Redundancy for localized frames

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Abstract

Redundancy is the qualitative property which makes Hilbert space frames so useful in practice. However, developing a meaningful quantitative notion of redundancy for infinite frames has proven elusive. Though quantitative candidates for redundancy exist, the main open problem is whether a frame with redundancy greater than one contains a subframe with redundancy arbitrarily close to one. We will answer this question in the affirmative for ℓ^1-localized frames. We then specialize our results to Gabor multi-frames with generators in $M^1(\mathbf{R}^d)$, and Gabor molecules with envelopes in $W(C, l^1)$. As a main tool in this work, we show there is a universal function $g(x)$ so that for every $\epsilon > 0$, every Parseval frame $\{f_i\}_{i=1}^M$ for an N-dimensional Hilbert space H_N has a subset of fewer than $(1+\epsilon)N$ elements which is a frame for H_N with lower frame bound $g(\epsilon/(2\frac{M}{N}-1))$. This work provides the first meaningful quantative notion of redundancy for a large class of infinite frames. In addition, the results give compelling new evidence in support of a general definition of redundancy given in [5].

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1 Introduction

A basis $\{x_i\}_{i\in I_0}$ for a Hilbert space H (finite or infinite) with an index set I_0 provides a decomposition of any element $x \in H$ as a *unique* linear combination of the basis elements: $x = \sum_{i\in I_0} c_i x_i$. For many applications, this uniqueness of decomposition is the feature that makes bases such a useful structure. However, there are fundamental signal processing issues for which the uniqueness of the coefficients $\{c_i\}_{i\in I_0}$ for a given element $x \in H$ is not a desired quality. These include the following two tasks: a) finding ways to represent elements when some of the coefficients c_i are going to be subject to loss or noise, and b) finding ways to compactly represent a meaningful approximation $x' \approx x$, i.e. finding an approximation $x' = \sum_i c'_i x_i$ that has few non-zero coefficients. For both these tasks, one observes that choosing to express x in terms of a larger set $\{f_i\}_{i\in I}$ that is overcomplete in H has potential advantages. With this setup, any vector $x \in H$ can be written as $\sum_{i\in I} c_i f_i$ in many different ways, and this freedom is advantageous for either of the above tasks. It can allow for a choice of $\{c_i\}_{i\in I}$ with additional structure which can be used in the first task to counter the noise on the coefficients as well as transmission losses. This same freedom of choice of $\{c_i\}_{i\in I}$ yields many more candidates for a compact meaningful approximation x' of the element x.

These overcomplete sets $\{f_i\}_{i\in I}$ (with some added structure when I is infinite) are known as *frames*. They are defined as follows: let H be a separable Hilbert space and I a countable index set. A sequence $\mathcal{F} = \{f_i\}_{i\in I}$ of elements of H is a *frame* for H if there exist constants A, $B > 0$ such that

$$A\,\|h\|^2 \le \sum_{i\in I} |\langle h, f_i\rangle|^2 \le B\,\|h\|^2, \quad \text{for all } h \in H. \tag{1}$$

The numbers A, B are called *lower* and *upper frame bounds*, respectively. When $A = B = 1$ the frame is said to be *Parseval.* The *frame operator* is the operator $S : H \to H$, $S(x) = \sum_{i\in I} \langle x, f_i \rangle f_i$, which is bounded and invertible when $\{f_i\}_{i\in I}$ is a frame.

Frames were first introduced by Duffin and Schaeffer [10] in the context of nonharmonic Fourier series, and today frames play important roles in many applications in mathematics, science, and engineering. We refer to the monograph [9], or the research-tutorial [7] for basic properties of frames.

Central, both theoretically and practically, to the interest in frames has been their overcomplete nature; the strength of this overcompleteness is the ability of a frame to express arbitrary vectors as a linear combination in a "redundant" way. For infinite dimensional frames, quantifying overcompleteness or redundancy has proven to be challenging. What has been missing are results that connect redundancy of a frame to the ability to remove large numbers of elements from the frame and still have the remaining elements form a frame. More formally, when imagining a measure of redundancy for infinite frames, an essential desired property would be a version of the following:

$\mathbf{P_1}$: Any frame with redundancy bigger than one would contain in it a frame with redundancy arbitrarily close to one.

In this work, we show that for two large classes of frames – a broad class of Gabor systems, and l^1 localized frames – the density of certain sets associated to the frame, termed the *frame density*, has property P_1. When combined with other work, this establishes the frame density for these classes of frames as a legitimate quantitative definition of redundancy. Furthermore, it provides an additional piece of evidence in support of a more general definition of frame redundancy given in [5] which applies to frames even when a notion of density is not apparent.

1.1 Results: finding subframes of density close to 1 for Gabor and localized frames.

A well studied important class of frames are the so called Gabor Frames. A *Gabor Frame* is defined to be a frame $\mathcal{F}$, generated from time-frequency shifts of a *generator* function $f \in L^2(\mathbf{R}^d)$. Specifically, given $f \in L^2(\mathbf{R}^d)$ along with a subset $\Lambda \subset \mathbf{R}^{2d}$:

$$\mathcal{F} = \{f_\lambda\}_{\lambda\in\Lambda} \text{ where for } \lambda = (\alpha, \beta), \quad f_\lambda(x) = e^{2\pi i\langle\alpha,x\rangle} f(x - \beta).$$

The structure of the set Λ, more specifically various measures of the density of Λ (see Sections 2 and 6) has been crucial in the study of Gabor frames.

Over the last 40 years (since H.J. Landau [23] gave a density condition for Gabor frames whose generators were certain entire functions), partial progress towards a quantitative notion of redundancy has occurred for both lattice and general Gabor frames. Many works have connected essential features of the frames to quantities related to the density Λ of the associated set of time and frequency shifts (See [20] and references therein). As dynamic as these results were, they could not be used to show that the obvious choice for redundancy, namely the density of Λ, satisfied any version of property P_1.

Additional results about redundancy of arbitrary frames or results relating to property P_1 for Gabor frames have remained elusive. Recent work, however, has made significant advances in quantifying redundancy of infinite frames. Progress began with the work in [1, 2, 3, 4] which examined and explored the notion of *excess* of a frame, i.e. the maximal number of frame elements that could be removed while keeping the remaining elements a frame for the same span. This work, however, left open many questions about frames with infinite excess (which include, for example, Gabor frames that are not Riesz bases).

A quantitative approach to a large class of frames with infinite excess (including Gabor frames) was given in [3, 4] which introduced a general notion of *localized* frames (see also [19] and then [17] that independently introduced a similar notion and started a seminal discussion of frame localization). The notion of localization is between two frames $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{E} = \{e_j\}_{j\in G}$ (G a discrete abelian group), and describes the decay of the expansion of the elements of $\mathcal{F}$ in terms of the elements of $\mathcal{E}$ via a map $a\colon I \to G$. With this set up, the density of the set $a(I)$ in G is a crucial quantity. For irregular sets $a(I)$, the density of $a(I)$ in G is not a single number but takes on different values depending on additional choices, related to a finite decomposition of G and ultrafilters, which are described Section 6. For the purposes of this introduction, we imagine these choices have been made and use the term *frame* *density* to refer to the resulting density of the set $a(I)$ in G. Among other results, [3, 4] shows that in the localized setting, the frame density can be used to provide nice quantitative measures of frames. A weak partial result related to property P_1 was given in [3, 4] where it was shown that for any localized frame $\mathcal{F}$ with frame density equal to d there exists an $\epsilon > 0$ and a subframe of $\mathcal{F}$ with corresponding frame density $d - \epsilon$. It is conjectured in [3] that a version of property P_1 should hold for the frame density, and that such a result would establish frame density as a quantitative measure of redundancy.

In this paper, we prove this conjecture: we show that for l^1 localized frames, the frame density has property P_1. We show that for any $0 < \epsilon < 1$ every l^1 localized frame with frame density $d > 1$ has a subframe with frame density smaller than $1+\epsilon$. Precisely, we show (see Section 2 for notation and definitions):

Theorem 1.1. *Assume $\mathcal{F} = \{f_i\,;\, i \in I\}$ is a frame for H, $\mathcal{E} = \{e_k\,;\, k \in G\}$ is a l^1-self localized frame for H, with G a discrete countable abelian group, $a : I \to G$ a localization map of finite upper density so that $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized and has finite upper density. Then for every $\varepsilon > 0$ there exists a subset $J = J_\varepsilon \subset I$ so that $D^+(a; J) \leq 1 + \varepsilon$ and $\mathcal{F}[J] = \{f_i; i \in J\}$ is frame for H.*

When specialized to Gabor frames, the result reads

Theorem 1.2. *Assume $\mathcal{G}(g;\Lambda)$ is a Gabor frame for $L^2(\mathbf{R}^d)$ with $g \in M^1(\mathbf{R}^d)$. Then for every $\varepsilon > 0$ there exists a subset $J_\varepsilon \subset \Lambda$ so that $\mathcal{G}(g; J_\varepsilon)$ is a Gabor frame for $L^2(\mathbf{R}^d)$ and its upper Beurling density satisfies $D_B^+(J_\varepsilon) \leq 1 + \varepsilon$.*

This result admits generalizations to both Gabor multi-frame and Gabor molecule settings (see Section 5).

The work hinges on a fundamental finite dimensional result that is of independent interest. For Parseval frames, the result says that an M-element Parseval frame for H_N (an N dimensional Hilbert space) contains a subframe of less than $(1+\epsilon)N$ elements with lower frame bound a function of $g(\epsilon, M/N)$, where g is a universal function. The precise statement of the general result is given in Lemma 3.2:

Lemma 3.2 (Finite dimensional removal)**.** *There exists a monotonically increasing function $g : (0,1) \to (0,1)$ with the following property. For any set $\mathcal{F} = \{f_i\}_{i=1}^{M}$ of M vectors in a Hilbert space of dimension N, and for any $0 < \epsilon < \frac{M}{N} - 1$ there exists a subset $\mathcal{F}_\epsilon \subset \mathcal{F}$ of cardinality at most $(1+\epsilon)N$ so that:*

$$S_{\mathcal{F}_\varepsilon} \geq g\left(\frac{\epsilon}{2\frac{M}{N}-1}\right) S_{\mathcal{F}} \tag{2}$$

where $S_{\mathcal{F}} f = \sum_{f\in\mathcal{F}} \langle \cdot, f\rangle f$ and $S_{\mathcal{F}_\varepsilon} f = \sum_{f\in\mathcal{F}_\epsilon} \langle \cdot, f\rangle f$ are the frame operators associated to $\mathcal{F}$ and $\mathcal{F}_\varepsilon$, respectively.

1.2 Consequences: Redundancy

These results complete a nice picture of redundancy for two large classes of frames: a broad class of Gabor systems, and l^1 localized frames. When imagining a measure of redundancy for infinite frames, in addition to property P_1 that is the focus of this work, a wish list of desired properties would include:

- $\mathbf{P_2}$: The redundancy of any frame for the whole space would be greater than or equal to one.
- $\mathbf{P_3}$: The redundancy of a Riesz basis would be exactly one.
- $\mathbf{P_4}$: The redundancy would be additive on finite unions of frames.

Combining Theorem 5.5 with some of the results in [3, 4] establishes that for a large class of Gabor frames, the density of the set Λ is a legitimate quantative measure of redundancy (see Theorem 6.4 in Section 6 for a formal statement).

Theorem 1.3. *For a Gabor molecule with envelope in $W(C, l^1)$, the Beurling density of its label set satisfies the properties of redundancy specified in P_1-P_4.*

What about similar results to Theorem 6.4 for localized frames? In this case we fix a frame $\mathcal{E}$ indexed by a countable abelian group and consider the class of all frames $\mathcal{F}$ that are l^1 localized with respect to $\mathcal{E}$. If $\mathcal{E}$ is a Riesz basis, as in the Gabor setting the frame density can be shown to satisfy the four desired redundancy properties. If $\mathcal{E}$ is a frame but not necessarily a Riesz basis, two of the desired properties are satisfied:

Theorem 1.4. *For frames $\mathcal{F}$ that are l^1 localized with respect to a fixed frame $\mathcal{E}$ indexed by a countable abelian group, the frame density of $\mathcal{F}$ satisfies the properties P_1 and P_4. If $\mathcal{E}$ is a Riesz basis then the frame density satisfies all properties $P_1 - P_4$.*

The significance of Theorems 1.3 and 1.4 is that they provide, for the first time, quantitative notions of redundancy for two large classes of frames that satisfy all four of the desired properties listed above.

We remark that there are at least two potentially fruitful ways to view these results. The first is to view frame density as *the* measure of redundancy. From this point of view natural questions include defining notions of density for other classes of frames and proving comparable results.

The second point of view, which we elaborate upon here, is to view these results in the context of the work [5] which quantified overcompleteness for all frames that share a common index set. In this context, frame density should not be thought of as redundancy but rather as a computational tool for computing redundancy in the class of frames treated here. Specifically, we begin by remarking that in contrast to the Gabor molecule case, the density of a localized frame $\mathcal{F}$ depends on the frame $\mathcal{E}$ that it is localized with respect to. When $\mathcal{E}$ is a Riesz basis, the density is "normalized" and as a result it satisfies two properties P_2 and P_3 that fail to hold in the "unnormalized" case of $\mathcal{E}$ an arbitrary frame. Even when $\mathcal{E}$ is a Riesz basis, the frame density is not an intrinsic property of the frame $\mathcal{F}$ and could have different values when localized with respect to different Riesz bases. This dependence on the frame that $\mathcal{F}$ is localized with respect to can be viewed as problematic for an optimal definition of redundancy. In contrast, [5] defines an intrinsic notion of redundancy that applies to all frames that share a common index set. The essential tool there was the so called *frame measure function* which is a function of certain averages of $\langle f_i, \tilde{f}_i \rangle$, the inner product of the frame element with its corresponding dual frame element $\tilde{f}_i$. A *redundancy function* for infinite frames was defined to be the reciprocal of the frame measure function. In the case of l^1 localized frames this redundancy function satisfies all properties $P_1 - P_4$. (see Section 7 for a more complete discussion).

1.3 Organization

The work is organized as follows. We begin by reviewing the definition of localized frames. In Section 3 we prove the above mentioned fundamental finite dimensional result (Lemma 3.2). We then prove a "truncation" result which is used later to reduce the infinite dimensional case to a sequence of finite dimensional cases. Section 4 contains the proof of Theorem 1.1. We first prove Theorem 1.1 for ℓ^1-localized Parseval frames and then generalize this to arbitrary ℓ^1-localized frames. In Section 5 we apply this result to Gabor Multi-frames with generators in $M^1(\mathbf{R}^d)$, and Gabor molecules with envelopes in $W(C, l^1)$ and get as a Corollary Theorem 1.2. In Section 6 we formally define the frame density and prove Theorems 1.3 and 1.4. Finally in Section 7 we discuss consequences in terms of the redundancy function introduced in [5].

2 Notation: localized frames

The idea of localized frames in the way it is used here, was introduced in [3]. A very similar notion of frame localization was introduced by Gröchenig in his seminal paper [19] and then studied further e.g. in [17]. For this paper, the starting point will be a Hilbert space H, along with two frames for H: $\mathcal{F} = \{f_i \ , \ i \in I\}$ indexed by the countable set I, and $\mathcal{E} = \{e_k \ ; \ k \in G\}$ indexed by a discrete countable abelian group G. Here we will assume $G = \mathbf{Z}^d \times \mathbf{Z}_D$ for some integers $d, D \in \mathbf{N}$, where $\mathbf{Z}_D = \{0, 1, 2, \ldots, D-1\}$ is the cyclic group of size D.

We relate the frames $\mathcal{F}$ and $\mathcal{E}$ by introducing a map $a : I \to G$ between their index sets. Following [19, 17, 3] we say $(\mathcal{F}, a, \mathcal{E})$ is l^p *localized* if

$$\sum_{k \in G} \sup_{i \in I} |\langle f_i, e_{a(i)-k} \rangle|^p < \infty \tag{3}$$

Here $1 \le p < \infty$.

We shall denote by $\mathbf{r} = (r(g))_{g \in G}$ the localization sequence for $\mathcal{F}$ with respect to $\mathcal{E}$, i.e.

$$r(g) = \sup_{i \in I, k \in G, a(i)-k=g} |\langle f_i, e_k \rangle|.$$

Thus $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized if and only if the localization sequence $\mathbf{r}$ is in $l^1(G)$. That is,

$$\|r\|_1 = \sum_{k \in G} r(k) < \infty \tag{4}$$

Similarily, the set $\mathcal{E}$ is said to be *l^1-self localized* if

$$\sum_{k \in G} \sup_{g \in G} |\langle e_{k+g}, e_g \rangle| < \infty \tag{5}$$

In other words, $\mathcal{E}$ is l^1-self localized if and only if $(\mathcal{E}, i, \mathcal{E})$ is l^1-localized, where $i : G \to G$ is the identity map. We denote by $\mathbf{s} = (s(g))_{g \in G}$ the self-localization sequence of $\mathcal{E}$, that is $s(g) = \sup_{k,l \in G, k-l=g} |\langle e_k, e_l \rangle|$.

An important quantity will be the l^1 norm of the tail of $\mathbf{r}$, namely

$$\Delta(R) := \sum_{|k| \geq R} r(k), \tag{6}$$

and thus if $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized, $\lim_{R \to \infty} \Delta(R) = 0$.

The *upper and lower densities* of a subset $J \subset I$ with respect to the map $a : I \to G$ are defined by

$$D^+(a; J) = \limsup_{N \to \infty} \sup_{k \in G} \frac{|a^{-1}(B_N(k)) \cap J|}{|B_N(0)|} \tag{7}$$

$$D^-(a; J) = \liminf_{N \to \infty} \inf_{k \in G} \frac{|a^{-1}(B_N(k)) \cap J|}{|B_N(0)|} \tag{8}$$

where $B_N(k) = \{g \in G\, ;\, |g - k| \leq N\}$ is the box of radius N and center k in G, and $|Q|$ denotes the number of elements in the set Q. Note that $|B_N(k)| = |B_N(k')|$ for all $k, k' \in G$ and $N > 0$. When $J = I$ we simply call $D^\pm(a; I)$ the *densities of* I, or the *densities of the map* a, and we denote them by $D^\pm(I)$ or $D^\pm(a)$. The map a (or, equivalently, the set I) is said to have *finite upper density* if $D^+(I) < \infty$. As proved in Lemma 2 of [3], if a has finite upper density, then there is $K_a \geq 1$ so that

$$|a^{-1}(B_N(k))| \leq K_a |B_N(0)| \tag{9}$$

for all $k \in G$ and $N > 0$. The finiteness of upper density is achieved when frame vectors have norms uniformly bounded away from zero (see Theorem 4 of [3]).

3 Two important lemmas

In this section we will prove two lemmas (Lemma 3.2 and Lemma 3.5) that will be the essential ingredients for the proof of the main result (Theorem 1.1).

3.1 Finite dimensional removal

Here we consider the following: a finite frame $\mathcal{F} = \{f_i\}_{i=1}^M$ of M vectors on an N dimensional space H. We are interested in finding a subset of $\mathcal{F}$ of small size that remains a frame for H. As the following example illustrates, if we insist that the subset be of size exactly N, we can always find a subframe, however the lower frame bound can be very poor.

Example 3.1. *Denote by $\{e_1, \dots, e_N\}$ an orthonormal basis for H_N. Let $\mathcal{F}$ consist of $\{e_1, \dots e_{N-1}\}$ along with N copies of $\frac{1}{\sqrt{N}} e_N$. Thus $\mathcal{F}$ is a Parseval frame with $M = 2N-1$ elements. However, a subframe with N elements must be the set $\{e_1, \dots e_{N-1}, \frac{1}{\sqrt{N}} e_N\}$ which has lower frame bound $\frac{1}{N} = \frac{1}{M-N+1}$ which goes to zero as N grows even though the ratio M/N stays bounded above by 2 and below by 1.5 (when $N \geq 2$).*

However, as we now show, if we allow the subset to be a little fraction larger than N, i.e. of size $(1+\epsilon)N$, then we are able to find a subframe whose lower frame bound does not depend on N but rather on M/N and ϵ:

Lemma 3.2 (Finite dimensional removal)**.** *There exists a monotonically increasing function $g : (0,1) \to (0,1)$ with the following property. For any set $\mathcal{F} = \{f_i\}_{i=1}^M$ of M vectors in a Hilbert space H_N of dimension N, and for any $0 < \epsilon < \frac{M}{N} - 1$ there exists a subset $\mathcal{F}_\epsilon \subset \mathcal{F}$ of cardinality at most $(1+\epsilon)N$ so that:*

$$S_{\mathcal{F}_\varepsilon} \geq g\left(\frac{\epsilon}{2\frac{M}{N}-1}\right) S_{\mathcal{F}} \tag{10}$$

where $S_{\mathcal{F}} f = \sum_{f \in \mathcal{F}} \langle \cdot, f \rangle f$ and $S_{\mathcal{F}_\varepsilon} f = \sum_{f \in \mathcal{F}_\epsilon} \langle \cdot, f \rangle f$ are the frame operators associated to $\mathcal{F}$ and $\mathcal{F}_\varepsilon$, respectively.

Example 3.1 shows that the reliance of the lower frame bound of the subframe on $(\frac{M}{N})^{-1}$ is necessary in Lemma 3.2.

An estimate of the function g is given below.

To prove Lemma 3.2 we will use Lemma 3.4 which is adapted from Theorem 4.3 of Casazza [6] (See Vershynin [29] for a generalization of this result which removes the assumption that the norms of the frame vectors are bounded below.) Recall that a family $\{f_i\}_{i\in I}$ is a *Riesz basic sequence* in a Hilbert space H with (*upper*, respectively *lower*) Riesz bounds A, B if for all families of scalars $\{a_i\}_{i\in I}$ we have:

$$A\sum_{i\in I}|a_i|^2 \le \|\sum_{i\in I} a_i f_i\|^2 \le B\sum_{i\in I}|a_i|^2.$$

Now, for the convenience of the reader we recall Theorem 4.3 in [6].

Theorem 3.3 (Theorem 4.3 in Casazza, [6])**.** *There is a function $\rho(v,w,x,y,z) : \mathbf{R}^5 \to \mathbf{R}^+$ with the following property: Let $(f_i)_{i=1}^M$ be any frame for an N-dimensional Hilbert space H_N with frame bounds A, B, $\alpha \le \|F_i\| \le \beta$, for all $1 \le i \le M$, and let $0 < \varepsilon < 1$. Then there is a subset $\sigma \subset \{1, 2, \ldots, M\}$, with $|\sigma| \ge (1-\varepsilon)N$ so that $(f_i)_{i\in\sigma}$ is a Riesz basis for its span with Riesz basis constant $\rho(\varepsilon, A, B, \alpha, \beta)$.*

We remind the reader that the Riesz basis constant is the larger between the upper Riesz basis bound, and the reciprocal of the lower Riesz basis bound.

Lemma 3.4. *There is a monotonically increasing function $h : (0,1) \to (0,1)$ with the following property: Let $\{f_i\}_{i=1}^M$ be any Parseval frame for an N-dimensional Hilbert space H_N with $\frac{1}{2} \le \|f_i\|^2$, for all $1 \le i \le M$. Then for any $0 < \varepsilon < 1$ there is a subset $\sigma \subset \{1, 2, \ldots, M\}$, with $|\sigma| \ge (1-\varepsilon)N$ so that $\{f_i\}_{i\in\sigma}$ is a Riesz basis for its span with lower Riesz basis bound $h(\varepsilon)$.*

Proof: The only part of this result which is not proved in Theorem 3.3 is that h may be chosen to be monotonically increasing. So let ρ satisfy Theorem 4.3 in [6] and define for $0 < \epsilon_0 < 1$:

$$h(\epsilon_0) = \sup_{0<\epsilon\le\epsilon_0} \frac{1}{\rho(\epsilon, 1, 1, \frac{1}{\sqrt{2}}, 1)}.$$

Then h is monotonically increasing. Let $\{f_i\}_{i=1}^M$ be any Parseval frame for a N-dimensional Hilbert space H_N with $\frac{1}{2} \le \|f_i\|^2$ for all $1 \le i \le M$ and fix

$0 < \epsilon < 1$. There exists a sequence $\{\epsilon_n\}_{n=1}^{\infty}$ (not necessarily distinct) with $0 < \epsilon_n \le \epsilon$ so that

$$h(\epsilon) = 1/\lim_{n\to\infty} \rho(\epsilon_n, 1, 1, \frac{1}{\sqrt{2}}, 1).$$

By Theorem 3.3, for every $n \in \mathbf{N}$ there is a subset $\mathcal{F}_n = \{f_i\}_{i\in I_n}$ of $\mathcal{F}$ so that

$$|\mathcal{F}_n| \ge (1-\epsilon_n)N,$$

and $\{f_i\}_{i\in I_n}$ is a Riesz basis for its span with lower Riesz basis bound $1/\rho(\epsilon_n, 1, 1, \frac{1}{\sqrt{2}}, 1)$. Since the number of subsets of $\mathcal{F}$ is finite, there exists at least one subset $\mathcal{G} \subset \mathcal{F}$ that appears infinitely often in the sequence $\{\mathcal{F}_n\}_n$. Thus $|\mathcal{G}| \ge (1-\epsilon_n)N$ and $\mathcal{G}$ has a lower frame bound greater than equal to $1/\rho(\epsilon_n, 1, 1, \frac{1}{\sqrt{2}}, 1)$ for n belonging to an infinite subsequence of the positive integers. Taking the limit along this subsequence yields $|\mathcal{G}| \ge lim_{n\to\infty}(1-\epsilon_n)N = (1-\epsilon)N$ and $\mathcal{G}$ has a lower frame bound greater than or equal to

$$1/\lim_{n\to\infty} \rho(\epsilon_n, 1, 1, \frac{1}{\sqrt{2}}, 1) = h(\epsilon_0).$$

□

Proof of Lemma 3.2

Step 1. We first assume that the frame $\{f_i\}_{i=1}^{M}$ is a Parseval frame for its span H_N, and each vector satisfies $\|f_i\|^2 \le \frac{1}{2}$. Therefore, by embedding H_N in a M-dimensional Hilbert space and using Naimark's dilation theorem [7] (or the super-frame construction [5]), we find an orthonormal basis $\{e_i\}_{i=1}^{M}$ and a projection P of rank N so that $f_i = Pe_i$. Let $f_i' = (1-P)e_i$. Then $\{f_i'\}_{i=1}^{M}$ is a Parseval frame for its span and $\|f_i'\|^2 \ge \frac{1}{2}$. Notice that we have for any set of coefficients $(c_i)_{i=1}^{M}$:

$$\sum_{i=1}^{M} |c_i|^2 = \|\sum_{i=1}^{M} c_i e_i\|^2 = \|\sum_{i=1}^{M} c_i f_i\|^2 + \|\sum_{i=1}^{M} c_i f_i'\|^2. \qquad (11)$$

For a $\delta > 0$ (that we will specify later), we now apply Lemma 3.4 to the frame $\{f_i'\}_{i=1}^{M}$ (not $\{f_i\}$) to get a subset $\sigma \in \{1, \ldots, M\}$ with $|\sigma| \ge (1-\delta)(M-N)$ such that $\{f_j'\}_{j\in\sigma}$ is a Riesz basis for its span with lower Riesz bound greater than or equal to $h(\delta)$.

Thus for any set of coefficients $(c_j)_{j\in\sigma}$ we have

$$\|\sum_{j\in\sigma} c_j f'_j\|^2 \geq h(\delta)\sum_{j\in\sigma}|c_j|^2.$$

Combining this with equation (11) and a choice of $(c_i)_{i=1}^M$ with the property that $c_i = 0$ if $i \notin \sigma$ we have

$$\|\sum_{j\in\sigma} c_j f_j\|^2 \leq (1-h(\delta))\sum_{j\in\sigma}|c_j|^2. \tag{12}$$

This equation is equivalent to saying that the operator $S_\sigma = \sum_{j\in\sigma}\langle\cdot, f_j\rangle f_j \leq (1-h(\delta))\mathbf{1}$. Therefore, setting $J = I\backslash\sigma$, we have

$$S_J = \sum_{j\in J}\langle\cdot, f_j\rangle f_j = \mathbf{1} - S_\sigma \geq h(\delta)\mathbf{1}.$$

Notice that $|J| \leq M-(1-\delta)(M-N) = N+\delta(M-N) = (1+\delta(\frac{M}{N}-1))N$. Thus any choice of $\delta \leq \epsilon/(\frac{M}{N}-1)$ produces a set J of cardinality $|J| \leq (1+\epsilon)N$ such that $S_J \geq h(\delta)\mathbf{1}$. Setting $\delta = \frac{\epsilon}{2M/N-1}$ and $g = h$ gives the desired result.

Step 2. Assume now that $\{f_i\}_{i=1}^M$ is a Parseval frame without constraints on the norms of f_i. The upper frame bound 1 implies $\|f_i\| \leq 1$, for every $1 \leq i \leq M$. Apply the previous result to the Parseval frame $\{f_{i,1}\}_{i=1}^M \cup \{f_{i,2}\}_{i=1}^M$ where $f_{i,1} = f_{i,2} = \frac{1}{\sqrt{2}}f_i$ for every $1 \leq i \leq M$. Thus we obtain a set $J_1 \subset \{1,2,\ldots,M\}\times\{1,2\}$, $|J_1| \leq (1+\epsilon)N$, so that

$$\sum_{(i,k)\in J_1}\langle\cdot, f_{i,k}\rangle f_{i,k} \geq h\left(\frac{\epsilon}{\frac{2M}{N}-1}\right)\mathbf{1}$$

Let $J = \{i \ : \ 1 \leq i \leq M,$ such that $(i,1)\in J_1$ or $(i,2)\in J_1\}$ Notice $|J| \leq |J_1| \leq (1+\epsilon)N$ and

$$\sum_{i\in J}\langle\cdot, f_i\rangle f_i \geq \sum_{(i,k)\in J_1}\langle\cdot, f_{i,k}\rangle f_{i,k} \geq h\left(\frac{\epsilon}{2\frac{M}{N}-1}\right)\mathbf{1}$$

which again produces the desired result with $g = h$.

Step 3. For the general case, assume S is the frame operator asssociated to $\{f_i\}_{i=1}^M$. Then $\{g_i := S^{-1/2}f_i\}_{i=1}^M$ is a Parseval frame with the same span.

Applying the result of step 2 to this frame, we conclude there exists a subset $J \subset \{1, 2, \ldots, M\}$ of cardinality $|J| \leq (1 + \epsilon)N$ so that $\{g_i; i \in J\}$ is frame such that

$$\sum_{i \in J} \langle \cdot, g_i \rangle g_i \geq h\left(\frac{\epsilon}{2\frac{M}{N} - 1}\right) \mathbf{1}.$$

It follows that

$$\sum_{i \in J} \langle \cdot, f_i \rangle f_i = S^{1/2} \left(\sum_{i \in J} \langle \cdot, g_i \rangle g_i\right) S^{1/2} \geq h\left(\frac{\epsilon}{2\frac{M}{N} - 1}\right) S$$

which is what we needed to prove. □

Remark: We provide the following estimate for the function g. Let

$$b = \left(\frac{\varepsilon}{2} \frac{A}{B} \frac{\alpha}{\beta}\right)^2$$

and choose a natural number m so that

$$(1 - b)^m \leq \varepsilon,$$

then a careful examination of the proof of Theorem 4.3 in [6] combined with the better constants computed in [28] yields that

$$\rho(\varepsilon, A, B, \alpha, \beta) \leq \frac{1}{b^{m+2}} \tag{13}$$

Then an estimate for the function g is given by:

$$g(\varepsilon) \geq \left(\frac{\varepsilon}{2\sqrt{2}}\right)^{2 + \frac{ln(\varepsilon)}{ln(1 - \frac{\varepsilon^2}{8})}} \geq \left(\frac{\varepsilon^2}{8}\right)^{1 + \frac{4}{\varepsilon^2} ln\left(\frac{1}{\varepsilon}\right)}. \tag{14}$$

3.2 Truncation

In this subsection we assume $\mathcal{E}$ is a l^1 self-localized Parseval frame for H indexed by G, and $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized. We let $\mathbf{r}$ denote the localization sequence of $\mathcal{F}$, and we let $\mathbf{s}$ denote the self-localization sequence of $\mathcal{E}$. Further, we denote by

$$f_{i,R} = \sum_{k \in G, |k - a(i)| < R} \langle f_i, e_k \rangle e_k \tag{15}$$

the *truncated* expansion of f_i with respect to $\mathcal{E}$. Clearly $f_{i,R} \to f_i$ as $R \to \infty$. But does this convergence imply convergence of the corresponding frame operators for $\{f_{i,R}\}_{i \in G}$? The answer is that it does as we now show. Specifically, for a subset $J \subset I$ we denote $\mathcal{F}[J] = \{f_i \; ; \; i \in J\}$ and $\mathcal{F}_R[J] = \{f_{i,R} \; ; \; i \in J\}$. Similarly we denote by S_J and $S_{R,J}$ the frame operators associated to $\mathcal{F}[J]$ and $\mathcal{F}_R[J]$, respectively. The following Lemma shows that the truncated frames well approximate the original frames:

Lemma 3.5. *Choose R_0 so that for all $R \geq R_0$, $\Delta(R) \leq (K_a \|\mathbf{s}\|_1)^{-1}$ (See equation (6)) and let S_J and $S_{R,J}$ be as above. Then*

$$\|S_J - S_{R,J}\| \leq E(R), \tag{16}$$

where $E(R) = 3K_a \Delta(R) \|\mathbf{s}\|_1$.

Proof: First denote by $T_J : H \to l^2(J)$, and $T_{R,J} : H \to l^2(J)$ the analysis maps:

$$T_J(x) = \{\langle x, f_i \rangle\}_{i \in J} \quad , \quad T_{R,J}(x) = \{\langle x, f_{i,R} \rangle\}_{i \in J}$$

Since $\mathcal{E}$ is a Parseval frame, $Q : H \to l^2(G)$, $Q(x) = \{\langle x, e_k \rangle\}_{k \in G}$ is an isometry, and

$$\|T_J - T_{R,J}\| = \|(T_J - T_{R,J})^*\| = \|Q(T_J - T_{R,J})^*\|$$

The operator $M = Q(T_J - T_{R,J})^* : l^2(J) \to l^2(G)$ is described by a matrix which we also denote by M. In the canonical bases of $l^2(J)$ and $l^2(G)$, the (k, i) element of M is given by

$$M_{k,i} = \langle f_i - f_{i,R}, e_k \rangle = \sum_{g \in G, |g - a(i)| \geq R} \langle f_i, e_g \rangle \langle e_g, e_k \rangle,$$

and thus

$$|M_{k,i}| \leq \sum_{g \in G, |g - a(i)| \geq R} r(g - a(i)) s(g - k) \tag{17}$$

We bound the operator norm of M using Schur's criterion [25, 22]

$$\|M\| \leq max(\sup_{i \in J} \sum_{k \in G} |M_{k,i}|, \sup_{k \in G} \sum_{i \in J} |M_{k,i}|)$$

It follows from (17) that

$$\begin{aligned}\sum_{k\in G} |M_{k,i}| &\leq \Delta(R)\|\mathbf{s}\|_1 \\ \sum_{i\in J} |M_{k,i}| &\leq K_a\Delta(R)\|\mathbf{s}\|_1.\end{aligned}$$

Thus we obtain $\|M\| \leq K_a\Delta(R)\|\mathbf{s}\|_1$ and hence

$$\|T_J - T_{R,J}\| = \|(T_J - T_{R,J})^*\| \leq K_a\Delta(R)\|\mathbf{s}\|_1$$

It follows that

$$\begin{aligned}\|S_J - S_{R,J}\| &= \|(T_J - T_{R,J})^* T_J + (T_{R,J})^*(T_J - T_{R,J})\| \\ &\leq (\|T_J\| + \|T_{R,J}\|)K_a\Delta(R)\|\mathbf{s}\|_1 \\ &\leq 3K_a\Delta(R)\|\mathbf{s}\|_1,\end{aligned}$$

the last inequality coming from $\|T_J\| \leq 1$ and

$$\|T_{R,J}\| \leq \|T_J\| + \|T_{R,J} - T_J\| \leq 1 + K_a\Delta(R)\|\mathbf{s}\|_1 \leq 2,$$

since $\Delta(R) < \frac{1}{K_a\|\mathbf{s}\|_1}$, for $R > R_0$. □

4 Proof of the main result

In this section we prove the main result of the paper, Theorem 1.1.

The core of the proof is contained in subsection 4.1 which proves Theorem 1.1 for the special case when both $\mathcal{F}$ and $\mathcal{E}$ are Parseval frames. In subsection 4.2 we show how to generalize this special case.

We begin by giving a brief description of the argument of subsection 4.1. Our starting point is the Parseval frame $\mathcal{F}$ that is localized with respect to another Parseval frame $\mathcal{E}$. Our goal is to produce a subset $\mathcal{F}' \subset \mathcal{F}$ which is a frame for the whole space and which has density not much larger than 1. An outline of the steps is as follows:

1. Using Lemma 3.5, we move from the frame $\mathcal{F}$ to a truncated frame $\mathcal{F}_R$.

2. Based on the localization geometry, we decompose I as the union of disjoint finite boxes $Q_N(k)$, with k taking values in an infinite lattice.

3. For each k, $\mathcal{F}_R[Q_N(k)]$ is a finite dimensional frame. We apply Lemma 3.2 to get subsets $J_{k,N,R}$ of smaller size such that $\mathcal{F}_R[J_{k,N,R}]$ remains a frame with frame operator greater than or equal to a small constant times the frame operator for $\mathcal{F}_R[Q_N(k)]$. Thus we have constructed a set $J = \cup_k J_{k,N,R}$ for which $\mathcal{F}_R[J]$ is a frame for the whole space.

4. We then use our choice of R along with Lemma 3.5 to conclude that the set $\mathcal{F}[J]$ is also frame for the whole space.

5. Finally, we show that our choice of N is large enough for the frame $\mathcal{F}[J]$ to have small density.

4.1 The case when $\mathcal{F}$ and $\mathcal{E}$ are Parseval

In this subsection we prove the result for the special case of Parseval frames.

Lemma 4.1. *Let $\mathcal{F} = \mathcal{F}[I]$ be a Parseval frame for H indexed by I, and let $\mathcal{E}$ be a l^1-self localized Parseval frame for H indexed by the discrete abelian group $G = \mathbf{Z}^d \times \mathbf{Z}_D$ so that $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized with respect to a localization map $a : I \to G$ of finite upper density. Then for every $\varepsilon > 0$ there exists a subset $J = J_\varepsilon \subset I$ so that $D^+(a; J) \leq 1 + \varepsilon$ and $\mathcal{F}' = \mathcal{F}[J]$ is frame for H.*

We begin by recalling some notation. For $k \in G$, $N \in \mathbf{N}$, $B_N(k) = \{g \in G : |g - k| \leq N\}$ is the elements of G in the ball with center k and radius N. Define $Q_N(k) = \{i \in I : |a(i) - k| \leq N\} = a^{-1}(B_N(k))$. Since $D^+(a) < \infty$, there exists $K_a \geq 1$ so that $|a^{-1}(B_N(k))| \leq K_a |B_N(k)|$. Recall that we assumed $\mathcal{F}$ and $\mathcal{E}$ are Parseval frames for H, $\mathcal{E}$ is l^1-self localized, and $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized. Denote by $\mathbf{r}$ the localization sequence for $(\mathcal{F}, a, \mathcal{E})$, denote by $\mathbf{s}$ the self-localization sequence for $\mathcal{E}$, and recall $E(R) = 3K_a \|\mathbf{s}\|_1 \sum_{k \in G, |k| > R} r(k)$ decays to 0 as $R \to \infty$. Let $g : (0,1) \to (0,1)$ denote the universal function of Lemma 3.2 and let C_ε denote the positive quantity:

$$C_\varepsilon = g\left(\frac{\varepsilon}{2(2K_a - 1)}\right). \tag{18}$$

We now fix $\varepsilon > 0$. For the duration we will fix two large integers R and N as follows. First R is chosen so that

$$E(R) < \frac{C_\varepsilon}{2(1+C_\varepsilon)} \tag{19}$$

Then N is chosen to be an integer larger than R so that

$$(1+\frac{\varepsilon}{2})\frac{|B_{N+R}(0)|}{|B_N(0)|} \leq 1+\varepsilon. \tag{20}$$

Such an N exists since $|B_M(0)| = D(2M+1)^d$ for $M > D$ and thus $\lim_{N\to\infty} \frac{|B_{N+R}(0)|}{|B_N(0)|} = 1$.

Step 1. Define $\mathcal{F}_R = \{f_{i,R}\,;\, i \in I\}$ to be the truncated frame given by Lemma 3.5 when it is applied to $\mathcal{F}$ and the given R. Let S_R be the frame operator associated to $\mathcal{F}_R$. Notice that since $\mathcal{F}$ is a Parseval frame (and hence its frame operator is $\mathbf{1}$) we have $\|I - S_R\| \leq E(R)$ and consequently

$$(1+E(R))\mathbf{1} \geq S_R \geq (1-E(R))\mathbf{1}. \tag{21}$$

Step 2. We let L be the sublattice $(2N\mathbf{Z})^d \times \{0\} \subset G$. For each $k \in L$ and integer M let $V_{M,k} = span\{e_j\,;\, j \in B_M(k)\}$. Notice $dim(V_{M,k}) \leq |B_M(k)|$. Let $r_{k,N,R} = dim\,span\{f_{i,R}\,;\, i \in Q_N(k)\}$. Since $span\,\{f_{i,R}\,;\, i \in Q_N(k)\} \subset V_{N+R,k}$ we obtain $r_{k,N,R} \leq |B_{N+R}(0)|$.

If $|Q_N(k)| \leq (1+\frac{\varepsilon}{2})|B_{N+R}(0)|$ then set $J_{k,N,R} = Q_N(k)$ so that

$$\sum_{i\in J_{k,N+R}} \langle \cdot, f_{i,R}\rangle f_{i,R} = \sum_{i\in Q_N(k)} \langle \cdot, f_{i,R}\rangle f_{i,R} \geq C_\varepsilon \sum_{i\in Q_N(k)} \langle \cdot, f_{i,R}\rangle f_{i,R} \tag{22}$$

where C_ε is defined in (18).

Assume now that $|Q_N(k)| > (1+\frac{\varepsilon}{2})|B_{N+R}(0)|$. We apply Lemma 3.2 to the set $\{f_{i,R}\,;\, i \in Q_N(k)\}$ (with $b = (1+\frac{\varepsilon}{2})\frac{|B_{N+R}(0)|}{r_{k,N,R}} - 1$ as the $\epsilon > 0$ in the lemma) and obtain a subset $J_{k,N,R} \subset Q_N(k)$ of size $|J_{k,N,R}| \leq (1+\frac{\varepsilon}{2})|B_{N+R}(k)|$ so that

$$\sum_{i\in J_{k,N.R}} \langle \cdot, f_{i,R}\rangle f_{i,R} \;\geq\; g\left(\frac{b}{(2|Q_N(k)|/r_{k,N,R})-1}\right) \sum_{i\in Q_N(k)} \langle \cdot, f_{i,R}\rangle f_{i,R} \tag{23}$$

$$\geq\; C_\varepsilon \sum_{i\in Q_N(k)} \langle \cdot, f_{i,R}\rangle f_{i,R} \tag{24}$$

where the last inequality follows from the monotonicity of g and the fact that

$$\frac{b}{2|Q_N(k)|/r_{k,N,R}-1} \geq \frac{\varepsilon}{2(2K_a-1)}.$$

In either case

$$|J_{k,N,R}| \leq (1+\frac{\varepsilon}{2})|B_{N+R}(k)| \leq (1+\varepsilon)|B_N(0)|$$

due to (20).

Step 3. Set

$$J_{N,R} = \cup_{k\in L} J_{k,N,R}. \tag{25}$$

Denote by $S_{R,N}$ the frame operator for $\{f_{i,R}\,;\, i \in J_{N,R}\}$. We then have

$$\begin{aligned} S_{R,N} &= \sum_{k\in L} \sum_{i\in J_{k,N,R}} \langle \cdot, f_{i,R}\rangle f_{i,R} \\ &\geq \sum_{k\in L} C_\varepsilon \sum_{i\in Q_N(k)} \langle \cdot, f_{i,R}\rangle f_{i,R} = C_\varepsilon S_R \qquad (26) \\ &\geq C_\varepsilon(1-E(R))\mathbf{1} \qquad (27) \end{aligned}$$

where the last lower bound comes from (21). This means $\mathcal{F}_{R,N} := \{f_{i,R}\,;\, i \in J_{N,R}\}$ is frame for H with lower frame bound $C_\varepsilon(1-E(R))$.

Step 4. We again apply Lemma 3.5 with $J = J_{N,R}$ to obtain that S_J, the frame operator associated to $\mathcal{F}[J] = \{f_i\,;\, i \in J\}$, is bounded below by

$$S_J \geq S_{R,N} - E(R)\mathbf{1} \geq (C_\varepsilon(1-E(R)) - E(R))\,\mathbf{1} \geq \frac{1}{2}C_\varepsilon\mathbf{1} \tag{28}$$

where the last inequality follows from (19). This establishes that $\mathcal{F}[J]$ is frame for H with lower frame bound at least $\frac{1}{2}C_\varepsilon$.

It remains to show that $J_{N,R}$ has the desired upper density.

Step 5. The upper density of $J = J_{N,R}$ is obtained as follows. First, in each box $B_N(k)$, $k \in L$, we have

$$\frac{|a^{-1}(B_N(k)) \cap J|}{|B_N(k)|} = \frac{|J_{k,N,R}|}{|B_N(k)|} \leq (1+\frac{\varepsilon}{2})\frac{|B_{N+R}(k)|}{|B_N(k)|} \leq 1+\varepsilon \tag{29}$$

Then, by an additive argument one can easily derive that

$$\limsup_{M\to\infty} \sup_{k\in G} \frac{|a^{-1}(J)\cap B_M(k)|}{|B_M(k)|} \leq 1+\varepsilon \tag{30}$$

which means $D^+(a;J) \leq 1+\varepsilon$. □

4.2 Generalizing

We now show how to remove the constraints that both $\mathcal{F}$ and $\mathcal{E}$ are Parseval in Lemma 4.1 . We begin by outlining the argument: starting with the frames $\mathcal{F}$ and $\mathcal{E}$ we show there are canonical Parseval frames $\mathcal{F}^{\#}$ and $\mathcal{E}^{\#}$ that have the same localization properties as $\mathcal{F}$ and $\mathcal{E}$. We then apply Lemma 4.1 to these frames to get a subframe of $\mathcal{F}^{\#}$ that is a frame for the whole space with the appropriate density. Finally, we show that the corresponding subframe of $\mathcal{F}$ has the desired frame and density properties.

A well known canonical construction (see [9]) begins with an arbitrary frame $\mathcal{F} = \{f_i\}$ and produces the canonical Parseval frame

$$\mathcal{F}^{\#} = \{f_i^{\#} = S^{-1/2} f_i\}, \tag{31}$$

where S is the frame operator associated to $\mathcal{F}$.

In our situation we have two frames $\mathcal{F} = \{f_i\,;\, i\in I\}$ and $\mathcal{E} = \{e_k\,;\, k\in G\}$ along with $a: I \to G$ such that $(\mathcal{F}, a, \mathcal{E})$ is l^1-localized and $\mathcal{E}$ is a l^1-self localized. As in (31) we define two Parseval frames $\mathcal{F}^{\#}$ and $\mathcal{E}^{\#}$ corresponding to $\mathcal{F}$ and $\mathcal{E}$ respectively.

Lemma 2.2 from [17] and Theorem 2 from [3] can be used to show that $\mathcal{F}^{\#}$ and $\mathcal{E}^{\#}$ inherit the localization properties of $\mathcal{F}$ and $\mathcal{E}$, namely

Lemma 4.2. *Given $\mathcal{F}^{\#}$ and $\mathcal{E}^{\#}$ as above, if $(\mathcal{F}, a, \mathcal{E})$ is l^1-localized and $\mathcal{E}$ is l^1-self localized then $(\mathcal{F}^{\#}, a, \mathcal{E}^{\#})$ is l^1-localized and $\mathcal{E}^{\#}$ is l^1-self localized.*

Proof

First, if $\mathcal{E}$ is l^1-self localized then by Theorem 2,(c) in [3] it follows that $\mathcal{E}^{\#}$ is l^1-self localized. Furthermore, by Theorem 2, (b) in the aforementioned paper it follows that $(\tilde{\mathcal{E}})$ is l^1-self localized, where $\tilde{\mathcal{E}} = \{\tilde{e_k}\ ;\ k\in G\}$ is the canonical dual of $\mathcal{E}$. This implies the existence of a sequence $s \in l^1(G)$ so that

$$|\langle \tilde{e_k}, \tilde{e_j} \rangle| \leq s(k-j) \quad , \quad \text{for all } k,j \in G. \tag{32}$$

Next assume additionally that $(\mathcal{F}, a, \mathcal{E})$ is l^1-localized. This means there exists a sequence $r \in l^1(G)$ so that

$$|\langle f_i, e_k \rangle| \leq r(a(i)-k), \text{ for every } i \in I \text{ and } k \in G. \tag{33}$$

Since $\tilde{e_k} = \sum_{j \in G} \langle \tilde{e_k}, \tilde{e_j} \rangle e_j$ it follows that

$$|\langle f_i, \tilde{e_k} \rangle| = |\sum_{j \in G} \langle f_i, e_j \rangle \langle \tilde{e_j}, \tilde{e_k} \rangle| \leq \sum_{j \in G} r(a(i)-j) s(j-k) = (r \star s)(a(i)-k),$$

(where $\star$ denotes convolution) and thus $(\mathcal{F}, a, \tilde{\mathcal{E}})$ is also l^1-localized. By Lemma 3 in [3] it follows that $(\mathcal{F}, a)$ is l^1-self localized.

Again Theorem 2, (b) implies now that $(\tilde{\mathcal{F}}, a)$ is l^1-self localized. Therefore there exists a sequence $t \in l^1(G)$ so that

$$|\langle \tilde{f_i}, \tilde{f_j} \rangle| \leq t(a(i)-a(j)) \quad , \quad \text{for every } i,j \in I. \tag{34}$$

We will show that $(\mathcal{F}, a)$ is l^1-self localized implies that $(\mathcal{F}^{\#}, a)$ is l^1 localized with respect to $(\mathcal{F}, a)$, meaning that there exists a sequence $u \in l^1(G)$ so that

$$|\langle f_i^{\#}, f_j \rangle| \leq u(a(i)-a(j)) \quad , \quad \text{for every } i,j \in I \tag{35}$$

Let $\mathbf{G} : l^2(I) \to l^2(I)$ be the Gramm operator associated to the frame $\mathcal{F}$, $\mathbf{G} = TT^*$, where $T : H \to l^2(I)$ is the analysis operator $T(x) = \{\langle x, f_i \rangle\}_{i \in I}$ and $T^* : l^2(I) \to H$, $T^*(c) = \sum_{i \in I} c_i f_i$ is the synthesis operator. Let $\delta_i \in l^2(I)$ denote the sequence of all zeros except for one entry 1 in the i^{th} position. The set $\{\delta_i,\ i \in I\}$ is the canonical orthonormal basis of $l^2(I)$. Since $\mathcal{F}$ is a frame, $\mathbf{G}$ is a bounded operator with closed range, and T^* is surjective (onto). Let $\mathbf{G}^\dagger$ denote the (Moore-Penrose) pseudoinverse of $\mathbf{G}$. Thus $P = \mathbf{G}\mathbf{G}^\dagger = \mathbf{G}^\dagger\mathbf{G}$ is the orthonormal projection onto the range of T in $l^2(I)$. A simple exercise shows that $\tilde{f_i} = T^*\mathbf{G}^\dagger\delta_i$, and $f_i^{\#} = T^*(\mathbf{G}^\dagger)^{1/2}\delta_i$. Using the notation from Appendix A of [3], we get $\mathbf{G} \in \mathcal{B}_1(I,a)$, the algebra of operators that have l^1 decay. Using Lemma A.1 and then the holomorphic calculus as in the Proof of Theorem 2 of the aforementioned paper, we obtain that $\mathbf{G}$ and all its powers $\mathbf{G}^q$, $q>0$ are in $\mathcal{B}_1(I,a)$. In particular, $\mathbf{G}^{1/2}\in\mathcal{B}_1(I,a)$ implying the existence of a sequence $u\in l^1(G)$ so that

$$|\langle \mathbf{G}^{1/2}\delta_i,\delta_j\rangle|\leq u(a(i)-a(j)).$$

Then:

$$\langle f_i^{\#},f_j\rangle=\langle T^*(\mathbf{G}^\dagger)^{1/2}\delta_i,T^*\delta_j\rangle=\langle \mathbf{G}(\mathbf{G}^\dagger)^{1/2}\delta_i,\delta_j\rangle=\langle \mathbf{G}^{1/2}\delta_i,\delta_j\rangle$$

which yields (35).

The same proof applied to $(\mathcal{E},Id)$, where Id is the identity map, implies that if $(\mathcal{E},Id)$ is l^1-self localized then $(\mathcal{E}^{\#},Id,\mathcal{E})$ is l^1-localized (which is to say, equivalently, that $(\mathcal{E}^{\#},Id)$ is l^1-localized with respect to $(\mathcal{E},Id)$). Explicitely this means there exists a sequence $v\in l^1(G)$ so that

$$|\langle e_k^{\#},e_n\rangle|\leq v(k-n)\quad,\quad \text{for every } k,n\in G \tag{36}$$

Putting together (32-36) we obtain:

$$\langle f_i^{\#},e_k^{\#}\rangle=\sum_{j,l\in I}\sum_{m,n\in G}\langle f_i^{\#},f_j\rangle\langle\tilde{f}_j,\tilde{f}_l\rangle\langle f_l,e_m\rangle\langle\tilde{e_m},\tilde{e_n}\rangle\langle e_n,e_k^{\#}\rangle$$

Hence

$$\begin{aligned}|\langle f_i^{\#},e_k^{\#}\rangle| &\leq \sum_{j,l\in I}\sum_{m,n\in G}u(a(i)-a(j))t(a(j)-a(l))r(a(l)-m)s(m-n)v(n-k)\\ &\leq K_a^2(u\star t\star r\star s\star v)(a(i)-k)\end{aligned}$$

where K_a is as in (9), and the convolution sequence $u\star t\star r\star s\star v\in l^1(G)$. This means $(\mathcal{F}^{\#},a,\mathcal{E}^{\#})$ is l^1 localized. □

We can now prove Theorem 1.1:

Proof of Theorem 1.1

As above we let $\mathcal{F}^{\#}$ and $\mathcal{E}^{\#}$ be the canonical Parseval frames associated with $\mathcal{F}$ and $\mathcal{E}$. By Lemma 4.2 we have $(\mathcal{F}^{\#},a,\mathcal{E}^{\#})$ is l^1 localized and $\mathcal{E}^{\#}$ is l^1-self localized. Given $\varepsilon>0$ we apply Lemma 4.1 to get a subset $J\subset I$ such that $D^+(a;J)\leq 1+\varepsilon$ and $\mathcal{F}^{\#}[J]$ is a frame for H.

To complete the proof, we now show that $\mathcal{F}[J]$ is also a frame for H. This follows from the following lemma:

Lemma 4.3. *Assume $\mathcal{F} = \{f_i\,;\, i \in I\}$ is frame for H with frame bounds $A \leq B$. Let $\mathcal{F}^\#$ be the canonical Parseval frame associated to $\mathcal{F}$. If $J \subset I$ is such that $\{f_i^\#, i \in J\}$ is frame for H with bounds $A' \leq B'$, then $\mathcal{F}[J] = \{f_i, i \in J\}$ is also frame for H with bounds AA' and BB'.*

Proof: Let S be the frame operator associated to $\mathcal{F}$ and so $A\mathbf{1} \leq S \leq B\mathbf{1}$. Now we have the following operator inequality

$$
\begin{aligned}
AA'\mathbf{1} &\leq A'S = S^{1/2}(A'\mathbf{1})S^{1/2} &(37)\\
&\leq S^{1/2}\left(\sum_{i \in J}\langle \cdot, f_i^\# \rangle f_i^\#\right)S^{1/2} &(38)\\
&\leq S^{1/2}(B'\mathbf{1})S^{1/2} = B'S \leq BB'\mathbf{1}. &(39)
\end{aligned}
$$

Notice however that the frame operator for $\mathcal{F}[J]$ satisfies

$$
\sum_{i \in J}\langle \cdot, f_i \rangle f_i = S^{1/2}\left(\sum_{i \in J}\langle \cdot, f_i^\# \rangle f_i^\#\right)S^{1/2}.
$$

Substituting this equality into the middle term of the string of inequalities (38) gives the desired result:

$$
AA'\mathbf{1} \leq \sum_{i \in J}\langle \cdot, f_i \rangle f_i \leq BB'\mathbf{1}.
$$

□

5 Application to Gabor Systems

In this section we specialize to Gabor frames and molecules the results obtained in previous section.

First we recall previously known results.

A (generic) *Gabor system* $\mathcal{G}(g;\Lambda)$ generated by a function $g \in L^2(\mathbf{R}^d)$ and a countable set of time-frequency points $\Lambda \subset \mathbf{R}^{2d}$ is defined by

$$
\mathcal{G}(g;\Lambda) = \{M_\omega T_x g\ ;\ (x,\omega) \in \Lambda\} = \{e^{2\pi i\langle \omega, t\rangle} g(t-x)\ ;\ (x,\omega) \in \Lambda\}. \quad (40)
$$

In general we allow Λ to be an irregular set of time-frequency points.

A *Gabor multi-system* $\mathcal{G}(g^1,\dots,g^n;\Lambda^1,\dots,\Lambda^n)$ generated by n functions $g^1,\dots,g^n$ and n sets of time-frequency points $\Lambda^1,\dots,\Lambda^n$ is simply the union of the corresponding Gabor systems:

$$\mathcal{G}(g^1,\dots,g^n;\Lambda^1,\dots,\Lambda^n) = \mathcal{G}(g^1;\Lambda^1)\cup\dots\cup\mathcal{G}(g^n;\Lambda^n). \tag{41}$$

A *Gabor molecule* $\mathcal{G}(\Gamma;\Lambda)$ associated to an enveloping function $\Gamma:\mathbf{R}^{2d}\to\mathbf{R}$ and a set of time-frequency points $\Lambda\subset\mathbf{R}^{2d}$ is a countable set of functions in $L^2(\mathbf{R}^d)$ indexed by Λ whose short-time Fourier transform (STFT) have a common envelope of concentration:

$$\begin{aligned}\mathcal{G}(\Gamma;\Lambda) \;=\; & \{g_{x,\omega}\;;\;\; g_{x,\omega}\in L^2(\mathbf{R}^d)\;\; : \qquad (42)\\ & |V_\gamma g_{x,\omega}(y,\xi)|\le\Gamma(y-x,\xi-\omega)\;,\;\forall(x,\omega)\in\Lambda\,,\;\forall(y,\xi)\in\mathbf{R}^{2d}\}\end{aligned}$$

where $\gamma(t)=2^{d/4}e^{-\pi\|t\|^2}$ and

$$V_\gamma h(y,\xi)=\int e^{-2\pi i\langle\xi,t\rangle}h(t)\gamma(t-y)dt. \tag{43}$$

Remark 5.1. *Note that Gabor systems (and multi-systems) are Gabor molecules, where the common localization function is the absolute value of the short-time Fourier transform of the generating function g, $\Gamma=|V_\gamma g|$ (or the sum of absolute values of STFTs of generating functions $g^1,\dots,g^n$, $\Gamma=|V_\gamma g^1|+\dots+|V_\gamma g^n|$).*

When a Gabor system, a Gabor multi-system, or a Gabor molecule, is a frame we shall simply call the set a Gabor frame, a Gabor multi-frame, or a Gabor molecule frame, respectively.

In this section the reference frame $\mathcal{E}$ is going to be the Gabor frame $\mathcal{E}=\mathcal{G}(\gamma;\alpha\mathbf{Z}^d\times\beta\mathbf{Z}^d)$ where γ is the Gaussian window $\gamma(t)=2^{d/4}e^{-\pi\|t\|^2}$ normalized so that its $L^2(\mathbf{R}^d)$ norm is one, and $\alpha,\beta>0$ are chosen so that $\alpha\beta<1$. As is well known (see [24, 26, 27]), for every such α and β, $\mathcal{G}(\gamma;\alpha\mathbf{Z}^d\times\beta\mathbf{Z}^d)$ is a frame for $L^2(\mathbf{R}^d)$.

The localization property introduced in Section 2 turns out to be equivalent to a joint concentration in both time and frequency of the generator(s) of a Gabor (multi-)system, or of the envelope of a Gabor molecule. The most natural measures of concentration are given by norms of the *modulation spaces*, which are Banach spaces invented and extensively studied by Feichtinger, with some of the main references being [11, 12, 13, 14, 16]. For a detailed development of the theory of modulation spaces and their weighted counterparts, we refer to the original literature mentioned above and to [18, Chapters 11–13].

For our purpose, two Banach spaces are sufficient: the modulation space M^1 and the *Wiener amalgam space* $W(C, l^1)$.

Definition 5.2. *The* modulation space $M^1(\mathbf{R}^d)$ *(also known as the* Feichtinger algebra S_0*) is the Banach space consisting of all functions f of $L^2(\mathbf{R}^d)$ so that*

$$\|f\|_{M^1} := \|V_\gamma f\|_{L^1} = \int\int_{\mathbf{R}^{2d}} |V_\gamma f(x,\omega)| dx d\omega \;<\; \infty \tag{44}$$

Definition 5.3. *The* Wiener amalgam space $W(C, l^1)$ *over* $\mathbf{R}^n$ *is the Banach space consisting of continuous functions* $F : \mathbf{R}^n \to \mathbf{C}$ *so that*

$$\|F\|_{W(C,l^1)} := \sum_{k\in\mathbf{Z}^n} \sup_{t\in[0,1]^n} |F(k+t)| \;<\; \infty \tag{45}$$

Note the Banach algebra $M^1(\mathbf{R}^d)$ is invariant under Fourier transform and is closed under both pointwise multiplication and convolution. Furthermore, a function $f \in M^1(\mathbf{R}^d)$ if and only if $V_\gamma f \in W(C, l^1)$ over $\mathbf{R}^{2d}$. In particular the Gaussian window $\gamma \in M^1(\mathbf{R}^d)$.

Consider now a Gabor molecule $\mathcal{G}(\Gamma;\Lambda)$ and define the localization map $a : \Lambda \to \alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d$ via $a(x,\omega) = \left(\alpha\lfloor\frac{1}{\alpha}x\rfloor, \beta\lfloor\frac{1}{\beta}\omega\rfloor\right)$, where $\lfloor\cdot\rfloor$ acts componentwise, and on each component, $\lfloor b\rfloor$ denotes the largest integer smaller than or equal to b.

For any set $J \subset \mathbf{R}^{2d}$, the *Beurling upper and lower density* are defined by

$$D_B^+(J) \;=\; limsup_{N\to\infty} \sup_{z\in\mathbf{R}^{2d}} \frac{|\{\lambda\in J : |\lambda - z| \le N\}|}{(2N)^{2d}} \tag{46}$$

$$D_B^-(J) \;=\; liminf_{N\to\infty} \inf_{z\in\mathbf{R}^{2d}} \frac{|\{\lambda\in J : |\lambda - z| \le N\}|}{(2N)^{2d}} \tag{47}$$

The relationship between the upper and lower densities of a subset $J \subset \Lambda$ and the corresponding Beurling densities are given by (see equation (2.4) in [4]):

$$D^+(a;J) = (\alpha\beta)^d D_B^+(J) \quad (48)$$

$$D^-(a;J) = (\alpha\beta)^d D_B^-(J) \quad (49)$$

We are now ready to state the main results of this section from which Theorem 1.2 follows as a Corollary:

Theorem 5.4. *Assume $\mathcal{G}(\Gamma;\Lambda) = \{g_\lambda \ ; \ \lambda \in \Lambda\}$ is a Gabor molecule that is frame for $L^2(\mathbf{R}^d)$ with envelope $\Gamma \in W(C,l^1)$. Then for any $\varepsilon > 0$ there exists a subset $J_\varepsilon \subset \Lambda$ so that $\mathcal{G}(\Gamma;J_\varepsilon) = \{g_\lambda \ ; \ \lambda \in J\}$ is frame for $L^2(\mathbf{R}^d)$ and $D_B^+(J_\varepsilon) \leq 1+\varepsilon$.*

Theorem 5.5. *Assume $\mathcal{G}(g^1,\dots,g^n;\Lambda^1,\dots,\Lambda^n)$ is a Gabor multi-frame for $L^2(\mathbf{R}^d)$ so that $g^1,\dots,g^n \in M^1(\mathbf{R}^d)$. Then for every $\varepsilon > 0$ there are subsets $J_\varepsilon^1 \subset \Lambda^1$, ...,$J_\varepsilon^n \subset \Lambda^n$, so that $\mathcal{G}(g^1,\dots,g^n;J_\varepsilon^1,\dots,J_\varepsilon^n)$ is a Gabor multi-frame for $L^2(\mathbf{R}^d)$ and $D_B^+(J_\varepsilon^1 \cup \cdots J_\varepsilon^n) \leq 1+\varepsilon$.*

Proof of theorem 5.4 Fix $0 < \varepsilon \leq \frac{1}{2}$. Choose $\alpha,\beta > 0$ so that $(\alpha\beta)^d = 1-\frac{\varepsilon}{2}$.

First by Theorem 2.d in [4], it follows that $(\mathcal{G}(\gamma,\alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d),i)$ is a l^1-self-localized frame for $L^2(\mathbf{R}^d)$.

Then by Theorem 8.a in [4] it follows that $(\mathcal{G}(\Gamma;\Lambda),a,\mathcal{G}(\gamma,\alpha\mathbf{Z}^d \times \beta\mathbf{Z}^d))$ is l^1-localized. Furthermore, by Theorem 9.a from the same reference, the Beurling upper density of Λ must be finite, hence $D^+(a) < \infty$.

Thus the hypotheses of Theorem 1.1 are satisfied and one can find a subset $J_\varepsilon \subset \Lambda$ so that $D^+(a;J_\varepsilon) \leq 1+\frac{\varepsilon}{4}$. Using 48,

$$D_B^+(J_\varepsilon) = \frac{D^+(a;J_\varepsilon)}{(\alpha\beta)^d} \leq \frac{1+\frac{\varepsilon}{4}}{1-\frac{\varepsilon}{2}} \leq 1+\varepsilon$$

which is what we needed to prove. □

Proof of Theorem 5.5

First note that $\mathcal{G}(g^1,\dots,g^n;\Lambda^1,\dots,\Lambda^n)$ is a Gabor molecule with envelope $\Gamma = |V_\gamma g^1| + \cdots + |V_\gamma g^n|$. Since each $g^1,\dots,g^n \in M^1(\mathbf{R}^d)$ we obtain $\Gamma \in W(C,l^1)$ and the conclusion follows from Theorem 5.4. □

In a private communication, K. Gröchenig pointed out to us that the Theorem 5.5 yields the following corollary:

Corollary 5.6. *For every $g \in M^1(\mathbf{R}^d)$ and $\varepsilon > 0$ there exists a countable subset $\Lambda_{\varepsilon,g}$ of $\mathbf{R}^{2d}$ with Beurling densities $1 \leq D_B^-(\Lambda_{\varepsilon,g}) \leq D_B^+(\Lambda_{\varepsilon,g}) \leq 1+\varepsilon$ so that $\mathcal{G}(g;\Lambda_{\varepsilon,g})$ is frame for $L^2(\mathbf{R}^d)$.*

Proof

Let g and ε be as in hypothesis. The general theory of coorbit spaces ([13, 14] and in particular Theorem 1 in [15]) implies that there exists a sufficiently dense lattice $\Sigma = \alpha\mathbf{Z}^{2d}$ of the phase space $\mathbf{R}^{2d}$ so that $\mathcal{G}(g;\alpha\mathbf{Z}^{2d})$ is frame for $L^2(\mathbf{R}^d)$. Next, Theorem 5.5 implies there exists a subset $\Lambda_{\varepsilon,g} \subset \alpha\mathbf{Z}^{2d}$ so that $\mathcal{G}(g;\Lambda_{\varepsilon,g})$ remains frame for $L^2(\mathbf{R}^d)$ and its upper Beurling density is bounded by $D_B^+(\Lambda_{\varepsilon,g}) \leq 1+\varepsilon$. Its lower Beurling density must be at least 1 by the general results of irregular Gabor frames (see, e.g. [8]). $\Box$

6 Frame density and the proofs of Theorems 1.3 and 1.4

The results presented so far have involved only lower and upper densities: $D^\pm(a;I)$ in the l^1 localized setting, and $D_B^\pm(\Lambda)$ in the Gabor setting. These lower and upper densities are only the extremes of the possible densities that we could naturally assign to I with respect to a. In particular, instead of taking the infimum or supremum over all possible centers as in (7),(8) we could choose one specific sequence of centers, and instead of computing the liminf or limsup we could consider the limit with respect to some ultrafilter. The different possible choices of ultrafilters and sequences of centers gives us a natural collection of definitions of density.

Definition 6.1. *For a free ultrafilter p and a sequence of centers $(k_n)_{n\geq 0}$ chosen in G define the frame* density *to be:*

$$D(p;J;a;(k_n)_{n\geq 0}) = p\text{-}\lim_n \frac{|a^{-1}(B_n(k_n)) \cap J|}{|B_n(0)|}. \tag{50}$$

with $a : I \to G$ and $J \subset I$.

We shall denote the set of free ultrafilters $\mathbf{N}^*$ (see [21] for more details on ultrafilters).

Definition 6.2. *For Gabor sets* (g, Λ) *or Gabor molecules* $\mathcal{G}(\Gamma; \Lambda)$ *the* Beurling density *of label set* Λ *with respect to a sequence of centers* $(k_n)_{n\geq 0}$ *and a free ultrafilter* $p \in \mathbf{N}^*$ *is given by*

$$D_B(p, \Lambda; (k_n)_{n\geq 0}) = p\text{-}\lim_n \frac{|\Lambda_n|}{(2n)^{2d}}, \tag{51}$$

where $\Lambda_n = \{\lambda \in \Lambda : |\lambda - k_n| \leq n\}$.

Fore more details regarding this type of density we refer the reader to [3].

With these definitions, density of a set is no longer a single value but rather a collection of values, one for each choice of centers k_n and ultrafilter p. We note that all these values lie between the upper and lower density and thus in the case where these are equal, all these values are the same.

From here on, we fix a choice of centers $(k_n)_{n\geq 0}$ in G. Thus the frame density becomes a function $D(p, J, a)$, or $D(p, J)$ when the localization map a is implicit. Similarly, the Beurling density becomes a function $D_B(p, \Lambda)$.

With these definitions, we prove the precise version of Theorems 1.3 and 1.4; the proofs are straightforward consequences of the results proved here and in [3, 4].

Theorem 6.3. *Assume frames* $\mathcal{F} = \{f_i; i \in I\}$, $\mathcal{F}_1 = \{f_i^1; i \in I_1\}$, $\mathcal{F}_2 = \{f_i^2; i \in I_2\}$ *for the same Hilbert space* H *are* l^1 *localized with respect to a frame* $\mathcal{E}$ *indexed by the countable abelian group* G, *with* $a : I \to G$, $a_1 : I_1 \to G$, $a_2 : I_2 \to G$ *being the localization maps all of finite upper density.*

1. For every $\varepsilon > 0$ *there exists a subset* $J_\varepsilon \subset I$ *such that* $\mathcal{F}[J_\varepsilon] = \{f_i; i \in J_\varepsilon\}$ *is frame for* H, *and* $D(p, J_\varepsilon) \leq 1 + \varepsilon$ *for all* $p \in \mathbf{N}^*$.

2. If $\mathcal{E}$ *is a Riesz basis for* H, *then* $D(p, I, a) \geq 1$ *for all* $p \in \mathbf{N}^*$.

3. If both $\mathcal{F}$ *and* $\mathcal{E}$ *are Riesz bases for* H, *then* $D(p, I, a) = 1$ *for all* $p \in \mathbf{N}^*$.

4. Denote by $\mathcal{F}' = \mathcal{F}_1 \dot{\sqcup} \mathcal{F}_2$ *the disjoint union of the two frames. Let* $I' = I_1 \dot{\sqcup} I_2$ *and set* $a' : I' \to G$ *the localization map of* $\mathcal{F}'$, *defined by* $a'(i) = a_1(i)$ *if* $i \in I_1$, *and* $a'(i) = a_2(i)$ *if* $i \in I_2$. *Then* $D(p, I', a') = D(p, I_1, a_1) + D(p, I_2, a_2)$.

Proof:

1. This comes directly from Theorem 1.1 since $D(p, J_\varepsilon) \leq D^+(J_\varepsilon)$.

2. l^1 localization implies l^2 localization, which in turn implies l^2-column and l^2-row decay (Theorem 1.g in [3]), which next implies strong HAP (Theorem 1.a in same) and weak HAP (Theorem 1.e), and finally that $D^-(I) \geq 1$ (Theorem 3.a in same). Consequently $D(p, I, a) \geq D^-(I) \geq 1$.

3. If both $\mathcal{F}$ and $\mathcal{E}$ are Riesz bases then l^1 localization implies also weak dual HAP (see again Theorem 1 in [3]) which in turn implies $D^+(I) \leq 1$ (Theorem 3.b in same). Hence $D(p, I, a) = 1$ for all $p \in \mathbf{N}^*$.

4. The assertion comes from

$$\frac{|a'^{-1}(B_n(k_n))|}{|B_n(0)|} = \frac{|a_1^{-1}(B_n(k_n))|}{|B_n(0)|} + \frac{|a_2^{-1}(B_n(k_n))|}{|B_n(0)|}$$

and the fact that p-lim is linear.

□

Theorem 6.4. *Assume $\mathcal{G}(\Gamma; \Lambda)$, $\mathcal{G}(\Gamma_1; \Lambda_1)$ and $\mathcal{G}(\Gamma_2; \Lambda_2)$ are Gabor molecules with envelopes in $W(, C, l^1)$. Then:*

1. If $\mathcal{G}(\Gamma; \Lambda)$ is frame for $L^2(\mathbf{R}^d)$ then for every $\varepsilon > 0$ there is a subset $J_\varepsilon \subset \Lambda$ such that $\mathcal{G}(\Gamma; J_\varepsilon)$ is frame for $L^2(\mathbf{R}^d)$ and $D_B(p, J_\varepsilon) \leq 1 + \varepsilon$ for every $p \in \mathbf{N}^*$.

2. If $\mathcal{G}(\Gamma; \Lambda)$ is frame for $L^2(\mathbf{R}^d)$ then $D(p, \Lambda) \geq 1$ for all $p \in \mathbf{N}^*$.

3. If $\mathcal{G}(\Gamma; \Lambda)$ is a Riesz basis then $D(p, \Lambda) = 1$ for all $p \in \mathbf{N}^*$.

4. Denote by $\mathcal{G}' = \mathcal{G}(\Gamma_1; \Lambda_1) \dot{\sqcup} \mathcal{G}(\Gamma_2; \Lambda_2)$ the disjoint union of the two Gabor molecules. Then $\mathcal{G}'$ is also a Gabor molecule with envelope $\Gamma' = \Gamma_1 + \Gamma_2$ and label set $\Lambda' = \Lambda_1 \dot{\sqcup} \Lambda_2$. Furthermore

$$D_B(p, \Lambda') = D_B(p, \Lambda_1) + D_B(p, \Lambda_2)$$

Proof:

1. This comes directly from Theorem 5.4 since $D_B(p, J_\varepsilon) \leq D_B^+(J_\varepsilon)$ for every $p \in \mathbf{N}^*$.

2. and 3. are consequences of Theorem 9(a) and (b) in [4] since $W(C,l^1) \subset W(C,l^2)$.

4. The statement is a direct consequence of

$$|\Lambda' \cap B_n(k_n)| = |\Lambda_1 \cap B_n(k_n)| + |\Lambda_2 \cap b_n(k_n)|$$

and linearity of p-limits.

□

Remark 6.5. *Theorem 9 in [4] implies that, in the more general case when the envelope is in $W(C,l^2)$, the density of that Gabor molecule satisfies the properties of redundancy specified in P_2-P_4, that are 2.-4. in Theorem 6.4.*

7 Consequences for the redundancy function

A quantification of overcompleteness for all frames that share a common index set was given in [5] and included a general definition for frame redundancy. Here we extract the relevant definitions and results for our setting.

The basic objects are a countable index set I together with a sequence of finite subsets $(I_n)_{n\geq 0}$ that covers I, that is $\cup_{n\geq 0} I_n = I$. For a subset $J \subset I$, the induced sequence of subsets $(J_n)_{n\geq 0}$ is given simply by $J_n = J \cap I_n$.

To any frame $\mathcal{F}$ indexed by I, $\mathcal{F} = \{f_i\}_{i\in I}$, we associate the following *redundancy function*:

$$R : \mathbf{N}^* \to \mathbf{R} \cup \{\infty\} \quad , \quad R(p;\mathcal{F},(I_n)_n) = \frac{1}{p\text{-}\lim_n \frac{1}{|I_n|}\sum_{i\in I_n}\langle f_i, \tilde{f}_i\rangle} \quad , \quad \forall p \in \mathbf{N}^* \tag{52}$$

where $\tilde{f}_i = S^{-1} f_i$ are the canonical dual frame vectors, and $\mathbf{N}^*$ denotes the compact space of free ultrafilters (see [5] for definitions). The limit with respect to ultrafilter p is always well-defined for bounded sequences, and since $0 \leq \langle f_i, \tilde{f}_i\rangle \leq 1$ it follows the denominator in (52) is a real number between 0 and 1.

If the sequence of finite subsets is given by the context, we use $R(p;\mathcal{F})$ to denote the redundancy function.

For Gabor frames $(f;\Lambda)$, the sequence of finite subsets $(\Lambda_n)_{n\geq 0}$ is defined by a sequence of centers $(k_n)_{n\geq 0}$ through $\Lambda_n = \{\lambda \in \Lambda\,;\, |\lambda - k_n| \leq n\}$. Then the redundancy function (52) becomes:

$$R : \mathbf{N}^* \to \mathbf{R} \cup \{\infty\}\ ,\ R(p) = \frac{1}{p\text{-}\lim_n \frac{1}{|\Lambda_n|} \sum_{\lambda\in\Lambda_n} \langle f_\lambda, \tilde{f}_\lambda\rangle}. \tag{53}$$

As proved in [4], in the case of Gabor frames, the redundancy function coincides with the density of the label set:

Theorem 7.1 (Theorem 3(b) in [4])**.** *Assume $\mathcal{G} = (g;\Lambda)$ is a Gabor frame in $L^2(\mathbf{R}^d)$. Then for any sequence of centers $(k_n)_{n\geq 0}$ in $\mathbf{R}^{2d}$ and free ultrafilter $p \in \mathbf{N}^*$,*

$$R(p;\mathcal{G}) = D(p;\Lambda) \tag{54}$$

For a l^1-localized frame $(\mathcal{F}, a, \mathcal{E})$ both $\mathcal{F}$ and $\mathcal{E}$ have their own redundancy function. Suppose we choose the sequences of finite subsets to be compatible with a in the following way: we choose a sequence of centers $(k_n)_{n\geq 0}$ in G and use the subsets $B_n(k_n) \subset G$ to define the redundancy function of $\mathcal{E}$ and $I_n = a^{-1}(B_n(k_n)) \subset I$ to define the redundancy function of $\mathcal{F}$:

$$R(p;\mathcal{F}) = \frac{1}{p\text{-}\lim_n \frac{1}{|I_n|} \sum_{i\in I_n} \langle f_i, \tilde{f}_i\rangle} \tag{55}$$

$$R(p;\mathcal{E}) = \frac{1}{p\text{-}\lim_n \frac{1}{|B_n(k_n)|} \sum_{j\in B_n(k_n)} \langle e_j, \tilde{e_j}\rangle} \tag{56}$$

There is a simple and important relation between the two redundancies and the density of the map a:

Theorem 7.2 (Theorem 5,(b) in [3])**.** *Assume $(\mathcal{F}, a, \mathcal{E})$ is l^2-localized and has finite upper density. Then*

$$R(p;\mathcal{F}) = D(p,a)R(p;\mathcal{E}) \tag{57}$$

for all $p \in \mathbf{N}^*$.

With these results in place, the main results of this work, Theorem 1.1 and 1.2, imply that a version of P_1 holds true for the redundancy function of l^1 localized frames and Gabor frames. Specifically

Theorem 7.3. *Assume $\mathcal{F} = \{f_i\,;\, i \in I\}$ is a frame for H, $\mathcal{E} = \{e_k\,;\, k \in G\}$ is a l^1-self localized frame for H, with G a discrete countable abelian group, $a : I \to G$ a localization map of finite upper density so that $(\mathcal{F}, a, \mathcal{E})$ is l^1 localized. Then for every $\varepsilon > 0$ there exists a subset $J = J_\varepsilon \subset I$ so that $\mathcal{F}[J] = \{f_i; i \in J\}$ is frame for H and*

$$R(p; \mathcal{F}[J]) \leq (1+\varepsilon) R(p; \mathcal{E}) \tag{58}$$

for all $p \in \mathbf{N}^*$.

When specialized to Gabor frames, this result reads:

Theorem 7.4. *Assume $\mathcal{G}(g; \Lambda)$ is a Gabor frame for $L^2(\mathbf{R}^d)$ with $g \in M^1(\mathbf{R}^d)$. Then for every $\varepsilon > 0$ there exists a subset $J_\varepsilon \subset \Lambda$ so that $\mathcal{G}' = \mathcal{G}(g; J_\varepsilon)$ is a Gabor frame for $L^2(\mathbf{R}^d)$ and its redundancy is upper bounded by $1 + \varepsilon$,*

$$R(p; \mathcal{G}') \leq 1 + \varepsilon$$

for all $p \in \mathbf{N}^*$.

By construction the redundancy function satisfies properties P_2 and P_3 regardless of any localization property: for any frame $\mathcal{F}$ indexed by I,

$$R(p; \mathcal{F}) \geq 1 \quad , \quad \forall p \in \mathbf{N}^*.$$

When $\mathcal{F}$ is a Riesz basis

$$R(p; \mathcal{F}) = 1 \quad , \quad \forall p \in \mathbf{N}^*.$$

Theorem 7.2 shows that in the setting of a frame $\mathcal{F}$ that is l^2 localized with respect to frame $\mathcal{E}$, the redundancy function of $\mathcal{F}$ is the product of the redundancy function for $\mathcal{E}$ *with* the frame density. The redundancy function of [5] is identically 1 for any Riesz basis and thus when $\mathcal{E}$ is a Riesz basis and $\mathcal{F}$ is l^2 localized with respect to $\mathcal{E}$, the redundancy function for $\mathcal{F}$ is *equal* to the frame density; consequently, for this case, the redundancy property satisfies the property P_4. Combining all these results, the redundancy function satisfies all four properties $P_1 - P_4$ in the case of a frame that is l^1 localized with respect to a family of frames of redundancy arbitrary close to 1:

Theorem 7.5. *Assume $\mathcal{E}_n$ be a sequence of l^1-self localized frames of H all indexed by the discrete abelian group G so that $liminf_n R(p, \mathcal{E}_n) = 1$ for all $p \in \mathbf{N}^*$. Assume $\mathcal{F} = \{f_i, i \in I\}$ is a frame for H and $(\mathcal{F}, a, \mathcal{E}_n)$ are all l^1-localized for all n, with respect to a localization map $a : I \to G$.*

1. For every $\varepsilon > 0$ there is a subset $J_\varepsilon \subset I$ so that $\mathcal{F}[J_\varepsilon] = \{f_i; i \in J_\varepsilon\}$ is frame for H and $R(p; \mathcal{F}[J_\varepsilon]) \leq 1 + \varepsilon$ for all $p \in \mathbf{N}^*$.

2. $R(p; \mathcal{F}) \geq 1$, for all $p \in \mathbf{N}^*$.

3. If $\mathcal{F}$ is a Riesz basis for H, then $R(p; \mathcal{F}) = 1$ for all $p \in \mathbf{N}^*$.

4. Assume $\mathcal{F}_1 = \{f_i^1, i \in I\}$ and $\mathcal{F}_2 = \{f_i^2, i \in I\}$ are two frames for H so that $(\mathcal{F}_k, a, \mathcal{E}_n)$ are l^1-localized for all n and $k = 1, 2$. Then

$$R(p; \mathcal{F}_1 \dot{\sqcup} \mathcal{F}_2) = R(p; \mathcal{F}_1) + R(p; \mathcal{F}_2)$$

for all $p \in \mathbf{N}^*$.

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References

[1] Radu Balan, Peter G. Casazza, Christopher Heil, and Zeph Landau. Deficits and excesses of frames. *Adv. Comput. Math.*, 18(2-4):93–116, 2003.

[2] Radu Balan, Peter G. Casazza, Christopher Heil, and Zeph Landau. Excesses of Gabor frames. *Appl. Comput. Harmon. Anal.*, 14(2):87–106, 2003.

[3] R. Balan, P.G. Casazza, C. Heil, and Z. Landau. Density, overcompleteness, and localization of frames I: Theory. *J. Fourier Anal. Appl.*, 12(2):105–143, 2006.

[4] R. Balan, P.G. Casazza, C. Heil, and Z. Landau. Density, overcompleteness, and localization of frames II: Gabor frames. *J. Fourier Anal. Appl.*, 12(3):307–344, 2006.

[5] R. Balan and Z. Landau. Measure functions for frames. *J. Funct. Anal.*, 252:630–676, 2007.

[6] P.G. Casazza. Local theory of frames and schauder bases for hilbert space. *Illinois Jour. Math.*, 43:291–306, 1999.

[7] P.G. Casazza. The Art of Frame Theory. *Taiwanese J. Math.*, 4:129–201, 2000.

[8] O. Christensen, B. Deng, and C. Heil. Density of Gabor frames. *Appl. Comput. Harmon. Anal.*, 7(3):292–304, 1999.

[9] Ole Christensen. *An Introduction to Frames and Riesz Bases.* Birkhäuser, Boston, 2003.

[10] R.J. Duffin and A.C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Am. Math. Soc.*, 72:341–366, 1952. reprinted in hewa06.

[11] H.G. Feichtinger. On a new Segal algebra. *Monatsh. Math.*, 92:269–289, 1981.

[12] H.G. Feichtinger. Atomic characterizations of modulation spaces through Gabor-type representations. In *Proc. Conf. Constructive Function Theory*, volume 19 of *Rocky Mountain J. Math.*, pages 113–126, 1989.

[13] H.G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86:307–340, 1989.

[14] H.G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, II. *Monatsh. Math.*, 108:129–148, 1989.

[15] H.G. Feichtinger and K. Gröchenig, *Non-Orthogonal wavelet and Gabor expansions, and group representations*, in "Wavelets and their applications", Beylkin, G. and Coifman, R. and Daubechies, I. Eds., (1992), pp. 353-376

[16] H.G. Feichtinger and K. Gröchenig. Gabor frames and time-frequency analysis of distributions. *J. Funct. Anal.*, 146(2):464–495, 1997.

[17] M. Fornasier and K. Gröchenig. Intrinsic localization of frames. *Constr. Approx.*, 22(3):395–415, 2005.

[18] K. Gröchenig. *Foundations of time-frequency analysis.* Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

[19] K. Gröchenig. Localization of Frames, Banach Frames, and the Invertibility of the Frame Operator. *J. Fourier Anal. Appl.*, 10(2):105–132, 2004.

[20] C. Heil. On the history and evolution of the density theorem for Gabor frames. Technical report, Georgia Institute of Technology, 2006.

[21] N. Hindman and D. Strauss, Algebra in the Stone-Čech Compactification, de Gruyter Expositions in Mathematics Vol. 27, Walter de Gruyter and Co., Berlin, 1998.

[22] R.V. Kadison and J.R. Ringrose. *Fundamentals of the Theory of Operator Algebras. I.* AMS Graduate Studies in Mathematics 15, 1997.

[23] H.J. Landau. *Necessary density conditions for sampling and interpolation of certain entire functions.* Acta Math. 117: 37-52, 1967.

[24] Yu.I. Lyubarskij. Frames in the Bargmann space of entire functions. In *Entire and subharmonic functions*, volume 11 of *Adv. Sov. Math.*, pages 167–180. American Mathematical Society (AMS), Providence, RI, 1992.

[25] F Riesz and B.S. Nagy. *Functional Analysis.* Dover Publications, 1990.

[26] Kristian Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. I. *J. Reine Angew. Math.*, 429:91–106, 1992.

[27] Kristian Seip and Robert Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

[28] D.S. Spielman and N. Srivastave, *An elementary proof of the restricted invertibility theorem*, preprint, http://arxiv.org/abs/0911.1114, Nov. 2009.

[29] R. Vershynin. Subsequences of frames. *Studia Mathematica*, 145:185–197, 2001.