Phi_M^+, h)|}$$

It remains to show

$$\frac{|e^{\rho}(\widetilde{g})|\widetilde{\chi}(\widetilde{g})}{|e^{\rho}(h)|\chi(h)} = \frac{|e^{\rho_M}(\widetilde{g})|\widetilde{\chi}(\widetilde{g})}{|e^{\rho_M}(h)|\chi_M(h)}. \tag{15.19}$$

Recall $\chi_M(h) = \chi(h)|e^{\rho_M-\rho}(h)|$. Inserting this it suffices to show

$$|e^{\rho-\rho_M}(\widetilde{g})| = |e^{\rho-\rho_M}(h)|^2. \tag{15.20}$$

which follows from the fact that $|e^{2\rho-2\rho_M}(h)| = |e^{\rho-\rho_M}(\phi(h))| = |e^{\rho-\rho_M}(\widetilde{g})|$.
□

We can rewrite the Lemma in terms of orbits.

Lemma 15.21 *Suppose $\widetilde{\mathcal{O}}_{\widetilde{M}} \in Orb(\widetilde{M})$ and $\mathcal{O}^{st}_{\overline{M}} \in Orb^{st}(\overline{M})$ are strongly regular and semisimple, and satisfy $\phi(\mathcal{O}^{st}_{\overline{M}}) = p(\widetilde{\mathcal{O}}_{\widetilde{M}})^{st}$.*

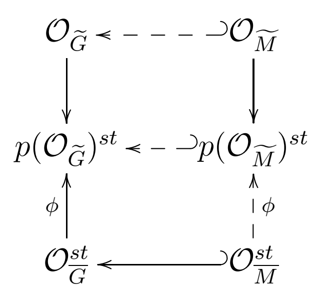
Let $\widetilde{\mathcal{O}}_{\widetilde{G}}$ be the unique $\widetilde{G}$-orbit containing $\widetilde{\mathcal{O}}_{\widetilde{M}}$ and let $\mathcal{O}^{st}_{\overline{G}}$ be the unique stable $\overline{G}$-orbit containing $\mathcal{O}^{st}_{\overline{M}}$. Then $\phi(\mathcal{O}^{st}_{\overline{G}}) = p(\widetilde{\mathcal{O}}_{\widetilde{G}})^{st}$, and

$$\Delta^{\widetilde{G}}_{\overline{G}}(\mathcal{O}^{st}_{\overline{G}}, O_{\widetilde{G}})\frac{|D_{\widetilde{G}}(\mathcal{O}_{\widetilde{G}})|^{\frac{1}{2}}}{|D_{\overline{G}}(\mathcal{O}^{st}_{\overline{G}})|^{\frac{1}{2}}} = \Delta^{\widetilde{M}}_{\overline{M}}(\mathcal{O}^{st}_{\overline{M}}, \mathcal{O}_{\widetilde{M}})\frac{|D_{\widetilde{M}}(\mathcal{O}_{\widetilde{M}})|^{\frac{1}{2}}}{|D_{\overline{M}}(\mathcal{O}^{st}_{\overline{M}})|^{\frac{1}{2}}}.$$

Lemma 15.22 *Suppose $\mathcal{O}_{\widetilde{G}} \in Orb(\widetilde{G})$ is strongly regular and $\mathcal{O}^{st}_{\overline{M}} \in Orb^{st}(\overline{M})$. Let $\mathcal{O}^{st}_{\overline{G}}$ be the unique stable orbit of $\overline{G}$ containing $\mathcal{O}^{st}_{\overline{M}}$. Then*

$$\phi(\mathcal{O}^{st}_{\overline{G}}) = p(\mathcal{O}_{\widetilde{G}})^{st} \Leftrightarrow \phi(\mathcal{O}^{st}_{\overline{M}}) = p(\mathcal{O}_{\widetilde{M}})^{st} \tag{15.23}$$

for some $\widetilde{M}$-orbit $\mathcal{O}_{\widetilde{M}} \subset \mathcal{O}_{\widetilde{G}}$. Furthermore, if $\mathcal{O}_{\widetilde{G}}$ is relevant, then $\mathcal{O}_{\widetilde{M}}$ is unique.



Proof. Choose $\overline{h} \in \overline{M}$, $\widetilde{g} \in \widetilde{G}$ so that $\mathcal{O}^{\mathrm{st}}_{\overline{M}} = \mathcal{O}^{\mathrm{st}}(\overline{M}, \overline{h})$ and $\mathcal{O}_{\widetilde{G}} = \mathcal{O}(\widetilde{G}, \widetilde{g})$. Then $\mathcal{O}^{\mathrm{st}}_{\overline{G}} = \mathcal{O}^{\mathrm{st}}(\overline{G}, \overline{h})$. Let $h = \phi(\overline{h})$ and $g = p(\widetilde{g})$. Note that $p(\mathcal{O}_{\widetilde{G}}) = \mathcal{O}(G, g)$ and $\phi(\mathcal{O}^{\mathrm{st}}_{\overline{M}}) = \mathcal{O}^{\mathrm{st}}(M, h)$.

Except for the final statement the result does not involve $\widetilde{G}$ or $\widetilde{M}$, and says

$$\mathcal{O}^{\mathrm{st}}(G, h) = \mathcal{O}^{\mathrm{st}}(G, g) \Leftrightarrow \mathcal{O}^{\mathrm{st}}(M, h) = \mathcal{O}^{\mathrm{st}}(M, y) \tag{15.24}$$

for some $y \in \mathcal{O}(G, g) \cap M$. The implication $\Leftarrow$ is obvious: $y \in \mathcal{O}(G, g)$ and $y \in \mathcal{O}^{\mathrm{st}}(M, h)$ implies $\mathcal{O}^{\mathrm{st}}(G, g) = \mathcal{O}^{\mathrm{st}}(G, y) = \mathcal{O}^{\mathrm{st}}(G, h)$.

Suppose $\mathcal{O}^{\mathrm{st}}(G, h) = \mathcal{O}^{\mathrm{st}}(G, g)$. By [24, Section 2]

$$\mathcal{O}^{\mathrm{st}}(G, h)/G \simeq \mathcal{O}^{\mathrm{st}}(M, h)/M. \tag{15.25}$$

More precisely if $\mathcal{O}^{\mathrm{st}}(M, h) = \cup_i \mathcal{O}(M, h_i)$ with $h_i \in M$, then $\mathcal{O}^{\mathrm{st}}(G, h) = \cup_i \mathcal{O}(G, h_i)$. Therefore $\mathcal{O}(G, g) = O(G, h_i)$ for some $h_i \in M$. Take $y = h_i$. This proves the implication $\Rightarrow$.

Assume that $\widetilde{g}$ is relevant. By the preceding paragaraph clearly $\mathcal{O}^{\mathrm{st}}(M, h)$ contains a unique M-orbit of the form $\mathcal{O}(M, y)$ with y G-conjugate to G. Choose $\widetilde{y} \in p^{-1}(y)$ so that $\widetilde{y}$ is $\widetilde{G}$-conjugate to $\widetilde{g}$, i.e. $\mathcal{O}(\widetilde{M}, \widetilde{y}) \subset \mathcal{O}_{\widetilde{G}}$. It is clear that $\phi(\mathcal{O}^{\mathrm{st}}_{\overline{M}}) = p(\mathcal{O}(\widetilde{M}, \widetilde{y}))^{\mathrm{st}}$, and the only other choice is $\mathcal{O}(\widetilde{M}, -\widetilde{y})$. By Lemma 9.4(3), since $\widetilde{y}$ is relevant, $\widetilde{y}$ is not $\widetilde{G}$-conjugate to $-\widetilde{y}$, so $\mathcal{O}(\widetilde{M}, -\widetilde{y})$ is not contained in $\mathcal{O}_{\widetilde{G}}$. $\square$

We continue to work with our given Levi subgroups $\widetilde{M}$ and $\overline{M}$. Let $\overline{MN}$ and $\widetilde{M}\widetilde{N}$ be parabolic subgroups of $\overline{G}$ and $\widetilde{G}$ with Levi components $\overline{M}$ and $\widetilde{M}$ respectively. Let $\Theta_{\overline{M}}$ be a stable character of $\overline{M}$ and let $\Theta_{\widetilde{M}}$ be a character of $\widetilde{M}$. Then the induced characters are independent of the choices of $\overline{N}$ and $\widetilde{N}$ so by abuse of notation we write

$$\mathrm{Ind}_{\overline{M}}^{\overline{G}}(\Theta_{\overline{M}}) = \mathrm{Ind}_{\overline{MN}}^{\overline{G}}(\Theta_{\overline{M}} \otimes 1), \quad \mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}}(\Theta_{\widetilde{M}}) = \mathrm{Ind}_{\widetilde{M}\widetilde{N}}^{\widetilde{G}}(\Theta_{\widetilde{M}} \otimes 1). \tag{15.26}$$

Theorem 15.27 *Let $\Theta_{\overline{M}}$ be a stable character of $\overline{M}$. Then*

$$\mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}}(\mathrm{Lift}_{\overline{M}}^{\widetilde{M}}(\Theta_{\overline{M}})) = \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\mathrm{Ind}_{\overline{M}}^{\overline{G}}(\Theta_{\overline{M}})). \tag{15.28}$$

Proof. Suppose $\mathcal{O}_{\widetilde{G}} \in \mathrm{Orb}(\widetilde{G})$ is strongly regular and semisimple. Let $\mathrm{Res}_{\widetilde{M}}^{\widetilde{G}} \mathcal{O}_{\widetilde{G}} = \{\mathcal{O}_{\widetilde{M}} \mid \mathcal{O}_{\widetilde{M}} \subset \mathcal{O}_{\widetilde{G}}\}$. Then for any character $\Theta_{\widetilde{M}}$ of $\widetilde{M}$, the induced character is given by (for example see [13])

$$\mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}}(\Theta_{\widetilde{M}})(\mathcal{O}_{\widetilde{G}}) = |D_{\widetilde{G}}(\mathcal{O}_{\widetilde{G}})|^{-\frac{1}{2}} \sum_{\mathcal{O}_{\widetilde{M}} \in \mathrm{Res}_{\widetilde{M}}^{\widetilde{G}}(\mathcal{O}_{\widetilde{G}})} |D_{\widetilde{M}}(\mathcal{O}_{\widetilde{M}})|^{\frac{1}{2}} \Theta_{\widetilde{M}}(\mathcal{O}_{\widetilde{M}}). \tag{15.29}$$

Therefore by (9.11)

$$\begin{aligned} \mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}}(\mathrm{Lift}_{\overline{M}}^{\widetilde{M}}(\Theta_{\overline{M}}))(\mathcal{O}_{\widetilde{G}}) &= \sum_{\mathcal{O}_{\widetilde{M}}} \frac{|D_{\widetilde{M}}(\mathcal{O}_{\widetilde{M}})|^{\frac{1}{2}}}{|D_{\widetilde{G}}(\mathcal{O}_{\widetilde{G}})|^{\frac{1}{2}}} \mathrm{Lift}_{\overline{M}}^{\widetilde{M}}(\Theta_{\overline{M}})(\mathcal{O}_{\widetilde{M}}) \\ &= c_{\overline{M}}^{-1} \sum_{\mathcal{O}_{\widetilde{M}}} \sum_{\mathcal{O}_{\overline{M}}^{\mathrm{st}}} \frac{|D_{\widetilde{M}}(\mathcal{O}_{\widetilde{M}})|^{\frac{1}{2}}}{|D_{\widetilde{G}}(\mathcal{O}_{\widetilde{G}})|^{\frac{1}{2}}} \Delta_{\overline{M}}^{\widetilde{M}}(\mathcal{O}_{\overline{M}}^{\mathrm{st}}, \mathcal{O}_{\widetilde{M}}) \Theta_{\overline{M}}(\mathcal{O}_{\overline{M}}^{\mathrm{st}}). \end{aligned} \tag{15.30}$$

The double sum is over

$$\{(\mathcal{O}_{\widetilde{M}}, \mathcal{O}_{\overline{M}}^{\mathrm{st}}) \,|\, \mathcal{O}_{\widetilde{M}} \in \mathrm{Res}_{\widetilde{M}}^{\widetilde{G}}(\mathcal{O}_{\widetilde{G}}), \phi(\mathcal{O}_{\overline{M}}^{\mathrm{st}}) = p(\mathcal{O}_{\widetilde{M}})^{\mathrm{st}}\} \tag{15.31}$$

Similarly, suppose $\mathcal{O}_{\overline{G}}^{\mathrm{st}} \in \mathrm{Orb}^{st}(\overline{G})$ is a semisimple orbit and let $\mathrm{Res}_{\overline{M}}^{\overline{G}}(\mathcal{O}_{\overline{G}}^{\mathrm{st}}) = \{\mathcal{O}_{\overline{M}}^{\mathrm{st}} \,|\, \mathcal{O}_{\overline{M}}^{\mathrm{st}} \subset \mathcal{O}_{\overline{G}}^{\mathrm{st}}\}$. It is well known that $\mathrm{Ind}_{\overline{M}}^{\overline{G}}(\Theta_{\overline{M}})$ is stable, and its has character formula

$$\mathrm{Ind}_{\overline{M}}^{\overline{G}}(\Theta_{\overline{M}})(\mathcal{O}_{\overline{G}}^{\mathrm{st}}) = |D_{\overline{G}}(\mathcal{O}_{\overline{G}}^{\mathrm{st}})|^{-\frac{1}{2}} \sum_{\mathcal{O}_{\overline{M}}^{\mathrm{st}} \in \mathrm{Res}_{\overline{M}}^{\overline{G}}(\mathcal{O}_{\overline{G}}^{\mathrm{st}})} |D_{\overline{M}}(\mathcal{O}_{\overline{M}}^{\mathrm{st}})|^{\frac{1}{2}} \Theta_{\overline{M}}(\mathcal{O}_{\overline{M}}^{\mathrm{st}}). \tag{15.32}$$

This follows from the previous induced character formula and (15.25). Therefore

$$\begin{aligned} \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\mathrm{Ind}_{\overline{M}}^{\overline{G}}(\Theta_{\overline{M}})(\mathcal{O}_{\widetilde{G}}) &= c_{\overline{G}}^{-1} \sum_{\mathcal{O}_{\overline{G}}^{\mathrm{st}}} \Delta_{\overline{G}}^{\widetilde{G}}(\mathcal{O}_{\overline{G}}^{\mathrm{st}}, \mathcal{O}_{\widetilde{G}}) \mathrm{Ind}_{\overline{M}}^{\overline{G}}(\Theta_{\overline{M}}))(\mathcal{O}_{\overline{G}}^{\mathrm{st}}) \\ &= c_{\overline{G}}^{-1} \sum_{\mathcal{O}_{\overline{G}}^{\mathrm{st}}} \sum_{\mathcal{O}_{\overline{M}}^{\mathrm{st}}} \frac{|D_{\overline{M}}(\mathcal{O}_{\overline{M}}^{\mathrm{st}})|^{\frac{1}{2}}}{|D_{\overline{G}}(\mathcal{O}_{\overline{G}}^{\mathrm{st}})|^{\frac{1}{2}}} \Delta_{\overline{G}}^{\widetilde{G}}(\mathcal{O}_{\overline{G}}^{\mathrm{st}}, \mathcal{O}_{\widetilde{G}}) \Theta_{\overline{M}}(\mathcal{O}_{\overline{M}}^{\mathrm{st}}). \end{aligned} \tag{15.33}$$

In this case the double sum is over

$$\{(\mathcal{O}_{\overline{G}}^{\mathrm{st}}, \mathcal{O}_{\overline{M}}^{\mathrm{st}}) \,|\, \phi(\mathcal{O}_{\overline{G}}^{\mathrm{st}}) = p(\mathcal{O}_{\widetilde{G}})^{\mathrm{st}}, \mathcal{O}_{\overline{M}}^{\mathrm{st}} \in \mathrm{Res}_{\overline{M}}^{\overline{G}}(\mathcal{O}_{\overline{G}}^{\mathrm{st}})\} \tag{15.34}$$

Suppose that $\mathcal{O}_{\widetilde{G}}$ is not relevant. Then (15.33) is zero since there are no orbits $\mathcal{O}_{\overline{G}}^{\mathrm{st}}$ satisfying $\phi(\mathcal{O}_{\overline{G}}^{\mathrm{st}}) = p(\mathcal{O}_{\widetilde{G}})^{\mathrm{st}}$. Then for all $\mathcal{O}_{\widetilde{M}} \in \mathrm{Res}_{\widetilde{M}}^{\widetilde{G}}(\mathcal{O}_{\widetilde{G}})$, $\mathcal{O}_{\widetilde{M}}$ is not relevant, so (15.30) is also zero.

Assume $\mathcal{O}_{\widetilde{G}}$ is relevant. Since a maximally split Cartan subgroup $\overline{H}_s$ of $\overline{M}$ is also one for $\overline{G}$, $c_{\overline{M}} = c_{\overline{G}}$. By Lemma 15.22 both sums are over the same set of orbits $\mathcal{O}^{\mathrm{st}}_{\overline{M}}$. For each such orbit, the equality of the corresponding terms is then given by Lemma 15.21. $\square$

By Theorem 15.27, we see that to understand lifting of standard representations of G, it suffices to understand lifting of discrete series representations of its cuspidal Levi subgroups.

16 Modified Character Data

We describe data which parametrizes L-packets for $\overline{G}$, and also data for irreducible representations of $\widetilde{G}$. These are provided by *character data* of Vogan. For $\overline{G}$ see [26, Definition 6.6.1], and for $\widetilde{G}$ see [28, Section 2]. For our purposes it is convenient to use a modified version of this data.

First let G be the real points of a connected, complex reductive group. Following [26, Definition 6.6.1] we define a *regular character* of G to be a triple $\gamma = (H, \Gamma, \lambda)$ consisting of a θ-stable Cartan subgroup H of G, a character Γ of H, and an element $\lambda \in \mathfrak{h}(\mathbb{C})^*$. We assume $\langle \lambda, \alpha^\vee \rangle \in \mathbb{R}^\times$ for all $\alpha \in \Phi_i$. Define $\Phi_i^+ = \{\alpha \in \Phi^i \mid \langle \lambda, \alpha^\vee \rangle > 0\}$ and let $\Phi_{i,c}^+ = \Phi_i \cap \Phi_{i,c}$ (cf. Section 2). Define $\rho_i = \frac{1}{2}\sum_{\Phi_i^+} \alpha$ and $\rho_{i,c} = \frac{1}{2}\sum_{\Phi_{i,c}^+} \alpha$. We assume

$$d\Gamma = \lambda + \rho_i - 2\rho_{i,c}. \tag{16.1}$$

Write $H = TA$ and let $M = \mathrm{Cent}_G(A)$ as usual. Let P be any parabolic subgroup containing M. Associated to $\gamma = (H, \Gamma, \lambda)$ is a relative discrete series representation $\pi_M = \pi_M(\gamma)$ of M, and a standard module $\pi_G(\gamma) = \pi(\gamma) = \mathrm{Ind}_P^G(\pi_M)$.

The central character of $\pi_M(\gamma)$ (respectively $\pi(\gamma)$) is $\Gamma|_{Z(M)}$ (resp. $\Gamma|_{Z(G)}$). The Harish-Chandra parameter of $\pi_M(\gamma)$ is λ, and the infinitesimal character of $\pi_M(\gamma)$ and $\pi(\gamma)$ is λ.

The definition of character data is designed to make the lowest K-types of $\pi(\gamma)$ evident. For example if H is a compact Cartan subgroup then Γ is the highest weight of a lowest K-type of $\pi(\gamma)$. We are interested in character formulas, so our needs are somewhat different. We modify this data appropriately.

Definition 16.2 *A modified regular character for G is a triple $\gamma = (H, \Gamma, \lambda)$ where H is a θ-stable Cartan subgroup of G, Γ is a character of H, and $\lambda \in \mathfrak{h}(\mathbb{C})^*$. We assume*

$$\langle \lambda, \alpha^\vee \rangle \in \mathbb{R}^\times \quad \text{for all } \alpha \in \Phi_i. \tag{16.4}$$

Let $\Phi_i^+(\lambda) = \{\alpha \in \Phi_i \mid \langle \lambda, \alpha^\vee \rangle > 0\}$ and $\rho_i(\lambda) = \frac{1}{2}\sum_{\alpha \in \Phi_i^+(\lambda)} \alpha$. We assume

$$d\Gamma = \lambda - \rho_i(\lambda). \tag{16.5}$$

Let $CD(G)$ be the set of modified regular characters. If H is fixed let $CD(G, H)$ be the set of modified regular characters (H, Γ, λ). If H is given we write $(\Gamma, \lambda) = (H, \Gamma, \lambda)$.

Write $H = TA$ and let $M = \mathrm{Cent}_G(A)$. Note that Φ_i is the set of roots of $\mathfrak{h}(\mathbb{C})$ in $\mathfrak{m}(\mathbb{C})$. Since the definition of modified regular characters only refers to the imaginary roots, $CD(G, H) = CD(M, H)$.

Associated to γ is a relative discrete series representation $\pi_M(\gamma)$, with Harish-Chandra parameter λ and central character $\Gamma|_{Z(M)}$. Let $\Theta_M(\gamma)$ be the character of $\pi_M(\gamma)$. Then for $h \in H'$,

$$\Theta_M(\gamma)(h) = (-1)^{q_M} \Delta^0(\Phi_i^+(\lambda), h)^{-1} \sum_{w \in W(M,H)} \epsilon(w) e^{w\rho_i(\lambda) - \rho_i(\lambda)}(h) \Gamma(w^{-1}h) \tag{16.6}$$

where $q_M = \frac{1}{2}\dim(M_d/K \cap M_d)$ and Δ^0 is given by (5.1).

Let $P = MN$ be any parabolic subgroup containing M, and define $\pi(\gamma) = \pi_G(\gamma) = \mathrm{Ind}_P^G(\pi_M(\gamma))$. Since we are interested only in characters the choice of P is not important. Let $\Theta_G(\gamma)$ be the character of $\pi_G(\gamma)$:

$$\Theta_G(\gamma) = \mathrm{Ind}_P^G(\Theta_M(\gamma)). \tag{16.7}$$

Suppose $\gamma = (H, \Gamma, \lambda)$ is a modified character. Note that (with the obvious notation) $2\rho_i(\lambda) - 2\rho_{i,c}(\lambda)$ is a sum of roots, so $e^{2\rho_i(\lambda) - 2\rho_{i,c}(\lambda)}$ is a well defined character of $H(\mathbb{C})$, and by restriction of H. This character is trivial on $Z(M)$. It follows easily that $(H, \Gamma e^{2\rho_i(\lambda) - 2\rho_{i,c}(\lambda)}, \lambda)$ is a regular character in the sense of Vogan, and defines the same relative discrete series representation of M and standard representation of G. This construction is clearly a bijection between modified character data and character data in the sense of Vogan.

There is a natural action of G on $CD(G)$ by conjugation. The preceding bijection is G-equivariant. By [26] or [28, Theorem 2.9] we conclude:

Lemma 16.8 *Suppose $\gamma, \gamma' \in CD(G)$. Then $\Theta_G(\gamma) = \Theta_G(\gamma')$ if and only if $\gamma = g\gamma'$ for some $g \in G$.*

We don't obtain every irreducible representation of $\overline{G}$ this way, since we don't allow "limit" characters, but this won't matter since we are only interested in stable virtual characters. Note that $W_i = W(\Phi_i)$ acts on H and this induces an action on $CD(G,H)$: $w(H,\Gamma,\lambda) = (H, w\Gamma, w\lambda)$ where $w\Gamma(h) = \Gamma(w^{-1}h)$. Note that $\pi_G(w\gamma)$ and $\pi_G(\gamma)$ have the same infinitesimal and central character.

Definition 16.9 *Suppose $\gamma = (H,\Gamma,\lambda)$ is a modified regular character. Let*

$$\pi_M^{st}(\gamma) = \sum_{w \in W(M,H)\backslash W_i} \pi_M(w\gamma). \tag{16.10}$$

and

$$\pi_G^{st}(\gamma) = \sum_{w \in W(M,H)\backslash W_i} \pi_G(w\gamma). \tag{16.11}$$

Let $\Theta_M^{st}(\gamma)$ and $\Theta_G^{st}(\gamma)$ be the characters of $\pi_M^{st}(\gamma)$ and $\pi_G^{st}(\gamma)$, respectively.

The character formula for $\Theta_M^{\mathrm{st}}(\gamma)$ on H is the same as (16.6) with $W(M,H)$ replaced by W_i.

Lemma 16.12 ([24], Lemma 5.2) *Fix $\gamma \in CD(G,H)$. Then $\Theta_M^{st}(\gamma)$ and $\Theta_G^{st}(\gamma)$ are stable characters.*

Suppose Θ is a stable virtual character of G. Then there exist $\gamma_1, \dots, \gamma_n \in CD(G)$ and integers $a_1, \dots, a_n$ so that $\Theta = \sum a_i \Theta^{G,st}(\gamma_i)$.

Stable characters were defined before Definition 9.10. While the second part of the lemma is not stated in [24] it follows easily from the proof. See Definition 18.9 and Lemmas 18.10 and 18.11 of [7].

Now let $\widetilde{G}$ be an admissible cover of G. The definition of regular character extends naturally to $\widetilde{G}$. See [11, Section 27] and [28, Section 2]. In this setting Γ is an irreducible representation of $\widetilde{H}$, or (by Lemma 10.8) a character of $Z(\widetilde{H})$.

Definition 16.13 *A genuine modified regular character of $\widetilde{G}$ is a triple $(\widetilde{H}, \widetilde{\Gamma}, \lambda)$. Here $\widetilde{H}$ is a Cartan subgroup of $\widetilde{G}$, $\widetilde{\Gamma}$ is a genuine one-dimensional* *representation of $Z(\widetilde{H})$, and $\lambda \in \mathfrak{h}(\mathbb{C})^*$. As in Definition 16.2. we require $\langle \lambda, \alpha^\vee \rangle \in \mathbb{R}^\times$ for all $\alpha \in \Phi_i$, and $d\widetilde{\Gamma} = \lambda - \rho_i(\lambda)$. Let $CD_g(\widetilde{G})$ be the set of genuine modified regular characters of $\widetilde{G}$, and $CD_g(\widetilde{G}, \widetilde{H})$ the subset with given Cartan subgroup $\widetilde{H}$.*

Let $H = TA$ be the image of $\widetilde{H}$ in G, $M = \mathrm{Cent}_G(A)$, and $\widetilde{M} = p^{-1}(M)$. Associated to $\gamma = (\widetilde{H}, \widetilde{\Gamma}, \lambda)$ is a relative discrete series representation $\pi_{\widetilde{M}}(\gamma)$ of $\widetilde{M}$. The formula for the character $\Theta_{\widetilde{M}}(\gamma)$ is (cf. 16.6)

(16.14)

$$\Theta_{\widetilde{M}}(\gamma)(h) = (-1)^{q_M} \Delta^0(\Phi_i^+(\lambda), h)^{-1} \sum_{w \in W(\widetilde{M}, \widetilde{H})} \epsilon(w) e^{w\rho_i(\lambda) - \rho_i(\lambda)}(h) \mathrm{Tr}(\widetilde{\tau}(\widetilde{\Gamma})(w^{-1}h))$$

for $h \in \widetilde{H}'$. Here $\widetilde{\tau}(\widetilde{\Gamma})$ is the irreducible representation of $\widetilde{H}$ associated to $\widetilde{\Gamma}$ by Lemma 10.8.

As before we also associate to γ the standard module $\pi_{\widetilde{G}}(\gamma)$ for $\widetilde{G}$ induced from $\pi_{\widetilde{M}}(\gamma)$. The central character of $\pi_{\widetilde{M}}(\gamma)$ is the restriction of $\widetilde{\Gamma}$ to the center of $\widetilde{M}$, and similarly for $\pi_{\widetilde{G}}(\gamma)$.

Lemma 16.15 *Suppose $\gamma, \gamma' \in CD_g(\widetilde{G})$. Then $\pi_{\widetilde{G}}(\gamma) \simeq \pi_{\widetilde{G}}(\gamma')$ if and only if $\gamma' = g\gamma$ for some $g \in \widetilde{G}$. Every irreducible genuine representation of $\widetilde{G}$ is isomorphic to $\pi_{\widetilde{G}}(\gamma)$ for some $\gamma \in CD_g(\widetilde{G})$.*

Proof.

The first statement is a special case of [28, Theorem 2.9].

The second statement amounts to the fact that $\widetilde{M}$ has no genuine limits of (relative) discrete series. This is because every irreducible genuine representation of $\widetilde{G}$ is of the form $\pi_{\widetilde{G}}(\gamma)$ where γ is *final* limit data as in [28, Definition 2.4]. For final limit data we allow $\langle \lambda, \alpha^\vee \rangle = 0$ if α is a noncompact imaginary root. But if $\widetilde{\Gamma}$ is genuine, then by [3, Lemma 6.11] $\langle d\widetilde{\Gamma}, \alpha^\vee \rangle \in \mathbb{Z} + \frac{1}{2}$ for all noncompact imaginary roots. This also holds for λ, so $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all noncompact imaginary roots. Therefore every genuine final limit data for $\widetilde{G}$ is in fact regular. □

There is no natural definition of stable distribution for $\widetilde{G}$, and we do not define analogues of $\pi^{\mathrm{st}}_G(\gamma)$ and $\pi^{\mathrm{st}}_G(\gamma)$ for $\widetilde{G}$.

We will make frequent use of formal sums of modified regular characters (for example in Lemma 16.12). So if $\gamma_i \in CD(G)$ and $a_i \in \mathbb{Z}$ $(i \leq n)$ we define $\Theta_G(\sum_i a_i \gamma_i) = \sum_i a_i \Theta_G(\gamma_i)$, and similar notation applies to $\widetilde{G}$.

17 Formal Lifting of Modified Character Data

Fix an admissible triple $(\widetilde{G}, G, \overline{G})$. Choose a maximally split Cartan subgroup H_s of G, and $(\widetilde{\chi}_s, \chi_s) \in \mathcal{S}(H_s)$. This data determines transfer factors for $\overline{G}$ and $\widetilde{G}$ (Section 7) and $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}$ is defined (Section 9). Let $\mu = \mu(\widetilde{\chi}_s, \chi_s) \in \mathfrak{z}^*$ as in (6.36)(c).

Suppose H is a θ-stable Cartan subgroup of G and $\gamma = (\overline{H}, \Gamma, \lambda) \in CD(\overline{G}, \overline{H})$ (Definition 16.2). We define the lift of γ to $CD_g(\widetilde{G}, \widetilde{H})$. We first choose transfer factors for lifting from $\overline{H}$ to $\widetilde{H}$. These differ from the obvious choice by a ρ-shift.

Write $H = TA$ and let $M = \mathrm{Cent}_G(A)$ as usual. Choose an arbitrary genuine character $\widetilde{\chi}$ of $Z(\widetilde{H})$ and a special set Φ^+ of positive roots of H in G. We assume

$$\Phi^+ \cap \Phi_i = \{\alpha \in \Phi_i \,|\, \langle \lambda, \alpha^\vee \rangle > 0\}. \tag{17.1}$$

Let χ_0 be the character of $\overline{H}$ so that $(\widetilde{\chi}, \chi_0) \in \mathcal{S}(H, \Phi^+, \widetilde{\chi}_s, \chi_s)$ (Definition 6.35). Define

$$\chi = |e^{\rho_i - \rho}| \chi_0. \tag{17.2}$$

Let ϕ_H be ϕ restricted to H, and use it to define

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\Gamma) = \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi}, \chi, \Gamma) \tag{17.3}$$

(cf. 10.15).

Lemma 17.4 *χ is independent of the choice of special positive roots Φ^+ satisfying* (17.1), *and $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}$ is independent of the choice of $\widetilde{\chi}$.*

Proof. This follows from Lemmas 6.15 and 6.47. $\square$

Definition 17.5 *If $\Gamma|_{Ker(\phi_H)} \neq \chi|_{Ker(\phi_H)}$ then $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\Gamma) = 0$, and we define $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma) = \emptyset$ and $\Theta_{\widetilde{G}}(\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)) = 0$.*

Otherwise write

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\Gamma) = \sum_{i=1}^{n} \widetilde{\tau}(\widetilde{\Gamma}_i) \tag{17.6}$$

where $\widetilde{\tau}(\widetilde{\Gamma}_i)$ *is given by Lemma 10.8,* $n = |p(Z(\widetilde{H}))/\phi(\overline{H})|$*, and each* $\widetilde{\Gamma}_i$ *is a genuine one-dimensional representation of* $Z(\widetilde{H})$*. For* $1 \leq i \leq n$ *let* $\widetilde{\gamma}_i = (\widetilde{H}, \widetilde{\Gamma}_i, \frac{1}{2}(\lambda - \mu))$*; this is a genuine regular character of* $\widetilde{G}$*. Define*

$$\text{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma) = \{\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n\}. \tag{17.7}$$

and

$$\Theta_{\widetilde{G}}(\text{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)) = \sum_{i=1}^{n} \Theta_{\widetilde{G}}(\widetilde{\gamma}_i). \tag{17.8}$$

It is important to keep in mind that $\text{Lift}_{\overline{H}}^{\widetilde{H}}$ depends on χ, and therefore on λ and γ (cf. Lemma 17.4).

The next Lemma follows from the definitions.

Lemma 17.9 *Let* $\widetilde{g} \in \widetilde{G}$ *and define* $g = \overline{p}(p(\widetilde{g})) \in \overline{G}$*. Then* $\text{Lift}_{\overline{G}}^{\widetilde{G}}(g\gamma) = \widetilde{g}\text{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$*.*

The following Lemma is an easy consequence of Lemma 17.4 and Corollary 10.16.

Lemma 17.10 *Fix* $\widetilde{\gamma} = (\widetilde{H}, \widetilde{\Gamma}, \lambda) \in CD_g(\widetilde{G})$*. Then there is a unique* $\gamma \in CD(\overline{G})$ *such that* $\widetilde{\gamma}$ *occurs in* $\text{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$*. It is given by* $\gamma = (H, \Gamma, 2\lambda + \mu)$ *where* Γ *is the unique character of* $\overline{H}$ *such that* $\widetilde{\tau}(\widetilde{\Gamma})$ *occurs in* $\text{Lift}_{\overline{H}}^{\widetilde{H}}(\Gamma)$ *(cf. Corollary 10.16).*

The reason for the shift (17.2) in the definition of $\text{Lift}_{\overline{H}}^{\widetilde{H}}$ is that it makes the next Lemma hold. Fix a cuspidal Levi subgroup M of G. Recall (Section 15) $(\widetilde{M}, M, \overline{M})$ is an admissible triple, where $\widetilde{M} = p^{-1}(M)$ and $\overline{M} = \overline{p}(M)$, and $\text{Lift}_{\overline{M}}^{\widetilde{M}}$ was defined in Section 15 (following Lemma 15.4).

Lemma 17.11

$$\text{Ind}_{\widetilde{M}}^{\widetilde{G}}(\Theta_{\widetilde{M}}(\text{Lift}_{\overline{M}}^{\widetilde{M}}(\gamma))) = \Theta_{\widetilde{G}}(\text{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)). \tag{17.12}$$

Proof. Fix a genuine character $\widetilde{\chi}$ of $Z(\widetilde{H})$.

Let χ_0 be the unique character of $\overline{H}$ so that $(\widetilde{\chi}, \chi_0) \in \mathcal{S}(H, \Phi^+, \widetilde{\chi}_s, \chi_s)$ and set $\chi_1 = \chi_0|e^{\rho_i - \rho}|$. Let $\mu = \mu(\widetilde{\chi}_s, \chi_s) = d\chi_0 - 2d\widetilde{\chi} - \rho(\Phi^+)$. Then the right hand side is the sum of terms $\mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}}(\Theta_{\widetilde{M}}(\widetilde{H},\widetilde{\Gamma},\frac{1}{2}(\lambda-\mu))$ where $\widetilde{\Gamma}\in \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi},\chi_1,\Gamma)$.

On the other hand let χ_2 be the unique character of $\overline{H}$ so that $(\widetilde{\chi},\chi_2)\in \mathcal{S}(H,\Phi^+_M,\widetilde{\chi}_s,\chi_{M,s})$. Let $\mu_M=\mu(\widetilde{\chi}_s,\chi_{M,s})=d\chi_2-2d\widetilde{\chi}-\rho(\Phi^+_i)$. Then (cf. Lemma 15.4) the left hand side is the sum of terms $\mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}}(\Theta_{\widetilde{M}}(\widetilde{H},\widetilde{\Gamma},\frac{1}{2}(\lambda-\mu_M)))$ where $\widetilde{\Gamma}\in \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi},\chi_2,\Gamma)$.

It is enough to show $\chi_1=\chi_2$, i.e. $(\widetilde{\chi},\chi_0)\in\mathcal{S}(H,\Phi^+,\widetilde{\chi}_s,\chi_s)$ implies $(\widetilde{\chi},\chi_0|e^{\rho_i-\rho}|)\in\mathcal{S}(H,\Phi^+_M,\widetilde{\chi}_s,\chi_{M,s})$. This is Lemma 15.4(2). □

Corollary 17.13 *Suppose $\widetilde{\gamma}=(\widetilde{H},\widetilde{\Gamma},\widetilde{\lambda})\in CD_g(\widetilde{G},\widetilde{H})$. Then there is a character Γ of $\overline{H}$ such that $\gamma=(\overline{H},\Gamma,2\widetilde{\lambda}+\mu)\in CD(\overline{G},\overline{H})$, and $\widetilde{\gamma}$ is a summand of $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$.*

Proof. This follows easily from Corollary 10.16: take $\Gamma(h)=\chi(h)(\widetilde{\Gamma}/\widetilde{\chi}^{-1})(\phi(h))$. We leave the details to the reader. □

Given $\gamma=(\overline{H},\Gamma,\lambda)\in CD(\overline{G},\overline{H})$ it is helpful to know when $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$ is non-zero for some choice of lifting data.

Lemma 17.14 *We can choose lifting data so that $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)\neq 0$ if and only if*

$$\Gamma(h)=\begin{cases}(\widetilde{\chi}^2e^\rho)(h) & h\in Ker(\phi_H)\cap\overline{H}^0_d,\\ \zeta_{cx}(h) & h\in\Gamma_r(\overline{H}).\end{cases}\tag{17.15}$$

Here, as usual, $\widetilde{\chi}$ is any genuine character of $\widetilde{H}$, and ρ is defined with respect to a special set of positive roots (Definition 2.1).

Proof. Fix lifting data $(\chi_s,\widetilde{\chi}_s)$, and let χ_0 be the character of $\overline{H}$ such that $(\widetilde{\chi},\chi_0)\in S(H,\Phi^+,\widetilde{\chi}_s,\chi_s)$. Then the lift is non-zero if $\Gamma(h)=|e^{\rho_i-\rho}(h)|\chi_0(h)$ for all $h\in Ker(\phi_H)$. It is easy to see $|e^{\rho_i-\rho}(h)|=1$ for all $h\in\mathrm{Ker}(\phi_H)$, so the condition is

$$\Gamma(h)=\chi_0(h)\quad(h\in\mathrm{Ker}(\phi_H)).\tag{17.16)(a}$$

Using (6.2) the condition in (17.15) is equivalent to

$$\Gamma(h)=\chi_0(h)\quad(h\in\overline{G}_d\cap\mathrm{Ker}(\phi_H)).\tag{17.16)(b}$$

This condition is obviously necessary; we need to show it is sufficient.

Let λ be a character of $\overline{H}$ satisfying

$$\lambda(h) = 1 \quad (h \in \overline{H} \cap \overline{G}_d) \tag{17.16)(c}$$

$$\lambda(h) = \Gamma(h)\chi_0^{-1}(h) \quad (h \in \mathrm{Ker}(\phi_H)). \tag{17.16)(d}$$

This is possible since $\overline{H} \cap \overline{G}_d \cap \mathrm{Ker}(\phi_H) = \overline{G}_d \cap \mathrm{Ker}(\phi_H)$, and $\Gamma\chi_0^{-1} = 1$ on this group by (b). Then λ extends uniquely to a one dimensional representation of $\overline{HG_d}$, and then (possibly not uniquely) to a one dimensional representation ψ of $\overline{G}$.

The result follows from Lemma 7.15: the lift is non-zero if we replace $\Delta_{\overline{G}}^{\widetilde{G}}(\chi_s, \widetilde{\chi}_s)$ with $\psi\Delta_{\overline{G}}^{\widetilde{G}}(\chi_s, \widetilde{\chi}_s)$. $\square$

18 Lifting Stable Discrete Series

Assume that $(\widetilde{G}, G, \overline{G})$ is an admissible triple. Fix lifting data $(\widetilde{\chi}_s, \chi_s)$ (Definition 7.1). This data determines choices of transfer factors for $\overline{G}$ and $\widetilde{G}$ (Section 6) and $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}$ is defined (Section 7). We assume for this section only that G has a relatively compact Cartan subgroup H, i.e. such that $H \cap G_d$ is compact, and hence $\widetilde{G}, G$, and $\overline{G}$ have relative discrete series representations. We consider lifting of the relative discrete series from $\overline{G}$ to $\widetilde{G}$. See Section 12 for the case when $G = \overline{G}$ is connected and semisimple.

Fix a relatively compact Cartan subgroup H of G with roots Φ. Since H is relatively compact $\Phi = \Phi_i$. Fix $\gamma = (\overline{H}, \Gamma, \lambda) \in CD(\overline{G}, \overline{H})$, and let $\Theta_{\overline{G}}^{\mathrm{st}}(\gamma)$ be the associated stable discrete series character as in Section 16. We want to compute $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\overline{G}}^{\mathrm{st}}(\gamma))$.

Define $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\Gamma)$ and $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma) = (\widetilde{H}, \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\Gamma), \frac{1}{2}(\lambda - \mu))$ as in Definition 17.5. In this case by (17.3)

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\Gamma) = \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi}, \chi, \Gamma) \tag{18.1}$$

where $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+(\lambda), \widetilde{\chi}_s, \chi_s)$ (since $\Phi = \Phi_i$, Φ^+ is uniquely determined and $\chi = \chi_0$). Fix $(\widetilde{\chi}, \chi) \in \mathcal{S}(H, \Phi^+(\lambda), \widetilde{\chi}_s, \chi_s)$.

Let $W = W(\Phi) = W_i$. For $w \in W$ let $w\gamma = (\overline{H}, w\Gamma, w\lambda)$. Then $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(w\gamma) = (\widetilde{H}, \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(w\Gamma), \frac{1}{2}w(\lambda - \mu))$ is defined. It is important to keep in mind that the lifting used in defining $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(w\Gamma)$ depends on w: by the preceding discussion

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(w\Gamma) = \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi}_w, \chi_w, w\Gamma) \tag{18.2}$$

where $(\widetilde{\chi}_w, \chi_w) \in \mathcal{S}(H, \Phi^+(w\lambda), \widetilde{\chi}_s, \chi_s)$. One possibility is to take

$$(\widetilde{\chi}_w, \chi_w) = (\widetilde{\chi}, \chi e^{w\rho-\rho}). \tag{18.3}$$

To see this, write $h = \gamma h_0$ where $\gamma \in \Gamma(\overline{H})$ and $h_0 \in \overline{H}^0$. By Lemma 6.47 we can take $\widetilde{\chi}_w = \widetilde{\chi}$ and $\chi_w(h) = \chi(h)e^{w\rho-\rho}(h_0)$. Since $\overline{H}$ is relatively compact $\Gamma(\overline{H}) \subset Z(\overline{G})$, so $e^{w\rho-\rho}(h_0) = e^{w\rho-\rho}(h)$.

We will also use the fact that if $x \in W(\widetilde{G}, \widetilde{H}), y \in W$ then

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(xy\Gamma)(\widetilde{h}) = \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(y\Gamma)(x^{-1}\widetilde{h}), \tag{18.4}$$

or more explicitly

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi}_{xy}, \chi_{xy}, xy\Gamma)(\widetilde{h}) = \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi}_y, \chi_y, y\Gamma)(x^{-1}\widetilde{h}). \tag{18.5}$$

This follows from the fact we can take $(\widetilde{\chi}_y, \chi_y)$ as in (18.3) and $(\widetilde{\chi}_{xy}, \chi_{xy}) = (x\widetilde{\chi}_y, x\chi_y)$. Then

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi}_{xy}, \chi_{xy}, xy\Gamma)(\widetilde{h}) = \mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(x\widetilde{\chi}_y, x\chi_y, xy\Gamma)(\widetilde{h}) \tag{18.6}$$

and it follows immediately from the definitions that this equals

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi}_y, \chi_y, y\Gamma)(x^{-1}\widetilde{h}). \tag{18.7}$$

Proposition 18.8 *Let $\gamma \in CD(\overline{G}, \overline{H})$. Then*

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\overline{G}}^{st}(\gamma)) = C_{\overline{G}}(H) \sum_{w \in W(\overline{G},\overline{H})\backslash W} \Theta_{\widetilde{G}}(\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)). \tag{18.9}$$

Proof. Let $\overline{\Theta} = \Theta_{\overline{G}}^{\mathrm{st}}(\gamma)$. By (16.6)

$$\overline{\Theta}(h) = (-1)^q \Delta^0(\Phi^+, h)^{-1} \sum_{w \in W} \epsilon(w) e^{w\rho-\rho}(h)\Gamma(w^{-1}h) \quad (h \in H') \tag{18.10}$$

where $q=q_G$. Fix a strongly regular element $\widetilde{h}\in\widetilde{H}$. Since H has no real roots we have

$$\Delta_{\overline{G}}^{\widetilde{G}}(h,\widetilde{h})=\frac{\Delta^0(\Phi^+,h)\widetilde{\chi}(\widetilde{h})}{\Delta^0(\Phi^+,\widetilde{h})\chi(h)}\quad (h\in X(\overline{H},\widetilde{h})).$$

By definition of lifting we have $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{h})=$

$$c^{-1}(-1)^q\Delta^0(\Phi^+,\widetilde{h})^{-1}\sum_{w\in W}\epsilon(w)\widetilde{\chi}(\widetilde{h})\sum_{h\in X(\overline{H},\widetilde{h})}\chi(h)^{-1}e^{w\rho-\rho}(h)w\Gamma(h).\tag{18.11}$$

If $h\in X(\overline{H},\widetilde{h})$ then $e^{w\rho-\rho}(\widetilde{h})=e^{2w\rho-2\rho}(h)$, so $e^{w\rho-\rho}(h)=e^{w\rho-\rho}(\widetilde{h})e^{\rho-w\rho}(h)$. Therefore

$$\begin{aligned}\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{h})=c^{-1}(-1)^q\Delta^0(\Phi^+,\widetilde{h})^{-1}\sum_{w\in W}\epsilon(w)e^{w\rho-\rho}(\widetilde{h})\times\\ \widetilde{\chi}(\widetilde{h})\sum_{h\in X(\overline{H},\widetilde{h})}\chi(h)^{-1}e^{\rho-w\rho}(h)w\Gamma(h).\end{aligned}\tag{18.12}$$

By 10.2

$$\widetilde{\chi}(\widetilde{h})\sum_{h\in X(\overline{H},\widetilde{h})}\chi(h)^{-1}e^{\rho-w\rho}(h)w\Gamma(h)=c(H)\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi},\chi e^{w\rho-\rho},w\Gamma)(\widetilde{h}).\tag{18.13}$$

By (18.3) $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi},\chi e^{w\rho-\rho},w\Gamma)=\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(w\Gamma)$. Recalling $C_{\overline{G}}(H)=C(H)=c(H)/c$ this gives

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{h})=C(H)(-1)^q\Delta^0(\widetilde{h},\Phi^+)^{-1}\sum_{w\in W}\epsilon(w)e^{w\rho-\rho}(\widetilde{h})\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(w\Gamma)(\widetilde{h})\tag{18.14}$$

Write the sum as

$$\sum_{y\in W(\widetilde{G},\widetilde{H})\backslash W}\epsilon(y)e^{y\rho-\rho}(\widetilde{h})\sum_{x\in W(\widetilde{G},\widetilde{H})}\epsilon(x)e^{xy\rho-y\rho}(\widetilde{h})\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(xy\Gamma)(\widetilde{h}).\tag{18.15}$$

By (18.5)

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(xy\Gamma)(\widetilde{h})=\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(y\Gamma)(x^{-1}\widetilde{h}).\tag{18.16}$$

Also, for all $y \in W$ we have

$$\Delta^0(\widetilde{h}, \Phi^+)^{-1}\epsilon(y)e^{y\rho-\rho} = \Delta^0(\widetilde{h}, y\Phi^+)^{-1}. \tag{18.17}$$

This gives

$$\begin{aligned}\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{h}) = C(H)(-1)^q \sum_{y\in W(\widetilde{G},\widetilde{H})\backslash W} \Delta^0(y\Phi^+, \widetilde{h})^{-1}\times \\ \sum_{x\in W(\widetilde{G},\widetilde{H})} \epsilon(x)e^{xy\rho-y\rho}(\widetilde{h})\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(y\Gamma)(x^{-1}\widetilde{h}).\end{aligned} \tag{18.18}$$

By Proposition 10.11 $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(y\Gamma) = 0$ unless $y \in W(\gamma)$ where

$$W(\gamma) = \{w \in W \mid \chi(h) = e^{\rho-w\rho}(h)w\Gamma(h) \text{ for all } h \in \mathrm{Ker}(\phi_H)\}. \tag{18.19}$$

Fix $y \in W(\gamma)$ and define

$$\widetilde{\Gamma}_y(\widetilde{h}) = \widetilde{\chi}(\widetilde{h})\chi^{-1}(h)e^{\rho-y\rho}(h)y\Gamma(h) \quad (h \in X(\overline{H}, \widetilde{h})). \tag{18.20}$$

Then by Proposition 10.11

$$\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(y\Gamma)(\widetilde{h}) = \sum_{\widetilde{\Gamma}\in X_g(\widetilde{H},\widetilde{\Gamma}_y)} \mathrm{Tr}(\widetilde{\tau}(\widetilde{\Gamma})(\widetilde{h})) \quad (\widetilde{h} \in \widetilde{H}). \tag{18.21}$$

Thus

$$\begin{aligned}&\sum_{x\in W(\widetilde{G},\widetilde{H})} \epsilon(x)e^{xy\rho-y\rho}(\widetilde{h})\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(y\Gamma)(x^{-1}\widetilde{h}) \\ &= \sum_{\widetilde{\Gamma}\in X_g(\widetilde{H},\widetilde{\Gamma}_y)} \sum_{x\in W(\widetilde{G},\widetilde{H})} \epsilon(x)e^{xy\rho-y\rho}(\widetilde{h})\mathrm{Tr}(\widetilde{\tau}(\widetilde{\Gamma})(x^{-1}\widetilde{h})).\end{aligned} \tag{18.22}$$

But for each $\widetilde{\Gamma} \in X_g(\widetilde{H}, \widetilde{\Gamma}_y)$ by (16.14) we have

(18.23)

$$(-1)^q\Delta^0(y\Phi^+, \widetilde{h})^{-1} \sum_{x\in W(\widetilde{G},\widetilde{H})} \epsilon(x)e^{xy\rho-y\rho}(\widetilde{h})\mathrm{Tr}(\widetilde{\tau}(\widetilde{\Gamma})(x^{-1}\widetilde{h})) = \Theta_{\widetilde{G}}(\widetilde{\gamma})(\widetilde{h})$$

where $\widetilde{\gamma} = (\widetilde{H}, \widetilde{\Gamma}, (y\lambda-\mu)/2)$ is the corresponding element of $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(y\gamma)$. Thus

$$Lift_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{h}) = C(H) \sum_{y\in W(\widetilde{G},\widetilde{H})\backslash W(\gamma)} \sum_{\widetilde{\gamma}\in \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(y\gamma)} \Theta_{\widetilde{G}}(\widetilde{\gamma})(\widetilde{h}). \tag{18.24}$$

In other words for every strongly regular element $\widetilde{h} \in \widetilde{H}$ we have

$$\text{(18.25)} \qquad \text{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})(\widetilde{h}) = C(H) \sum_{w \in W(\overline{G},\overline{H})\backslash W} \Theta_{\widetilde{G}}(\text{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma))(\widetilde{h}).$$

Since $\overline{\Theta}$ is a stable discrete series character of $\overline{G}$ with Harish-Chandra parameter λ, it is a stable invariant eigendistribution with infinitesimal character λ and is relatively supertempered. Thus by Theorems 10.32 and 10.35 $\text{Lift}_{\overline{G}}^{\widetilde{G}}(\overline{\Theta})$ is an invariant eigendistribution on $\widetilde{G}$ with infinitesimal character $(\lambda-\mu)/2$ and is relatively supertempered. Now each $\Theta_{\widetilde{G}}(\text{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)), w \in W$, is either zero or is a sum of discrete series characters which all have infinitesimal character $(\lambda-\mu)/2$, and hence is also a relatively supertempered invariant eigendistribution on $\widetilde{G}$ with infinitesimal character $(\lambda - \mu)/2$. Thus (18.25) holds for all $\widetilde{g} \in \widetilde{G}'$ by Harish-Chandra's Theorem 13.9. □

Not all of the terms in (18.9) are non-zero. For a precise description of which ones are non-zero see Proposition 19.15.

Example 18.26 Let $G = SL(2,\mathbb{R})$. A discrete series representation of $\widetilde{G}$ or G is determined by its Harish-Chandra parameter $\lambda \in \mathfrak{t}^*(\mathbb{C})$ where T is a compact Cartan subgroup of G. Write $\pi_G^{\text{st}}(\lambda)$ for the corresponding stable sum of discrete series of G, and $\pi_{\widetilde{G}}(\lambda)$ for a discrete series representation of $\widetilde{G}$. Let $P = \mathbb{Z}\langle\rho\rangle$ be the weight lattice. We compute $C(T) = 2$, and then

$$\text{(18.27)} \qquad \text{Lift}_G^{\widetilde{G}}(\pi_G^{\text{st}}(\lambda)) = \begin{cases} 2\pi_{\widetilde{G}}(\lambda/2) + 2\pi_{\widetilde{G}}(-\lambda/2) & \lambda \in 2P+\rho \\ 0 & \text{otherwise} \end{cases}$$

In the usual coordinates the lift is non-zero if and only if λ is an odd integer, in which case $\pm\lambda/2 \in \mathbb{Z} + \frac{1}{2}$.

If $\overline{G} = PSL(2,\mathbb{R}) \simeq SO(2,1)$ the result is the similar, except that $C(T) = 1$.

19 Lifting Standard Representations

Fix an admissible triple $(\widetilde{G}, G, \overline{G})$ and lifting data $(\widetilde{\chi}_s, \chi_s)$ for $\widetilde{G}$.

Suppose $\Theta_{\overline{G}}^{\text{st}}$ is a stable virtual character of $\overline{G}$. Recall (Lemma 16.12) that $\Theta_{\overline{G}}^{\text{st}}$ is a sum of standard characters. Therefore, assuming we can write $\Theta_{\overline{G}}^{\text{st}}$ as a sum of standard modules, it is enough to compute the lifting of an arbitrary standard character. This is the main result of this section (Theorem 19.1).

We recall some constructions from Section 16. Suppose γ is modified character data for $\overline{G}$ (Definition 16.2), with associated stable standard character $\Theta^{\mathrm{st}}_{\overline{G}}(\gamma)$. Write $\gamma = (\overline{H}, \Gamma, \lambda)$ and let $\overline{M}$ be the corresponding Levi factor of $\overline{G}$. Recall γ is also character data for $\overline{M}$, so the stable relative discrete series character $\Theta^{\mathrm{st}}_{\overline{M}}(\gamma)$ is defined, and $\Theta^{\mathrm{st}}_{\overline{G}}(\gamma) = \mathrm{Ind}^{\overline{G}}_{\overline{M}}(\Theta^{\mathrm{st}}_{\overline{M}}(\gamma))$. (As in the previous section we write $\mathrm{Ind}^{\overline{G}}_{\overline{M}}$ instead of $\mathrm{Ind}^{\overline{G}}_{\overline{MN}}$).

Recall (Section 16) $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\gamma)$ is a set of modified character data for $\widetilde{G}$. Recall $W_i = W(\Phi_i)$ acts on $CD(\overline{G}, \overline{H})$.

Theorem 19.1

$$\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\Theta^{st}_{\overline{G}}(\gamma)) = C_{\overline{G}}(H) \sum_{w \in W(\overline{M}, \overline{H})\backslash W_i} \Theta_{\widetilde{G}}(\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(w\gamma)). \tag{19.2}$$

Explicitly for $w \in W_i$ define $\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(w\gamma)$ by Definition 17.5, and define $\widetilde{\gamma}(w, i) \in CD_g(\widetilde{G})$ by:

$$\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(w\gamma) = \sum_{i=1}^{n_w} \widetilde{\gamma}(w, i) \tag{19.3}$$

(if the lift is empty take $n_w = 0$). Then

$$\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\Theta^{st}_{\overline{G}}(\gamma)) = C_{\overline{G}}(H) \sum_{w \in W(\overline{M}, \overline{H})\backslash W_i} \sum_{i=1}^{n_w} \Theta_{\widetilde{G}}(\widetilde{\gamma}(w, i)). \tag{19.4}$$

Proof. This is merely a question of assembling the pieces. Write $\sum_w$ for the sum over $W(\overline{M}, \overline{H})\backslash W_i$. Then

(19.5)

$$\begin{aligned}
\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\Theta^{\mathrm{st}}_{\overline{G}}(\gamma)) &= \mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(\mathrm{Ind}^{\overline{G}}_{\overline{M}}(\Theta^{\mathrm{st}}_{\overline{M}}(\gamma))) \quad \text{((16.7) and Definition (16.9)} \\
&= \mathrm{Ind}^{\widetilde{G}}_{\widetilde{M}}(\mathrm{Lift}^{\widetilde{M}}_{\overline{M}}(\Theta^{\mathrm{st}}_{\overline{M}}(\gamma))) \quad \text{(Theorem 15.27)} \\
&= C_{\overline{G}}(H)\mathrm{Ind}^{\widetilde{G}}_{\widetilde{M}}(\sum_w \Theta_{\widetilde{M}}(\mathrm{Lift}^{\widetilde{M}}_{\overline{M}}(w\gamma)) \quad \text{(Proposition 18.8)} \\
&= C_{\overline{G}}(H) \sum_w \Theta_{\widetilde{G}}(\mathrm{Lift}^{\widetilde{G}}_{\overline{G}}(w\gamma)) \quad \text{(Lemma 17.11).}
\end{aligned}$$

$\square$

It is important to know that the terms in the sum (19.4) are distinct.

Proposition 19.6 *In the sum* (19.4), $\pi_{\widetilde{G}}(\widetilde{\gamma}(w,i)) \simeq \pi_{\widetilde{G}}(\widetilde{\gamma}(w',i'))$ *if and only if* $W(\overline{M},\overline{H})w = W(\overline{M},\overline{H})w'$ *and* $i = i'$.

Proof. Suppose for $i = 1,2$, $w_i \in W_i$, and $\widetilde{\gamma}_i$ occurs in $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w_i\gamma)$. We may as well assume $w_1 = 1$, and write $\widetilde{\gamma}_1 = (\widetilde{H}, \widetilde{\Gamma}_1, \frac{1}{2}(\lambda - \mu)) \in \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$, $\widetilde{\gamma}_2 = (\widetilde{H}, \widetilde{\Gamma}_2, \frac{1}{2}(w\lambda - \mu)) \in \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)$ for some $w \in W_i$. We want to show

$$(\widetilde{H}, \widetilde{\Gamma}_2, \frac{1}{2}(w\lambda - \mu) = g(\widetilde{H}, \widetilde{\Gamma}_1, \frac{1}{2}(\lambda - \mu)) \tag{19.7}$$

for some $g \in \widetilde{G}$ implies $w = 1$ and $\widetilde{\Gamma}_1 = \widetilde{\Gamma}_2$. Since g normalizes $\widetilde{H}$ let u be the image of $\overline{p}(p(g))$ in $W(\overline{G},\overline{H})$ and set $v = w^{-1}u$. Both w and u are in W^θ, the elements of W fixed by θ, so $v \in W^\theta$. Also $u\lambda = w\lambda$, so $v\lambda = \lambda$ and therefore $v\Phi_i^+(\lambda) = \Phi_i^+(\lambda)$.

We now apply [27, Propositions 3.12 and 4.16]. By Proposition 3.12(c) $v \in W^\theta, v\Phi_i^+(\lambda) = \Phi_i^+(\lambda)$ implies $v \in W^q$, the Weyl group of the roots perpendicular to ρ_i. Then by Proposition 4.16(a) $v \in W^q \subset W(\overline{G},\overline{H})$. Therefore $w = uv^{-1} \in W(\overline{G},\overline{H})$. But $w \in W_i$ so $w \in W(\overline{G},\overline{H}) \cap W_i = W(\overline{M},\overline{H})$.

Since we are summing over cosets of $W(\overline{M},\overline{H})$ we may as well assume $w = 1$. Therefore $\widetilde{\gamma}_1 = (\widetilde{H}, \widetilde{\Gamma}_1, \frac{1}{2}(\lambda - \mu))$, $\widetilde{\gamma}_2 = (\widetilde{H}, \widetilde{\Gamma}_2, \frac{1}{2}(\lambda - \mu))$. Since these are both contained in $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ agree on $\widetilde{H}^0$. Since they are conjugate by $g \in \widetilde{G}$ they agree on $Z(\widetilde{G})$. Since $\widetilde{H} = Z(\widetilde{G})\widetilde{H}^0$ (Proposition 4.7) this proves they are equal, proving the Lemma. $\square$

Corollary 19.8 *In the setting of the Theorem, let* $\{\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_n\}$ *be the set of constituents of* $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)$ *as* w *runs over* W_i, *considered without multiplicity. Then*

$$\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\pi_{\overline{G}}^{st}(\gamma)) = C_{\overline{G}}(H) \sum_{i=1}^{n} \pi_{\widetilde{G}}(\widetilde{\gamma}_i). \tag{19.9}$$

Remark 19.10 It is a remarkable fact that not only are the constituents of (19.4) distinct, they have distinct central characters. See Remarks 11.16

and 12.9. As in [5] one can use this, together with Fourier inversion on $Z(\widetilde{G})$ to obtain a character formula for $\Theta^{\widetilde{G}}(\widetilde{\gamma}(w,i))$. Since this uses a number of structural results not needed elsewhere we omit the proof.

Corollary 19.11 *Suppose π is an admissible virtual representation of $\overline{G}$ and Θ_π is stable. Then $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_\pi)$ is the character of a genuine virtual representation of $\widetilde{G}$, or 0. If π is tempered and $\mu(\widetilde{\chi},\chi_s)\in i\mathfrak{z}$ then so is $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_\pi)$.*

Every genuine standard module for $\widetilde{G}$ occurs in some Lift.

Lemma 19.12 *Fix $\widetilde{\gamma}\in CD_g(\widetilde{G})$. Define $\gamma\in CD(\overline{G})$ as in Lemma 17.10, so that $\widetilde{\gamma}$ occurs in $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$. Then $\Theta_{\widetilde{G}}(\widetilde{\gamma})$ occurs in $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\overline{G}}^{st}(\gamma))$. Conversely, suppose $\Theta_{\widetilde{G}}(\widetilde{\gamma})$ occurs in $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\overline{G}}^{st}(\gamma'))$ for some $\gamma'\in CD(\overline{G})$. Then $\Theta_{\overline{G}}^{st}(\gamma)=\Theta_{\overline{G}}^{st}(\gamma')$.*

Proof. The first statement is an immediate consequence of Theorem 19.1. Now suppose that $\gamma'\in CD(\overline{G})$ such that $\Theta_{\widetilde{G}}(\widetilde{\gamma})$ occurs in $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\overline{G}}^{\mathrm{st}}(\gamma'))$. Then $\Theta_{\widetilde{G}}(\widetilde{\gamma})=\Theta_{\widetilde{G}}(\widetilde{\gamma}')$ where $\widetilde{\gamma}'$ occurs in $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma')$ for some $w\in W_i'$, the imaginary Weyl group for the Cartan subgroup associated to γ'. Since $\Theta_{\widetilde{G}}(\widetilde{\gamma})=\Theta_{\widetilde{G}}(\widetilde{\gamma}')$, by Lemma 16.15 there is $\widetilde{g}\in\widetilde{G}$ such that $\widetilde{\gamma}=\widetilde{g}\widetilde{\gamma}'$. Define $g=\overline{p}(p(\widetilde{g}))$. Then by Lemma 17.9 $\widetilde{\gamma}=\widetilde{g}\widetilde{\gamma}'$ occurs in $\widetilde{g}\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma')=\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(gw\gamma')$. But by Lemma 17.10, this implies that $\gamma=gw\gamma'$. Hence $\Theta_{\overline{G}}^{\mathrm{st}}(\gamma)=\Theta_{\overline{G}}^{\mathrm{st}}(\gamma')$. □

We now make the sum (19.4) more explicit. Assume $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\overline{G}}^{\mathrm{st}}(\gamma))\neq 0$. Then $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)\neq\emptyset$ for some $w\in W_i$. and after replacing γ with $w\gamma$ we assume $w=1$. We may then describe the set of $w\in W_i$ such that $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)$ is non-empty. See (12.5) for a special case.

Fix $\gamma=(\overline{H},\Gamma,\lambda)\in CD(\overline{G})$ and suppose $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\pi_{\overline{G}}^{\mathrm{st}}(\gamma))\neq 0$. Without loss of generality we may assume $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)\neq 0$. Then each component of $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)$ is of the form $(\widetilde{H},\widetilde{\Gamma},\frac{1}{2}(\lambda-\mu))$. In particular $\frac{1}{2}(\lambda-\mu)-\rho_i(\lambda)$ is the differential of a genuine character of $\widetilde{H}$.

Let

(19.13)(a) $$L=\{X\in\mathfrak{h}\,|\,\exp(X)\in\Gamma(H)\cap C\cap H^0\}.$$

Note that $\Gamma(H)\cap C\cap H^0$ is a finite central subgroup, and L is a lattice. Let

$$(19.13)(b)\qquad W_\# = \{w\in W_i\,|\,\exp((w\lambda/2-\lambda/2)(X))=1 \text{ for all } X\in L\}.$$

Here is an alternative description of $W_\#$ in terms of $M=\mathrm{Cent}_G(A)$. Let $P_M^\vee$ be the weight lattice of the derived group of M, and

$$(19.13)(c)\qquad L'=\{\gamma^\vee\in P_M^\vee\,|\,\exp(2\pi i\gamma^\vee)\in\Gamma(H)\cap C\cap H^0\}.$$

Note that $X_{M_*}\subset L'\subset P_M^\vee$ where $X_{M_*}=X_*(T(\mathbb{C})\cap M_d(\mathbb{C}))$. Then

$$(19.13)(d)\qquad W_\# = \{w\in W_i\,|\,\langle w\lambda/2-\lambda/2,\gamma^\vee\rangle\in\mathbb{Z} \text{ for all } \gamma^\vee\in L'\}.$$

Suppose $\alpha\in\Phi_i$. By [3, Lemma 6.11]) and the fact that $\frac{1}{2}(\lambda-\mu)-\rho_i(\lambda)$ is the differential of a genuine character of $\widetilde{H}$, we conclude $\langle\lambda/2,\alpha^\vee\rangle\in\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$ depending on whether α is compact or noncompact. It follows that $W_\#\subset\mathrm{Norm}_{W_i}(\Phi_{i,c})$, and it is easy to see $W(M,H)\subset W_\#$, so we have:

$$(19.14)\qquad W(\Phi_{i,c})\subset W(M,H)\subset W_\#\subset\mathrm{Norm}_{W_i}(\Phi_{i,c})$$

Compare [27, Proposition 4.16(d)].

Frequently $\Gamma(H)\cap C\cap H^0=1$, so L is the kernel of exp restricted to $\mathfrak{h}$ and $L'=X_{M_*}$.

Proposition 19.15 *Fix $\gamma\in CD(\overline{G})$ and suppose $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\gamma)\neq 0$. With $W_\#$ as in* (19.13) *we have*

$$(19.16)\qquad \mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(\Theta_{\overline{G}}^{st}(\gamma)) = C_{\overline{G}}(H)\sum_{w\in W(M,H)\backslash W_\#}\ \sum_{\widetilde{\gamma}\in\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)}\Theta_{\widetilde{G}}(\widetilde{\gamma}).$$

There are $|W(M,H)\backslash W_\#||Z_0(H)/\phi(\overline{H})|$ terms in the sum.

For example suppose γ is a discrete series parameter for $G=SL(2,\mathbb{R})$, and $\overline{G}=SL(2,\mathbb{R})$ or $PSL(2,\mathbb{R})$. Then $\phi(\overline{H})=Z_0(H)=H$, $W(G,H)=1$ and $W_\#=\mathbb{Z}/2\mathbb{Z}$. See Example 18.26.

Proof. Fix $w\in W_i$. It is enough to show $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma)\neq 0$ if and only if $w\in W_\#$, and by Lemma 17.11 this is equivalent to $\mathrm{Lift}_{\overline{M}}^{\widetilde{M}}(w\gamma)\neq 0$.

By assumption $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(\widetilde{\chi},\chi,\Gamma)\neq\emptyset$ where $(\widetilde{\chi},\chi)\in S(H,\Phi^+,\widetilde{\chi}_s,\chi_s)$. By Proposition 10.11 $\Gamma(h)=\chi(h)$ for all $h\in\mathrm{Ker}(\phi_H)$ where ϕ_H is the restriction of ϕ to $\overline{H}$. Then $w\gamma=(\overline{H},w\Gamma,w\lambda)$, and (cf. 18.3) $\mathrm{Lift}_{\overline{H}}^{\widetilde{H}}(w\Gamma)=$

$\mathrm{Lift}(\widetilde{\chi}, \chi_w, w\Gamma)$ where $\chi_w(h) = e^{w\rho-\rho}(h)\chi(h)$. Thus $\mathrm{Lift}_{\overline{G}}^{\widetilde{G}}(w\gamma) \neq 0$ if and only if $w\Gamma(h) = e^{w\rho-\rho}(h)\chi(h)$ for all $h \in \mathrm{Ker}(\phi_H)$. But $w\Gamma(h) = \Gamma(w^{-1}h) = \chi(w^{-1}h)$, so the condition is

$$\text{(19.17)(a)} \qquad \chi((w^{-1}h)h^{-1}) = e^{w\rho-\rho}(h) \quad \text{for all } h \in \mathrm{Ker}(\phi_H).$$

Write $\overline{H} = Z(\overline{M})(\overline{H} \cap \overline{M}_d)^0$, and $h = zh_0$ accordingly. Since $z \in Z(\overline{M})$ and $w \in W_i = W(M(\mathbb{C}), H(\mathbb{C}))$, $(w^{-1}h)h^{-1} \in (\overline{H} \cap \overline{M}_d)^0$. By (6.2)(a) we can write condition (a) as

$$\text{(19.17)(b)} \qquad (\widetilde{\chi}^2 e^{\rho})((w^{-1}h)h^{-1}) = e^{w\rho-\rho}(h) \quad \text{for all } h \in \mathrm{Ker}(\phi_H).$$

Let $H(\mathbb{C})_2 = \{h \in H(\mathbb{C}) \,|\, h^2 = 1\}$. Then $\mathrm{Ker}(\phi_H) = \overline{p}(H(\mathbb{C})_2) \cap \overline{H}$, so we can we can replace $\mathrm{Ker}(\phi_H)$ with $H(\mathbb{C})_2 \cap \overline{p}^{-1}(\overline{H})$ on the right hand side of (b), which then becomes equivalent to

$$\text{(19.17)(c)} \qquad \widetilde{\chi}^2((w^{-1}h)h^{-1}) = 1 \quad \text{for all } h \in H(\mathbb{C})_2 \cap \overline{p}^{-1}(\overline{H}).$$

We claim this is equivalent to

$$\text{(19.17)(d)} \qquad \widetilde{\chi}^2((w^{-1}h)h^{-1}) = 1 \quad \text{for all } h \in H^0, h^2 \in \Gamma(H) \cap C.$$

If h is in the set in (c), write $h = ta$ with $t \in \exp(\mathfrak{t}(\mathbb{C}))$ and $a \in \exp(\mathfrak{a}(\mathbb{C})) = A(\mathbb{C})$. Then t is in the set in (d), and $t = ha^{-1}$ with $a \in A(\mathbb{C})$. Conversely for h as in (d), suppose $h^2 = \exp(iX) \in \Gamma(H)$ with $X \in \mathfrak{a}$ (cf. 2.2) and let $a = \exp(iX/2) \in A(\mathbb{C})$. Then $x = ha$ satisfies the condition in (c). The claim follows since W_i acts trivially on $A(\mathbb{C})$.

For h in the set in (d) write $h = \exp(X/2)$ for $X \in \mathfrak{h}$. As in the discussion preceding the Proposition choose $\widetilde{\chi}$ to be a genuine character of $Z(\widetilde{H})$ with differential $\frac{1}{2}(\lambda - \mu) - \rho_i$. Then since $\mu \in \mathfrak{z}(\mathbb{C})^*$

$$\text{(19.18)} \qquad \begin{aligned} \widetilde{\chi}^2((w^{-1}h)h^{-1}) &= \exp((\lambda - \mu - 2\rho_i)(w^{-1}X/2 - X/2)) \\ &= \exp(w\lambda/2 - \lambda/2)(X)\exp((2\rho_i - w2\rho_i)(X/2)) \\ &= \exp(w\lambda/2 - \lambda/2)(X) \end{aligned}$$

The last equality follows from $\exp((2\rho_i - w\rho_i)(X/2)) = \exp((\rho_i - w\rho_i)(X)) = 1$ since $\exp(X) = h^2 \in \Gamma(H) \cap C \subset Z(G)$. This gives (19.13)(b). The alternative description of $W_{\#}$ is fairly standard. The kernel of the exponential map restricted to $\mathfrak{h}$ is contained in $\mathfrak{t}$, and now everything is taking place in M, and in fact in M_d. □

At least if $\phi(\overline{H}) = Z_0(H)$ the right hand side of (19.16), which is a sum over $W(M,H)\backslash W_\sharp$, is a reasonable candidate for an *L-packet* for $\widetilde{G}$. Unlike the linear case the terms in this sum have different central characters; this also happens for $Mp(2n,\mathbb{R})$ [4]. However $W_\#$ may depend on $\overline{G}$, so this sum is not canonical. We also note that there is no obvious notion of stability for genuine virtual characters of $\widetilde{G}$. Therefore the precise definition of L-packet for $\widetilde{G}$ remains to be determined.

20 Appendix

In this section we extend Hirai's Theorem on invariant eigendistributions to a class of groups containing all reductive groups of Harish-Chandra's class. We first give some definitions from [17]. Let G be a reductive Lie group with real Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_d(\mathbb{C})$ be the derived algebra of the complex Lie algebra $\mathfrak{g}(\mathbb{C})$. Let $G^*(\mathbb{C})$ be the connected complex adjoint group of $\mathfrak{g}_d(\mathbb{C})$.

Definition 20.1 *G satisfies condition A if the image of G under the adjoint map Ad is contained in $G^*(\mathbb{C})$.*

Recall [26] G is of Harish-Chandra's class if G has finitely many connected components, $Z(G_d)$ is finite, and condition A holds.

The kernel of Ad is $\mathrm{Cent}_G(G^0)$, so let $\mathrm{Ad}(G) = G/\mathrm{Cent}_G(G^0)$. Since $Z(G) \subset \mathrm{Cent}_G(G^0)$ there is a natural map $p : G/Z(G) \to \mathrm{Ad}(G)$.

Definition 20.2 *G satisfies condition B if it satisfies condition A and there is connected, complex group $G^1(\mathbb{C})$ with adjoint group $G^*(\mathbb{C})$ and an injective homomorphism ϕ making the following diagram commute:*

$$\begin{array}{ccc} G/Z(G) & \xrightarrow{\ \phi\ } & G^1(\mathbb{C}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Ad} \\ Ad(G) & \xrightarrow{\ Ad\ } & G^*(\mathbb{C}) \end{array}$$

For example G satisfies condition B if it satisfies condition A and $Z(G) = \mathrm{Cent}_G(G^0)$ (take $G^1(\mathbb{C}) = G^*(\mathbb{C})$). Note that $GL(2,\mathbb{R})$ satisfies condition B, but any admissible two-fold cover of $GL(2,\mathbb{R})$ (cf. Section 3) satisfies condition A (and is of Harish-Chandra's class) but not condition B. The failure of nonlinear groups to satisfy condition B requires us to extend Hirai's results.

Recall a connected complex Lie group is *acceptable* if ρ (one-half the sum of the positive roots) exponentiates to a character of a Cartan subgroup (cf. Section 2).

Definition 20.3 *G is acceptable if it satisfies condition A and there is an acceptable, connected, complex group $G^1(\mathbb{C})$ with adjoint group $G^*(\mathbb{C})$ and a homomorphism ϕ making the following diagram commute:*

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G^1(\mathbb{C}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Ad} \\ Ad(G) & \xrightarrow{Ad} & G^*(\mathbb{C}) \end{array}$$

This definition of acceptability for disconnected groups is very strong. For example, $GL(2,\mathbb{R})$ is not an acceptable group (there is no nontrivial homomorphism from $GL(2,\mathbb{R})$ to $SL(2,\mathbb{C})$), even though its identity component is acceptable.

Hirai's Theorem gives necessary and sufficient conditions for a class function on G' to be an invariant eigendistribution when G satisfies condition B and is acceptable. That the conditions are necessary requires only condition A and acceptability, and is a fairly straightforward extension of results of Harish-Chandra [10]. Hirai's main contribution is to show that the conditions are sufficient, and for this he requires condition B.

Let G be a reductive Lie group satisfying condition A. Then the algebra of all left and right invariant differential operators on G is equal to the center $\mathfrak{Z}$ of the universal envelopping algebra. In [16] Hirai showed how to extend the results of Harish-Chandra to show that when Θ is a G^0-invariant eigendistribution for the action of $\mathfrak{Z}$, then Θ is given by integration against a locally integrable function on G which we also call Θ. Further, Θ is analytic on G', the set of regular semisimple elements, and is a G^0-invariant function, that is $\Theta(xgx^{-1}) = \Theta(g), x \in G^0, g \in G'$.

We assume for the remainder of this appendix that G is a reductive Lie group satisfying condition A and use the notation of Section 13.

Definition 20.4 *Let ν be a character of $\mathfrak{Z}$ and let Θ be a G^0-invariant function on G'. We say $\Theta \in IE(\nu)$ if Θ is the analytic function on G' cor-*

responding to a G^0-invariant eigendistribution with infinitesimal character ν.

Let Θ be a G^0-invariant function on G'. Let H be a Cartan subgroup of G, Φ^+ a choice of positive roots, and use the notation of Section 9. Define (20.5)

$$\Psi^0(H,\Phi^+,h) = \Delta^0(\Phi^+,h)\Theta(h), \quad \Psi(H,\Phi^+,h) = \epsilon_r(\Phi^+,h)\Psi^0(H,\Phi^+,h).$$

Suppose that G is acceptable with $\phi : G \to G^1(\mathbb{C})$ satisfying the conditions of Definition 20.3. Let $\rho = \rho(\Phi^+)$ and write $\xi_\rho(h) = e^\rho(\phi(h)), h \in H$. For $h \in H'$ define
(20.6)

$$\Psi^0_\rho(H,\Phi^+,h) = \xi_\rho(h)\Psi^0(H,\Phi^+,h), \quad \Psi_\rho(H,\Phi^+,h) = \xi_\rho(h)\Psi(H,\Phi^+,h).$$

The functions $\Psi^0_\rho(H,\Phi^+)$ and $\Psi_\rho(H,\Phi^+)$ are the ones used by Hirai to state his conditions. They can only be defined for acceptable groups. We first restate Hirai's conditions in terms of the functions $\Psi^0(H,\Phi^+)$ and $\Psi(H,\Phi^+)$ which can be defined without the assumption of acceptability.

For $X \in \mathfrak{h}$, define D_X and D^ρ_X as in (13.1) and (13.2). The following lemma follows easily from the definitions.

Lemma 20.7 *Assume that G is acceptable. Let $F \in C^\infty(H')$ and define $F_\rho(h) = \xi_\rho(h)F(h), h \in H'$. Then*

$$DF_\rho(h) = \xi_\rho(h)D^\rho F(h), h \in H', D \in S(\mathfrak{h}(\mathbb{C})).$$

Let $F : H' \to \mathbb{C}$, and let ν be a character of $\mathfrak{Z}$. Define conditions (C1, Φ^+,ν) and (C2) as in Section 13. In the case that G is acceptable, we say F satisfies condition (CA1, ν) if it satisfies (C1, Φ^+,ν) with D^ρ replaced by D.

Lemma 20.8
1. *If $\Psi(H,\Phi^+)$ satisfies condition (C1, Φ^+,ν) for one choice of positive roots, then $\Psi(H,\Phi^+)$ satisfies condition (C1, Φ^+,ν) for any choice of positive roots.*
2. *Assume that G is acceptable. Then $\Psi_\rho(H,\Phi^+)$ satisfies condition (CA1, ν) if and only if $\Psi(H,\Phi^+)$ satisfies condition (C1, Φ^+,ν).*

Proof. (1) Let Φ^+ be any choice of positive roots. Since $\epsilon_r(\Phi^+,h)$ is locally constant on H', $\Psi(H,\Phi^+)$ satisfies condition (C1, Φ^+,ν) if and only if $\Psi^0(H,\Phi^+)$ satisfies condition (C1, Φ^+,ν). Let $w \in W(\Phi)$ so that $w\Phi^+$ is another choice of positive roots for Φ with $\rho(w\Phi^+) = w\rho$. Then $\Delta^0(w\Phi^+,h) =$

$\epsilon(w)e^{\rho-w\rho}(h)\Delta^0(\Phi^+,h), h\in H$, where $\epsilon(w)=\pm1$ is the determinant of w. Thus

$$\Psi^0(H,w\Phi^+,h)=\epsilon(w)e^{\rho-w\rho}(h)\Psi^0(H,\Phi^+,h), h\in H'. \tag{20.9}$$

Thus for any $X\in\mathfrak{h}, h\in H'$,

$$D^{w\rho}_X\Psi^0(H,w\Phi^+,h)=\epsilon(w)e^{\rho-w\rho}(h)D^{\rho}_X\Psi^0(H,\Phi^+,h).$$

Thus $\Psi^0(H,w\Phi^+)$ satisfies condition (C1, $w\Phi^+,\nu$) if and only if $\Psi^0(H,\Phi^+)$ satisfies condition (C1, Φ^+,ν).

(2) follows from Lemma 20.7 and the fact that $\xi_\rho(h)$ and $\xi_\rho^{-1}(h)$ are both real analytic on H. □

The proof of the following lemma is similar to that of Lemma 20.8 since $\epsilon_r(\Phi^+,h)$ is constant on each connected component of $H'(R)$.

Lemma 20.10
1. *If $\Psi(H,\Phi^+)$ satisfies condition (C2) for one choice of positive roots, then $\Psi(H,\Phi^+)$ satisfies condition (C2) for every choice of positive roots.*
2. *Assume that G is acceptable. Then $\Psi_\rho(H,\Phi^+)$ satisfies condition (C2) if and only if $\Psi(H,\Phi^+)$ satisfies condition (C2).*

Let $\alpha\in\Phi_r^+$ and define $J,\Phi_J^+=c\Phi^+,H(\alpha)$, and $\beta=c^*(\alpha)$ as in Section 13. Let $h\in H(\alpha)$ and assume that $\Psi(H,\Phi^+)$ satisfies condition (C1, Φ^+,ν) and (C2). Then $D^{\rho,\pm}_{\check\alpha}\Psi(H,\Phi^+,h)$ are defined as in (13.4). There are also well-defined limits

$$D^{\rho}_{\check\alpha}\Psi^0(H,\Phi^+,h)=\lim_{s\to0}D^{\rho}_{\check\alpha}\Psi^0(H,\Phi^+,h\exp(s\check\alpha)). \tag{20.11}$$

and when G is acceptable

$$D_{\check\alpha}\Psi^0_\rho(H,\Phi^+,h)=\lim_{s\to0}D_{\check\alpha}\Psi^0_\rho(H,h\exp(s\check\alpha)). \tag{20.12}$$

Further, $h\in J'(R)$, so that $D^{\rho_J}_{\check\beta}\Psi^0(J,\Phi_J^+,h)$ and $D^{\rho_J}_{\check\beta}\Psi(J,\Phi_J^+,h)$ are defined, and when G is acceptable, $D_{\check\beta}\Psi^0_{\rho_J}(J,\Phi_J^+,h)$ is defined.

Define

$$\epsilon_r^\alpha(\Phi^+,h)=sign\prod_{\gamma\in\Phi_r^+,\gamma\neq\alpha}(1-e^{-\gamma}(h)), h\in H(\alpha). \tag{20.13}$$

Lemma 20.14 *For all $h \in H(\alpha)$,*

$$D^{\rho,\pm}_{\check{\alpha}}\Psi(H,\Phi^+,h) = \pm\epsilon^\alpha_r(\Phi^+,h)D^\rho_{\check{\alpha}}\Psi^0(H,\Phi^+,h);$$

$$D^{\rho_J}_{\check{\beta}}\Psi(J,\Phi^+_J,h) = \epsilon_r(\Phi^+_J,h)D^{\rho_J}_{\check{\beta}}\Psi^0(J,\Phi^+_J,h).$$

Proof. Fix $h \in H(\alpha)$. The first equation holds because for all small $s \neq 0$,

$$\epsilon_r(\Phi^+, h\exp(s\check{\alpha})) = \epsilon^\alpha_r(\Phi^+,h)sign(s).$$

The second equation holds because $h \in J'(R)$ so that $\epsilon_r(\Phi^+_J,h)$ is constant in a neighborhood of h. □

Lemma 20.15 *Assume that Φ^+ is a choice of positive roots such that α is a simple root for Φ^+_r. Then $\epsilon^\alpha_r(\Phi^+,h) = \epsilon_r(\Phi^+_J,h)$ for all $h \in H(\alpha)$.*

Proof. Fix $h \in H(\alpha)$. Write $\Phi^+_r\backslash\{\alpha\} = \Phi_1 \cup \Phi_2$ where

$$\Phi_1 = \{\gamma \in \Phi^+_r :< \gamma, \check{\alpha} >= 0\}$$

and Φ_2 is its complement in $\Phi^+_r\backslash\{\alpha\}$. Then the positive real roots in Φ_J are given by $\Phi^+_{r,J} = c^*\Phi_1$. Thus

$$\epsilon_r(\Phi^+_J,h) = sign\prod_{\gamma\in\Phi_1}(1-e^{-c^*\gamma}(h)) = sign\prod_{\gamma\in\Phi_1}(1-e^{-\gamma}(h))$$

since $e^{c^*\gamma}(h) = e^\gamma(h)$ for all $\gamma \in \Phi, h \in H(\alpha)$. Thus

$$\epsilon^\alpha_r(\Phi^+,h) = \epsilon_r(\Phi^+_J,h)sign\prod_{\gamma\in\Phi_2}(1-e^{-\gamma}(h)).$$

Let $\gamma \in \Phi_2$. Then since α is simple for Φ^+_r, $\gamma \neq \alpha$, and $< \gamma, \check{\alpha} >\neq 0$, $s_\alpha\gamma \in \Phi_2, s_\alpha\gamma \neq \gamma$. But $e^{-s_\alpha\gamma}(h) = e^{-\gamma}(h)$. Thus

$$sign\prod_{\gamma\in\Phi_2}(1-e^{-\gamma}(h)) = 1.$$

□

Consider the following conditions.

(20.16) $$D_{\check{\alpha}}\Psi^0_\rho(H,\Phi^+,h) = D_{\check{\beta}}\Psi^0_{\rho_J}(J,\Phi^+_J,h), h \in H(\alpha).$$

$$D^{\rho}_{\check{\alpha}}\Psi^0(H,\Phi^+,h) = D^{\rho_J}_{\check{\beta}}\Psi^0(J,\Phi^+_J,h), h \in H(\alpha). \tag{20.17}$$

Condition (20.16) is the condition used by Hirai in [16] for acceptable groups, and (20.17) is the analogous condition for not necessarily acceptable groups. It is useful for our applications in Section 14 to replace (20.17) by condition (C3) of Section 13. The following lemma clarifies the relationship between these conditions.

Lemma 20.18 *1. If $\Psi^0(H,\Phi^+)$ satisfies (20.17) for one choice of positive roots, then $\Psi^0(H,\Phi^+)$ satisfies (20.17) for every choice of positive roots.*

2. Let Φ^+ be a choice of positive roots such that α is simple for Φ^+_r. Then $\Psi^0(H,\Phi^+)$ satisfies (20.17) if and only if $\Psi(H,\Phi^+)$ satisfies condition (C3).

3. Assume that G is acceptable. Then $\Psi^0_\rho(H,\Phi^+)$ satisfies (20.16) if and only if $\Psi^0(H,\Phi^+)$ satisfies (20.17).

Proof. Let $h \in H(\alpha)$ and let Φ^+ be any choice of positive roots.

(1) Let $w \in W(\Phi)$. Then $w_J = c^*w(c^*)^{-1} \in W(\Phi_J)$, $c^*w\Phi^+ = w_J\Phi^+_J$ and $w_J\rho_J = \rho(w_J\Phi^+_J)$. Further,

$$D^{w\rho}_{\check{\alpha}}\Psi^0(H,w\Phi^+,h) = \epsilon(w)e^{\rho-w\rho}(h)D^{\rho}_{\check{\alpha}}\Psi^0(H,\Phi^+,h);$$

$$D^{w_J\rho_J}_{\check{\beta}}\Psi^0(J,w_J\Phi^+_J,h) = \epsilon(w_J)e^{\rho_J-w_J\rho_J}(h)D^{\rho_J}_{\check{\beta}}\Psi^0(J,\Phi^+_J,h).$$

But $\epsilon(w)=\epsilon(w_J)$ and $e^{\rho_J-w_J\rho_J}(h) = e^{c^*(\rho-w\rho)}(h) = e^{\rho-w\rho}(h)$. Thus $\Psi^0(H,w\Phi^+)$ satisfies (20.17) when $\Psi^0(H,\Phi^+)$ does.

(2) follows from combining Lemmas 20.14 and 20.15.

(3) Using Lemma 20.7 we have

$$D_{\check{\alpha}}\Psi^0_\rho(H,\Phi^+,h) = \xi_\rho(h)D^{\rho}_{\check{\alpha}}\Psi^0(H,\Phi^+,h);$$

$$D_{\check{\beta}}\Psi^0_{\rho_J}(J,\Phi^+_J,h) = \xi_{\rho_J}(h)D^{\rho_J}_{\check{\beta}}\Psi^0(J,\Phi^+_J,h).$$

Recall that ξ_ρ, ξ_{ρ_J} are defined using a homomorphism $\phi: G \to G^1(\mathbb{C})$. Let $H^1(\mathbb{C})$ and $J^1(\mathbb{C})$ be the Cartan subgroups of $G^1(\mathbb{C})$ with Lie algebras $\mathfrak{h}(\mathbb{C})$ and $\mathfrak{j}(\mathbb{C})$. Then we can pick a representative of the Cayley transform $c \in G^1(\mathbb{C})$ with $cH^1(\mathbb{C})c^{-1} = J^1(\mathbb{C})$ and $chc^{-1} = h$ for all $h \in H^1(\mathbb{C}) \cap J^1(\mathbb{C})$. Then for $h \in H(\alpha)$,

$$\xi_{\rho_J}(h) = e^{\rho_J}(\phi(h)) = e^{c^*\rho}(\phi(h)) = e^{\rho}(c\phi(h)c^{-1}) = e^{\rho}(\phi(h)) = \xi_\rho(h),$$

since $\phi(h) \in H^1(\mathbb{C}) \cap J^1(\mathbb{C})$. Thus $\Psi^0_\rho(H, \Phi^+)$ satisfies (20.16) if and only if $\Psi^0(H, \Phi^+)$ satisfies (20.17). $\square$

Definition 20.19 *Θ satisfies condition $C(\nu)$ if for every Cartan subgroup H of G and choice of positive roots Φ^+, $\Psi(H, \Phi^+)$ satisfies conditions (C1,Φ^+, ν) and (C2), and $\Psi^0(H, \Phi^+)$ satisfies (20.17) for all $\alpha \in \Phi^+_r$.*

Hirai's necessary and sufficient conditions are (CA1, ν), (C2), and (20.16). Using Lemmas 20.8, 20.10, and 20.18, when G is acceptable, Θ satisfies $C(\nu)$ if and only if Θ satisfies Hirai's conditions. Thus we can state Hirai's Theorem as follows.

Theorem 20.20 *(Hirai, [17]) Let G be a reductive Lie group that satisfies condition B and is acceptable. Let Θ be a G^0-invariant function on G' and let ν be a character of $\mathfrak{Z}$. Then $\Theta \in IE(\nu)$ if and only if Θ satisfies $C(\nu)$.*

Remark 20.21 Suppose that Θ is a G-invariant function on G'. If $\Theta \in IE(\nu)$, then the corresponding eigendistribution is G-invariant. Thus Θ corresponds to a G-invariant eigendistribution if and only if Θ satisfies $C(\nu)$.

Remark 20.22 In Hirai's statement of Theorem 20.20 in §11 of [17], he starts with the functions $\Psi_\rho(H, \Phi^+)$ (which he calls κ^j) on each of a set of representatives H^j of G^0-conjugacy classes of Cartan subgroups. He then gives an extra condition that he calls ϵ-symmetric which guarantees that they can be patched together to give a G^0-invariant function on G' which is our Θ. We don't use this condition since we assume from the beginning that we have a G^0-invariant function on G'.

We will prove the following extension of Hirai's Theorem.

Theorem 20.23 *Let G be a reductive Lie group which satisfies condition A. Let Θ be a G^0-invariant function on G' and let ν be a character of $\mathfrak{Z}$. Then $\Theta \in IE(\nu)$ if and only if Θ satisfies $C(\nu)$.*

We will prove Theorem 20.23 by making a number of reductions and then applying Hirai's theorem. A number of routine lemmas are stated without proof.

Let G be a reductive Lie group which satisfies condition A. Let Θ be a G^0-invariant function on G' and let ν be a character of $\mathfrak{Z}$. Since G is a Lie group, G has at most countably many connected components. Write $G = \cup_i x_i G^0$, and let $\Theta_i = \Theta\chi_i$ where χ_i is the characteristic function of x_iG^0. Then each Θ_i is a G^0-invariant function on G' which is supported on x_iG^0.

Lemma 20.24 *$\Theta \in IE(\nu)$ if and only if $\Theta_i \in IE(\nu)$ for all i. Further, Θ satisfies $C(\nu)$ if and only if Θ_i satisfies $C(\nu)$ for all i.*

Lemma 20.25 *Each connected component of G has a representative x such that $x^2 \in C_G(G^0)$ and $< x > \cap C_G(G^0)G^0 \subset C_G(G^0)$ where $< x >$ denotes the cyclic subgroup generated by x.*

Proof. Let $G^*(\mathbb{C})$ denote the connected complex adjoint group with Lie algebra $\mathfrak{g}_d(\mathbb{C})$. Since G satisfies condition A there is a homomorphism $p : G \to G^*(\mathbb{C})$ with kernel $C_G(G^0)$ such that $Ad\ p(g) = Ad\ g, g \in G$. In fact, $p(G) \subset G^*(\mathbb{R})$, the real points of $G^*(\mathbb{C})$. Fix $x \in G$. Then $p(xG^0)$ is a connected component of $G^*(\mathbb{R})$. If $p(xG^0) = (G^*)^0$, then $xG^0 \subset C_G(G^0)G^0$ so we can pick our representative $x \in C_G(G^0)$. Otherwise, $p(xG^0) = t(G^*)^0$ where $t^2 = 1$ and $t \notin (G^*)^0$. We can pick our representative x so that $p(x) = t$. Then $x \notin C_G(G^0)G^0$, but $p(x^2) = 1$ so $x^2 \in C_G(G^0)$. Thus $< x > \cap C_G(G^0)G^0 =< x^2 > \subset C_G(G^0)$. □

First Reduction. Because of Lemmas 20.24 and 20.25 we may as well assume that Θ is supported on xG^0 where $x \in G$ such that $x^2 \in C_G(G^0)$ and $< x > \cap C_G(G^0)G^0 \subset C_G(G^0)$.

Fix $x \in G$ such that $x^2 \in C_G(G^0)$ and $< x > \cap C_G(G^0)G^0 \subset C_G(G^0)$. Define $\tau : G^0 \to G^0$ by $\tau(g) = xgx^{-1}, g \in G^0$. Since $x^2 \in C_G(G^0)$ we have $\tau^2 = 1$. Let $G_\tau =< \tau > \ltimes G^0$ be the semidirect product of $< \tau >= \{1, \tau\}$ and G^0. That is, if $x \in C_G(G^0)$, then $\tau = 1$ and $G_\tau = G^0$. Otherwise $G_\tau = G^0 \cup \tau G^0$ has two connected components.

Define $\phi : xG^0 \to \tau G^0$ by $\phi(xg) = \tau g, g \in G^0$. Then ϕ is a diffeomorphism with $Ad(xg) = Ad(\phi(xg))$ for all $g \in G^0$ and $\phi(hxgh^{-1}) = h\phi(xg)h^{-1}$ for all $h, g \in G^0$. Thus there is a bijection between G^0-invariant functions Θ on G' which are supported on xG^0 and G^0-invariant functions Θ_τ on G'_τ which are supported on τG^0 given by $\Theta(xg) = \Theta_\tau(\phi(xg)), g \in G^0$.

Lemma 20.26 *Let Θ be a G^0-invariant function on G' which is supported on xG^0, and let Θ_τ be the corresponding G^0-invariant function on G'_τ which is supported on τG^0. Then $\Theta \in IE(\nu)$ as a function on G' if and only if $\Theta_\tau \in IE(\nu)$ as a function on G'_τ. Further, Θ satisfies $C(\nu)$ if and only if Θ_τ satisfies $C(\nu)$.*

Lemma 20.27 $C_{G_\tau}(G^0) = Z(G^0)$.

Proof. Clearly $C_{G_\tau}(G^0) \cap G^0 = Z(G^0)$. Suppose there is $g \in G^0$ such that $\tau g \in C_{G_\tau}(G^0)$. Then $\tau \in C_{G_\tau}(G^0)G^0$ so that $x \in < x > \cap C_G(G^0)G^0 \subset C_G(G^0)$ and $\tau = 1$. Thus $\tau G^0 \cap C_{G_\tau}(G^0) = \emptyset$ unless $\tau G^0 = G^0$. In any case we have $C_{G_\tau}(G^0) = C_{G_\tau}(G^0) \cap G^0 = Z(G^0)$. □

Second Reduction. Because of Lemma 20.26 we may as well replace G by G_τ. Thus we assume there is $x \in G$ with $x^2 = 1$ and $G =< x > \ltimes G^0$. Moreover, because of Lemma 20.27 we have $C_G(G^0) = Z(G^0)$. We continue to write $\tau(g) = xgx^{-1}, g \in G^0$.

Let C be a closed discrete subgroup of $Z(G^0)$ such that $\tau(C) = C$. Then C is a normal subgroup of G. Define $G_1 = G/C$ and $x_1 = xC$. Then $G_1^0 = G^0/C, x_1^2 = 1$, and $G_1 =< x_1 > \ltimes G_1^0$.

Suppose that Θ_1 is a G_1^0-invariant function on G_1. Then Θ_1 lifts to a G^0-invariant function Θ on G'

Lemma 20.28 $\Theta \in IE(\nu)$ *if and only if* $\Theta_1 \in IE(\nu)$. *Further,* Θ *satisfies* $C(\nu)$ *if and only if* Θ_1 *satisfies* $C(\nu)$.

Lemma 20.29 *Let* $C = \{\tau(z)z^{-1} : z \in Z(G^0)\}$. *Then* C *is a closed discrete subgroup of* $Z(G^0)$ *with* $\tau(C) = C$. *Define* $G_1 = G/C$. *Then* $Z(G_1) = C_{G_1}(G_1^0) = Z(G_1^0)$.

Proof. Define $\psi : Z(G^0) \to Z(G^0)$ by $\psi(z) = \tau(z)z^{-1}$. Then ψ is a homomorphism so that $C = \phi(Z(G^0))$ is a subgroup of $Z(G^0)$. Write $\mathfrak{g} = \mathfrak{g}_d \oplus \mathfrak{z}$ where $\mathfrak{g}_d = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$. Then $Z(G^0) = Z(G_d^0)Z$ where $Z = \exp\mathfrak{z}$. Now $Z(G_d^0)$ is a closed discrete subgroup of $Z(G^0)$ and $\tau(z) = z$ for all $z \in Z$. Thus $C = \{\tau(z)z^{-1} : z \in Z(G_d^0)\} \subset Z(G_d^0)$ and so is a closed discrete subgroup of $Z(G^0)$. Further, since $\tau^2 = 1$, we have $\tau(\psi(z)) = \psi(z^{-1}) \in C$ for all $z \in Z(G^0)$. Thus $\tau(C) = C$.

Write $p : G \to G_1 = G/C$ for the projection. Then $Z(G_1^0) = p(Z(G^0))$. Let $z \in Z(G^0)$. Then $\tau(z)z^{-1} \in C$ so that $p(x)p(z)p(x)^{-1} = p(\tau(z)) = p(z)$. Thus $p(z) \in Z(G_1)$. Thus $Z(G_1^0) \subset Z(G_1) \subset C_{G_1}(G_1^0)$. Conversely, let $g \in G$ and suppose that $p(g) \in C_{G_1}(G_1^0)$. Then $Ad\ g = Ad\ p(g) = 1$ so that $g \in C_G(G^0) = Z(G^0)$ by Lemma 20.27. Thus $p(g) \in Z(G_1^0)$. Thus $C_{G_1}(G_1^0) \subset Z(G_1^0)$. □

Lemma 20.30 *Define C as in Lemma 20.29. Let Θ be a G^0-invariant function on G' which is supported on xG^0. Then*

$$\Theta(gc) = \Theta(g), g \in G', c \in C.$$

That is, Θ factors to a G_1^0-invariant function Θ_1 on G_1' supported on $x_1G_1^0$.

Proof. Since Θ is supported on xG^0 and $C \subset G^0$, the result is true if $g \notin xG^0$. Let $g \in G^0$ with $xg \in G'$ and let $c \in C$. Then there is $z \in Z(G^0)$ such that $c = \tau(z)z^{-1} = xzx^{-1}z^{-1} = x^{-1}zxz^{-1}$. Since Θ is G^0-invariant and $x^{-1}zx \in Z(G^0)$ we have

$$\Theta(xg) = \Theta(zxgz^{-1}) = \Theta(xx^{-1}zxgz^{-1}) = \Theta(xgx^{-1}zxz^{-1}) = \Theta(xgc).$$

□

Third Reduction. By Lemmas 20.28 and 20.30 we may as well assume that $G = G_1$. But by Lemma 20.29 we have $Z(G_1) = C_{G_1}(G_1^0) = Z(G_1^0)$. That is, we can assume that $Z(G) = C_G(G^0) = Z(G^0)$ and there is $x \in G$ with $x^2 = 1$ and $G = < x > \ltimes G^0$.

The following Lemma completes the proof of Theorem 20.23.

Lemma 20.31 *Let G be a reductive Lie group satisfying condition A. Assume that $Z(G) = C_G(G^0) = Z(G^0)$ and there is $x \in G$ with $x^2 = 1$ and $G = < x > \ltimes G^0$. Let Θ be a G^0-invariant function on G' and let ν be a character of $\mathfrak{Z}$. Then $\Theta \in IE(\nu)$ if and only if Θ satisfies $C(\nu)$.*

Proof. Write $\mathfrak{g} = \mathfrak{g}_d \oplus \mathfrak{z}$ as in Lemma 20.29. Then $G^0 = G_d^0 Z$ where G_d^0 and Z are the connected subgroups of G corresponding to $\mathfrak{g}_d$ and $\mathfrak{z}$ respectively. Since G_d^0 is a connected semisimple Lie group there is a connected finite central extension $p_1 : G_1 \to G_d^0$ such that G_1 is acceptable. That is, there is a connected acceptable complex Lie group $G_2(\mathbb{C})$ with Lie algebra $\mathfrak{g}_d(\mathbb{C})$ such that the inclusion map of $\mathfrak{g}_d \subset \mathfrak{g}_d(\mathbb{C})$ lifts to a homomorphism $\phi_1 : G_1 \to G_2(\mathbb{C})$. Let $K = ker\ p_1 \cap ker\ \phi_1 \subset Z(G_1)$ and define $\overline{G}_1 = G_1/K$. Then p_1 and ϕ_1 both factor to $\overline{G}_1$, so that we have $\overline{p}_1 : \overline{G}_1 \to G_d^0$ and $\overline{\phi}_1 : \overline{G}_1 \to G_2(\mathbb{C})$. Thus $\overline{G}_1$ is also an acceptable cover of G_d^0. Thus we may as well assume that $ker\ p_1 \cap ker\ \phi_1 = \{1\}$.

Since G satisfies condition A and $G_2(\mathbb{C})$ is a complex Lie group with Lie algebra $\mathfrak{g}_d(\mathbb{C})$ there is $x_2 \in G_2(\mathbb{C})$ such that $Ad\ x = Ad\ x_2$. Then $Ad\ x_2^2 = Ad\ x^2 = 1$ so that $x_2^2 \in Z(G_2(\mathbb{C}))$. Thus x_2 has finite order. Let

$p_2 : \widetilde{G}_1 \to G_1$ be the simply connected cover of G_1. Then there is a unique automorphism τ of $\widetilde{G}_1$ such that

$$\tau(\widetilde{\exp} X) = \widetilde{\exp}(Ad\ xX) = \widetilde{\exp}(Ad\ x_2 X), X \in \mathfrak{g}.$$

It satisfies $\tau^2 = 1$, and

$$p_1 p_2 \tau(\widetilde{g}) = x p_1 p_2(\widetilde{g}) x^{-1}, \ \ \phi_1 p_2 \tau(\widetilde{g}) = x_2 \phi_1 p_2(\widetilde{g}) x_2^{-1}, \ \widetilde{g} \in \widetilde{G}_1.$$

Suppose that $\widetilde{g} \in ker\ p_2$. Then

$$p_1 p_2 \tau(\widetilde{g}) = x p_1 p_2(\widetilde{g}) x^{-1} = 1, \ \ \phi_1 p_2 \tau(\widetilde{g}) = x_2 \phi_1 p_2(\widetilde{g}) x_2^{-1} = 1.$$

Thus $p_2\tau(\widetilde{g}) \in ker\ p_1 \cap ker\ \phi_1 = \{1\}$, so that $\tau(\widetilde{g}) \in ker\ p_2$. Thus τ descends to give a well-defined automorphism $\tau : G_1 \to G_1$ satisfying $\tau^2 = 1$ and

$$p_1(\tau(g_1)) = x p_1(g_1) x^{-1}, \quad \phi_1(\tau(g_1)) = x_2 \phi_1(g_1) x_2^{-1}, g_1 \in G_1. \tag{20.32}$$

Assume that $z_1 \in Z(G_1)$. Then $p_1(z_1) \in Z(G_d^0) \subset Z(G^0) = Z(G)$ by assumption. Further, $Ad(\phi_1(z_1)) = Ad\ z_1 = 1$. This implies that $\phi_1(z_1) \in Z(G_2(\mathbb{C}))$ since $G_2(\mathbb{C})$ is connected. Thus $p_1(\tau(z_1)) = x p_1(z_1) x^{-1} = p_1(z_1)$ and $\phi_1(\tau(z_1)) = x_2 \phi_1(z_1) x_2^{-1} = \phi_1(z_1)$. Thus $\tau(z_1) z_1^{-1} \in ker\ p_1 \cap ker\ \phi_1 = \{1\}$. That is, $\tau(z_1) = z_1$ for all $z_1 \in Z(G_1)$.

Let $m = 2m_2$ where m_2 is the order of x_2 and write $\mathbb{Z}_m$ for the additive group of integers mod m. Let $\widetilde{G} = \{(z,k,g) : z \in Z, k \in \mathbb{Z}_m, g \in G_1\}$. For $z_1, z_2 \in Z, k, q \in \mathbb{Z}_m, g_1, g_2 \in G_1$, define the product

$$(z_1, k, g_1)(z_2, q, g_2) = (z_1 z_2, k + q, \tau^{-q}(g_1) g_2). \tag{20.33}$$

Thus $\widetilde{G}$ is the direct product of Z with the semidirect product of $\mathbb{Z}_m$ and G_1 where $k \in \mathbb{Z}_m$ acts on G_1 by τ^k. $\widetilde{G}^0$ is isomorphic to the direct product of Z and G_1 and so $\widetilde{G}$ is a reductive Lie group with Lie algebra $\mathfrak{g}$. We calculate that

$$(z, k, g)(z_2, q, g_2)(z, k, g)^{-1} = (z_2, q, \tau^k(\tau^{-q}(g) g_2 g^{-1})). \tag{20.34}$$

Thus $Ad(z,k,g) = Ad(x^k p_1(g))$ so that $\widetilde{G}$ satisfies condition A. We will show that

$$Z(\widetilde{G}) = C_{\widetilde{G}}(\widetilde{G}^0) = \{(z,k,g) : z \in Z, x^k = 1, g \in Z(G_1)\}. \tag{20.35}$$

Using (20.34) and the fact that $\tau(g) = g$ for all $g \in Z(G_1)$, it is clear that $\{(z,k,g) : z \in Z, x^k = 1, g \in Z(G_1)\} \subset Z(\widetilde{G}) \subset C_{\widetilde{G}}(\widetilde{G}^0)$. Suppose that $(z,k,g) \in C_{\widetilde{G}}(\widetilde{G}^0)$. Then $Ad(x^k p_1(g)) = Ad(z,k,g) = 1$, so that $x^k p_1(g) \in C_G(G^0) = Z(G) = Z(G^0)$ by assumption. Thus $x^k \in < x > \cap G^0 = \{1\}$. Thus $x^k = 1$ and $p_1(g) \in Z(G) \cap G^0_d = Z(G^0_d)$ so that $g \in Z(G_1)$. Thus $C_{\widetilde{G}}(\widetilde{G}^0) \subset \{(z,k,g) : z \in Z, x^k = 1, g \in Z(G_1)\} \subset Z(\widetilde{G})$. This shows that $\widetilde{G}$ satisfies condition B.

Define $\phi : \widetilde{G} \to G_2(\mathbb{C})$ by $\phi(z,k,g) = x_2^k \phi_1(g)$. It is well-defined because m is a multiple of the order of x_2. It is easy to show it is a homomorphism using (20.32). It is a continuous homomorphism which induces canonically the natural injection of $\mathrm{Ad}(\widetilde{G})$ into Ad $(G_2(\mathbb{C}))$. Thus $\widetilde{G}$ is acceptable.

Define $p : \widetilde{G} \to G$ by $p(z,k,g) = zx^k p_1(g)$. The mapping is well-defined since m is even, and it satisfies $Ad\ p(z,k,g) = Ad(z,k,g)$. It is easy to show that it is homomorphism using (20.32), and it is clearly surjective. Let $(z,k,g) \in ker\ p$ so that $zx^k p_1(g) = 1$. Then $Ad(z,k,g) = Adp(z,k,g) = 1$ so that $(z,k,g) \in C_{\widetilde{G}}(\widetilde{G}^0) = Z(\widetilde{G})$. Thus $ker\ p \subset Z(\widetilde{G})$. Further, using (20.35) we have $Z(\widetilde{G})^0 = \{(z,0,1) : z \in Z\}$. Thus the identity component of $ker\ p$ is contained in $Z(\widetilde{G})^0 \cap ker\ p = \{(1,0,1)\}$ so that $ker\ p$ is a discrete subgroup of $Z(\widetilde{G})$.

Let Θ be a G^0-invariant function on G' and let ν be a character of $\mathfrak{Z}$. Lift Θ to a $\widetilde{G}^0$-invariant function $\widetilde{\Theta}$ on $\widetilde{G}'$. Then $\Theta \in IE(\nu)$ if and only if $\widetilde{\Theta} \in IE(\nu)$ and Θ satisfies $C(\nu)$ if and only if $\widetilde{\Theta}$ satisfies $C(\nu)$. But $\widetilde{G}$ is acceptable and satisfies condition B. Thus $\widetilde{\Theta} \in IE(\nu)$ if and only if $\widetilde{\Theta}$ satisfies $C(\nu)$ by Theorem 20.20. □

References

[1] J. Adams. Unitary shimura correspondence for split real groups. *Journal of the American Mathematical Society*, 20(3). preprint.

[2] J. Adams. Characters of non-linear groups. *Proceedings of Conf. on Representation Theory and Harmonic Analysis, Okayama, Japan*, 26:1–18, 2000.

[3] J. Adams and P. Trapa. Duality for nonlinear simply laced groups. preprint, http://www.math.umd.edu/~jda/preprints/duality.pdf

[4] Jeffrey Adams. Lifting of characters on orthogonal and metaplectic groups. *Duke Math. J.*, 92(1):129–178, 1998.

[5] Jeffrey Adams. Characters of covering groups of SL(n). *J. Inst. Math. Jussieu*, 2(1):1–21, 2003.

[6] Jeffrey Adams. Nonlinear covers of real groups. *Int. Math. Res. Not.*, (75):4031–4047, 2004.

[7] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr. *The Langlands classification and irreducible characters for real reductive groups*, volume 104 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.

[8] Y. Flicker D. A. Kazhdan. Metaplectic correspondence. 64:53–110, 1986.

[9] Y. Flicker. Automorphic forms on covering groups of GL(2). 57:119–182, 1980.

[10] Harish-Chandra. Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119:457–508, 1965.

[11] Harish-Chandra. Harmonic analysis on real reductive groups. I. The theory of the constant term. *J. Functional Analysis*, 19:104–204, 1975.

[12] Harish-Chandra. Supertempered distributions on real reductive groups. In *Studies in applied mathematics*, volume 8 of *Adv. Math. Suppl. Stud.*, pages 139–153. Academic Press, New York, 1983.

[13] Henryk Hecht and Wilfried Schmid. Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules. *Acta Math.*, 151(1-2):49–151, 1983.

[14] Rebecca A. Herb. Two-structures and discrete series character formulas. In *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, pages 285–319. Amer. Math. Soc., Providence, RI, 2000.

[15] Takeshi Hirai. Invariant eigendistributions of Laplace operators on real simple Lie groups. I. Case of SU(p,q).. *Japan. J. Math.*, 39:1–68, 1970.

[16] Takeshi Hirai. Supplements and corrections to my paper: "The characters of some induced representations of semisimple Lie groups" (J. Math. Kyoto Univ. **8** (1968), 313–363). *J. Math. Kyoto Univ.*, 15:237–250, 1975.

[17] Takeshi Hirai. Invariant eigendistributions of Laplace operators on real simple Lie groups. II. General theory for semisimple Lie groups. *Japan. J. Math. (N.S.)*, 2(1):27–89, 1976.

[18] J.-S. Huang. The unitary dual of the universal covering group of $GL(n, \mathbf{R})$. 61:705–745, 1990.

[19] J.-S. Huang J. Adams. Kazhdan-patterson lifting for $\mathrm{GL}(n, \mathbf{R})$. 89(3):423–444, 1997.

[20] D. A. Kazhdan and S. J. Patterson. Metaplectic forms. *Inst. Hautes Études Sci. Publ. Math.*, (59):35–142, 1984.

[21] D. A. Kazhdan and S. J. Patterson. Towards a generalized Shimura correspondence. *Adv. in Math.*, 60(2):161–234, 1986.

[22] A. Knapp. *Representation Theory of Semisimple Groups. An overview based on Examples*. Princeton University Press, Princeton, NJ, 1986.

[23] David Renard. Endoscopy for $\mathrm{Mp}(2n, \mathbf{R})$. *Amer. J. Math.*, 121(6):1215–1243, 1999.

[24] D. Shelstad. Characters and inner forms of a quasi-split group over $\mathbf{R}$. *Compositio Math.*, 39(1):11–45, 1979.

[25] D. Shelstad. l-indistinguishability for real groups. 259:385–430, 1982.

[26] D. Vogan. *Representations of Real Reductive Lie Groups*, volume 15 of *Progress in mathematics*. Birkhäuser, Boston, 1981.

[27] D. Vogan. Irreducible characters of semisimple Lie groups IV. character-multiplicity duality. 49, No. 4:943–1073, 1982.

[28] David A. Vogan, Jr. Unitarizability of certain series of representations. *Ann. of Math. (2)*, 120(1):141–187, 1984.