电动力学

第二十六讲

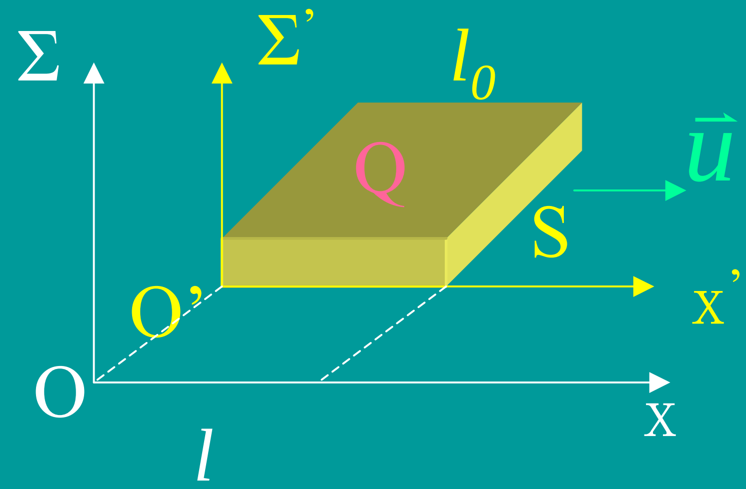
西安石油大学理学院
应用物理系

§5 电动力学的相对论不变性

一. 四维电流密度矢量

电荷为一洛仑兹标量

Σ系电荷密度



$$\rho=\frac{Q}{V}=\frac{Q}{lS}=\frac{Q}{Sl_0\sqrt{1-u^2/c^2}}$$

$$=\frac{Q/Sl_0}{\sqrt{1-u^2/c^2}}$$

静止电荷密度ρ_0

$$\rho=\rho_0/\sqrt{1-u^2/c^2}=\gamma_u\rho_0$$

结论：电荷密度与物体的运动速度有关

注意：物体的运动速度u可以朝向任意方向且不必是匀速

三维空间　　四维空间

$$j_4=\rho_0\gamma_\mu(ic)=ic\rho$$

$$\vec{j}=\rho\vec{u}=\rho_0\gamma_\mu\vec{u}$$

$$j_\mu=\rho_0U_\mu=\rho_0\gamma_\mu(\vec{u},ic)$$

第四维分量

四维位移 $x_\mu=(\vec{x},ict)$ ⟷ $j_u=(\vec{j},ic\rho)$ 四维电流密度

$$0=\nabla\bullet\vec{j}+\frac{\partial(ic\rho)}{\partial(ict)} \Rightarrow \frac{\partial j_1}{\partial x_1}+\frac{\partial j_2}{\partial x_2}+\frac{\partial j_3}{\partial x_3}+\frac{\partial j_4}{\partial x_4}=0 \Rightarrow \frac{\partial j_\mu}{\partial x_\mu}=0$$

电荷守恒定律协变

洛仑兹标量

注意：

- j和ρ统一为四维电流密度矢量说明，电流和电荷是一个统一的物理量的两个方面，静止时为电荷密度ρ_0，运动时为j，且电荷密度也变了
- 四维电流密度矢量也是用四维速度矢量定义的

二．麦克斯韦方程的协变性

洛仑兹标量算符

1．四维势矢量　洛仑兹规范　微分算符

$$\begin{cases}\nabla^2\vec{A}-\dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{j}\\ \nabla^2\varphi-\dfrac{1}{c^2}\dfrac{\partial^2\varphi}{\partial t^2}=-\dfrac{\rho}{\varepsilon_0}\\ \nabla\bullet\vec{A}+\dfrac{1}{c^2}\dfrac{\partial\varphi}{\partial t}=0\end{cases}$$

达朗贝尔方程

(洛仑兹条件)

$$\Box\equiv\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}=\nabla^2+\frac{\partial^2}{\partial(ict)^2}=\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu}$$

$$\Box\vec{A}=-\mu_0\vec{j}$$ → 激发矢势

$$\Box\varphi=-\frac{\rho}{\varepsilon_0}=-\mu_0c^2\rho=-(-ic)\mu_0 j_4$$ → 激发标势

$$\nabla\bullet\vec{A}+\frac{\partial\left(\frac{i}{c}\varphi\right)}{\partial(ict)}=0$$

四维位移矢量的第四个分量

$$A_\mu=(\vec{A},\frac{i\varphi}{c})$$

四维势矢量

$$A_\mu = (\vec{A}, \frac{i\varphi}{c})$$

协变式

洛仑兹条件

$$\nabla \bullet \vec{A} + \frac{\partial\left(\frac{i}{c}\varphi\right)}{\partial(ict)} = 0 \Rightarrow \frac{\partial A_\mu}{\partial x_\mu} = 0$$

$$\Box \vec{A} = -\mu_0 \vec{j}$$

$$\Box \varphi = -(-ic)\mu_0 j_4 \Rightarrow \Box\left(\frac{i}{c}\varphi\right) = -\mu_0 j_4$$

$$\Rightarrow \Box A_\mu = -\mu_0 j_\mu$$

协变式

结论： 由势表示的麦克斯韦方程组即表示成了协变式

2. 电磁场张量

$$x_4 = ict$$

$$\nabla = \hat{e}_x \frac{\partial}{\partial x_1} + \hat{e}_y \frac{\partial}{\partial x_2} + \hat{e}_z \frac{\partial}{\partial x_3} \qquad \vec{A} = A_1\hat{e}_x + A_2\hat{e}_y + A_3\hat{e}_z$$

电场用势表示

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} = (ic)\nabla\left(\frac{i}{c}\varphi\right) - (ic)\frac{\partial \vec{A}}{\partial x_4} = (ic)\left[\nabla A_4 - \frac{\partial \vec{A}}{\partial x_4}\right]$$

$$= (ic)\left[\left(\frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4}\right)\hat{e}_x + \left(\frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4}\right)\hat{e}_y + \left(\frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4}\right)\hat{e}_z\right]$$

$$-\frac{i}{c}E_1 = \frac{\partial A_4}{\partial x_1} - \frac{\partial A_1}{\partial x_4} \qquad -\frac{i}{c}E_2 = \frac{\partial A_4}{\partial x_2} - \frac{\partial A_2}{\partial x_4} \qquad -\frac{i}{c}E_3 = \frac{\partial A_4}{\partial x_3} - \frac{\partial A_3}{\partial x_4}$$

磁场用势表示

$$\vec{B} = \nabla \times \vec{A} = \hat{e}_x\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) - \hat{e}_y\left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right) + \hat{e}_z\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$= B_1 \qquad\qquad = B_2 \qquad\qquad = B_3$$

引入反对称张量 $$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$$

$$F_{12} = -F_{21} = B_3 \qquad F_{41} = -F_{14} = \frac{i}{c}E_1$$

$$F_{31} = -F_{13} = B_2 \qquad F_{42} = -F_{24} = \frac{i}{c}E_2$$

$$F_{23} = -F_{32} = B_1 \qquad F_{43} = -F_{34} = \frac{i}{c}E_3$$

$$F_{11} = F_{22} = F_{33} = F_{44} = 0$$

结论：通过四维势，该反对称张量将电场强度和磁场强度统一为一个物理量——电磁场张量

$$F_{12} = -F_{21} = B_3 \qquad F_{41} = -F_{14} = \frac{i}{c}E_1$$

$$F_{31} = -F_{13} = B_2 \qquad F_{42} = -F_{24} = \frac{i}{c}E_2$$

$$F_{23} = -F_{32} = B_1 \qquad F_{43} = -F_{34} = \frac{i}{c}E_3$$

$$F_{11} = F_{22} = F_{33} = F_{44} = 0$$

$$\nabla \bullet \vec{E} = \frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = \frac{c}{i}\frac{\partial F_{41}}{\partial x_1} + \frac{c}{i}\frac{\partial F_{42}}{\partial x_2} + \frac{c}{i}\frac{\partial F_{43}}{\partial x_3}$$

$$\mu_0 j_4 = ic\mu_0\varepsilon_0 \nabla \bullet \vec{E} = ic\mu_0\varepsilon_0\left[\frac{c}{i}\frac{\partial F_{41}}{\partial x_1} + \frac{c}{i}\frac{\partial F_{42}}{\partial x_2} + \frac{c}{i}\frac{\partial F_{43}}{\partial x_3}\right]$$

$$= \frac{\partial F_{41}}{\partial x_1} + \frac{\partial F_{42}}{\partial x_2} + \frac{\partial F_{43}}{\partial x_3} + \frac{\partial F_{44}}{\partial x_4} = \frac{\partial F_{4\nu}}{\partial x_\nu}$$

$$(F_{44} = 0)$$

$$F_{12} = -F_{21} = B_3 \qquad F_{41} = -F_{14} = \frac{i}{c}E_1$$

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$$F_{11} = F_{22} = F_{33} = F_{44} = 0$$

$$= \hat{e}_x\left(\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \mu_0\varepsilon_0(ic)\frac{\partial E_1}{\partial x_4}\right) + \hat{e}_y\left(\frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1} - \mu_0\varepsilon_0(ic)\frac{\partial E_2}{\partial x_4}\right)$$

$$+ \hat{e}_z\left(\frac{\partial B_2}{\partial x_1} - \frac{\partial B_1}{\partial x_2} - \mu_0\varepsilon_0(ic)\frac{\partial E_3}{\partial x_4}\right)$$

$$=\hat{e}_x\left(\frac{\partial F_{12}}{\partial x_2}+\frac{\partial F_{13}}{\partial x_3}+\mu_0\varepsilon_0(ic)\frac{c}{i}\frac{\partial F_{14}}{\partial x_4}\right)+\hat{e}_y\left(\frac{\partial F_{23}}{\partial x_3}+\frac{\partial F_{21}}{\partial x_1}+\mu_0\varepsilon_0(ic)\frac{c}{i}\frac{\partial F_{24}}{\partial x_4}\right)$$

$$+\hat{e}_z\left(\frac{\partial F_{31}}{\partial x_1}+\frac{\partial F_{32}}{\partial x_2}+\mu_0\varepsilon_0(ic)\frac{c}{i}\frac{\partial F_{34}}{\partial x_4}\right)$$

$$=\hat{e}_x\left(\frac{\partial F_{12}}{\partial x_2}+\frac{\partial F_{13}}{\partial x_3}+\frac{\partial F_{14}}{\partial x_4}+\frac{\partial F_{11}}{\partial x_1}\right)+\hat{e}_y\left(\frac{\partial F_{21}}{\partial x_1}+\frac{\partial F_{22}}{\partial x_2}+\frac{\partial F_{23}}{\partial x_3}+\frac{\partial F_{24}}{\partial x_4}\right)$$

$$+\hat{e}_z\left(\frac{\partial F_{31}}{\partial x_1}+\frac{\partial F_{32}}{\partial x_2}+\frac{\partial F_{33}}{\partial x_3}+\frac{\partial F_{34}}{\partial x_4}\right)$$

$$F_{11}=F_{22}=F_{33}=F_{44}=0$$

$$=\hat{e}_x\left(\frac{\partial F_{1\nu}}{\partial x_\nu}\right)+\hat{e}_y\left(\frac{\partial F_{2\nu}}{\partial x_\nu}\right)+\hat{e}_z\left(\frac{\partial F_{3\nu}}{\partial x_\nu}\right)=\mu_0\vec{j}$$

$$\nabla\bullet\vec{E}=\frac{\rho}{\varepsilon_0}\quad\Rightarrow\quad\frac{\partial F_{4\nu}}{\partial x_\nu}=\mu_0 j_4$$

$$\Rightarrow\quad\frac{\partial F_{\mu\nu}}{\partial x_\nu}=\mu_0 j_\mu$$

$$\left.\begin{aligned}\nabla\bullet\vec{B}&=0\\ \nabla\times\vec{E}&=-\frac{\partial\vec{B}}{\partial t}\end{aligned}\right\}\Rightarrow\quad \boxed{\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}+\frac{\partial F_{\lambda\mu}}{\partial x_\nu}=0}$$

结论： 麦克斯韦方程组已用电磁场张量表示成了协变形式

注意：

- **E**和**B**构成了电磁场张量的分量表明， **E**和**B**是同一物理量的两个方面。在参考系变换时，它们互相变化
- $F_{\mu\nu}$是洛仑兹协变张量

§6 相对论力学

本节的主要任务是将经典力学的基本规律——牛顿定律改写为相对论协变形式

一. 四维动量

1. 能量动量四维矢量

$$\vec{F} = \frac{d\vec{p}}{dt}$$

相对论下非协变

$$\vec{p} = m\vec{v}$$

四维速度矢量

$$U_\mu = \frac{dx_\mu}{d\tau} = \gamma \frac{dx_\mu}{dt}$$

定义四维动量

$$P_\mu = m_0 U_\mu$$

协变

静止质量

$$P_\mu = m_0 U_\mu = m_0 \gamma (u_i, ic)$$

$$\vec{p} = m_0 \gamma \vec{u} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\vec{u}$$

$$p_4 = m_0 \gamma (ic)$$

三维空间动量

m

物体运动速度

$$\vec{p} = m_0 \gamma \vec{u} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\vec{u} = m\vec{u}$$

经典动量 $m_0\vec{u}$

与能量相关

能量量纲

$$p_4 = m_0 \gamma (ic) = \frac{i}{c}\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{i}{c}(mc^2) = \frac{i}{c}W$$

相对论能量

$$= \frac{i}{c}\left(m_0 c^2 + \frac{1}{2} m_0 v^2 + \cdots\right)$$

物体动能

v<<c

静止能量　　动能

相对论能量　$W=m_0c^2+T$

质能关系　$W=mc^2$

静质能关系　$W_0=m_0c^2$

四维动量　$P_\mu=\left(\vec{p},\frac{i}{c}W\right)$

能量——动量四维矢量

2．静止能量

$$\left.\begin{aligned} &W=m_0c^2+T \\ &W=m_0c^2\Big/\sqrt{1-\tfrac{v^2}{c^2}} \end{aligned}\right\} \Rightarrow T=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}-m_0c^2$$

$$=mc^2-m_0c^2=(m-m_0)c^2$$

结论：物体的动能决定于物体的运动质量与静止质量，而物体的静止能量m_0c^2只与其静止质量有关，与物体结构无关。m_0体现了物质的内部运动

静质能关系 $W_0 = m_0c^2$

静质能关系的意义

- 它反映了作为惯性量度的质量与作为运动强度量度的能量之间的关系
- 揭示了静止物体（粒子）内部仍具有仍存在运动。一定质量的粒子具有一定的内部运动能量；对于粒子系统，它的静止能量$W_0=M_0c^2$，M_0为相对于质心静止时的总质量

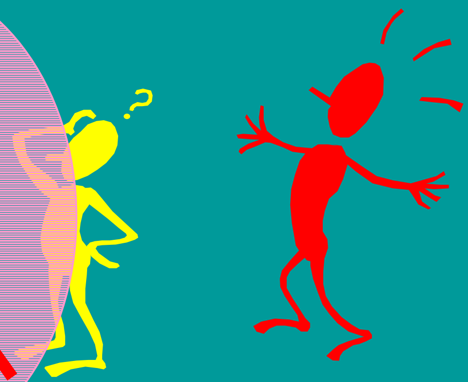
3．结合能

m_{0i} ⟺ $W_{0i}=m_{0i}c^2$

M_0 ⟺ $W_0=M_0c^2$

$\sum_i W_{0i} \overset{?}{=} W_0$

$\sum_i m_{0i} \overset{?}{=} M_0$



结合能　　　　质量亏损　　　　关系

$$\Delta W=\sum_i m_{0i}c^2-W_0$$

$$\Delta M=\sum_i m_{0i}-M_0$$

$$\Delta W=(\Delta M)c^2$$

物质是反映在其运动和转化过程之中的

4．动量，能量和质量的关系

$$P_\mu=\left(\vec{p},\frac{i}{c}W\right)$$

→ 不变量：$P_\mu P_\mu=p^2-W^2/c^2$

静止系：$\vec{p}=0$，$W=m_0c^2$ → $P_\mu P_\mu=-m_0^2c^2$

运动系：→ $P_\mu P_\mu=p^2-\dfrac{W^2}{c^2}$

$$\Rightarrow\ p^2-\frac{W^2}{c^2}=-m_0^2c^2$$

$$W=\sqrt{p^2c^2+m_0^2c^4}$$

二．相对论力学方程

1．力学方程

力学基本方程的协变形式

四维力矢量

维矢量

$$\vec{F}=\frac{d\vec{p}}{dt}$$

v<<c

$$\vec{K}=\frac{d\vec{p}}{d\tau}$$

思想推广

空间分量

$$K_\mu=\frac{dP_\mu}{d\tau}$$

洛仑兹标量

力学方程

$$\begin{cases} K_4=\dfrac{dp_4}{d\tau}=\dfrac{i}{c}\dfrac{dW}{d\tau} \\ W=\sqrt{p^2c^2+m_0^2c^4} \end{cases}$$

$$\begin{cases} \vec{K}=\dfrac{d\vec{p}}{d\tau} \\ \vec{K}\bullet\vec{v}=\dfrac{dW}{d\tau} \end{cases}$$

$$K_4=\frac{i}{c}\frac{c^2}{W}\vec{p}\bullet\frac{d\vec{p}}{d\tau}=\frac{i}{c}\frac{\vec{p}}{\gamma m_0}\bullet\frac{d\vec{p}}{d\tau}$$

$$=\frac{i}{c}\vec{v}\bullet\frac{d\vec{p}}{d\tau}=\frac{i}{c}\vec{K}\bullet\vec{v}$$

$W=\gamma m_0c^2$

四维力矢量

$$K_\mu=\left(\vec{K},\frac{i}{c}\vec{K}\bullet\vec{v}\right)$$

$$\begin{cases} \vec{K} = \dfrac{d\vec{p}}{d\tau} \\ \vec{K} \bullet \vec{v} = \dfrac{dW}{d\tau} \end{cases}$$

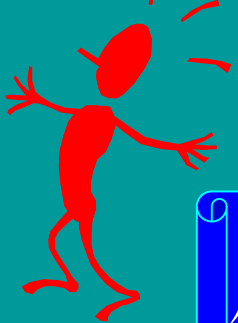
用参考系时间
dt=γdτ

定义力

$$\left. \begin{aligned} \sqrt{1-\frac{v^2}{c^2}}\vec{K} &= \frac{d\vec{p}}{dt} \\ \sqrt{1-\frac{v^2}{c^2}}\vec{K} \bullet \vec{v} &= \frac{dW}{dt} \end{aligned} \right\}$$

注意
2点

相对论力学方程



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不是四维力矢量的分量

$$\begin{cases} \vec{F} = \dfrac{d\vec{p}}{dt} \\ \vec{F} \bullet \vec{v} = \dfrac{dW}{dt} \end{cases}$$

1

相对论的动量与能量

2. 洛仑兹力

洛仑兹力公式 $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$

$$\vec{F} = \sqrt{1-\frac{v^2}{c^2}}\vec{K}$$

四维力矢量的空间分量

洛仑兹力公式 $\vec{K} = \gamma e(\vec{E} + \vec{v} \times \vec{B})$

- 四维力矢量的空间分量乘以γ即为洛仑兹力
- 由于**K**是四维力矢量的空间分量，因此，洛仑兹力满足协变性的要求