第三节 定积分的换元法和分部积分法

不定积分

$$\left\{\begin{array}{l}\text{换元积分法}\\ \text{分部积分法}\end{array}\right. \longrightarrow \text{定积分}\left\{\begin{array}{l}\text{换元积分法}\\ \text{分部积分法}\end{array}\right.$$

一、定积分的换元法

二、定积分的分部积分法

一、定积分的换元法

定理1. 设函数 $f(x)\in C[a,b]$, 单值函数 $x=\varphi(t)$ 满足:

1) $\varphi(t)\in C^1[\alpha,\beta]$, $\varphi(\alpha)=a,\varphi(\beta)=b$;

2) 在 $[\alpha,\beta]$ 上 $a\le\varphi(t)\le b$,

则 $$\int_a^b f(x)\,\mathrm{d}x=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t$$

证: 所证等式两边被积函数都连续, 因此积分都存在, 且它们的原函数也存在. 设 $F(x)$ 是 $f(x)$ 的一个原函数, 则 $F[\varphi(t)]$ 是 $f[\varphi(t)]\varphi'(t)$ 的原函数, 因此有

$$\int_a^b f(x)\,\mathrm{d}x=F(b)-F(a)=F[\varphi(\beta)]-F[\varphi(\alpha)]$$

$$=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t$$

$$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t$$

说明:

1) 当$\beta < \alpha$, 即区间换为$[\beta, \alpha]$时， 定理 1 仍成立 .

2) 必需注意换元必换限 , 原函数中的变量不必代回 .

3) 换元公式也可反过来使用 , 即

$$\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x \quad (\text{令 } x = \varphi(t))$$

或配元 $\int_\alpha^\beta f[\varphi(t)]\underline{\varphi'(t)\,\mathrm{d}t} = \int_\alpha^\beta f[\varphi(t)]\,\mathrm{d}\varphi(t)$

配元不换限

例**1.** 计算 $\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x \ (a > 0)$.

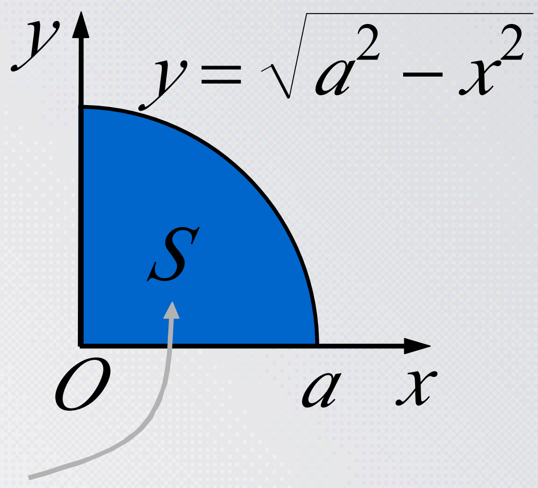
解：令 $x = a\sin t$, 则 $\mathrm{d}x = a\cos t\,\mathrm{d}t$, 且

当 $x = 0$ 时, $t = 0$; $x = a$ 时, $t = \frac{\pi}{2}$.

$$\therefore \quad \text{原式} = a^2 \int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t)\,\mathrm{d}t$$

$$= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}$$



例2. 计算 $\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x$.

解: 令 $t=\sqrt{2x+1}$, 则 $x=\frac{t^2-1}{2}$, $\mathrm{d}x=t\,\mathrm{d}t$, 且

当 $x=0$ 时, $t=1$; $x=4$ 时, $t=3$.

$$\therefore \quad 原式 = \int_1^3 \frac{\frac{t^2-1}{2}+2}{t}\, t\,\mathrm{d}t$$

$$= \frac{1}{2}\int_1^3 (t^2+3)\,\mathrm{d}t$$

$$= \frac{1}{2}\left(\frac{1}{3}t^3+3t\right)\Bigg|_1^3 = \frac{22}{3}$$

例3. 设 $f(x)\in C[-a,a]$,

偶倍奇零

(1) 若 $f(-x)=f(x)$, 则 $\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$

(2) 若 $f(-x)=-f(x)$, 则 $\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$

证: $\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x$

令 $x=-t$

$$= \int_{0}^{a} f(-t)\,\mathrm{d}t + \int_{0}^{a} f(x)\,\mathrm{d}x$$

$$= \int_{0}^{a} [f(-x)+f(x)]\,\mathrm{d}x$$

$$= \begin{cases} 2\int_{0}^{a} f(x)\,\mathrm{d}x, & f(-x)=f(x)\text{时} \\ 0, & f(-x)=-f(x)\text{时} \end{cases}$$

例4. 设 $f(x)$ 是连续的周期函数, 周期为 T, 证明:

(1) $\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x$

(2) $\int_a^{a+nT} f(x)\,\mathrm{d}x = n\int_0^T f(x)\,\mathrm{d}x \quad (n \in \mathbf{N})$, 并由此计算

$$I = \int_0^{n\pi} \sqrt{1+\sin 2x}\,\mathrm{d}x$$

解: (1) 记 $\Phi(a) = \int_a^{a+T} f(x)\,\mathrm{d}x$, 则

$$\Phi'(a) = f(a+T) - f(a) = 0$$

可见$\Phi(a)$与a无关, 因此$\Phi(a) = \Phi(0)$, 即

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x$$

(2) $\int_a^{a+nT} f(x)\mathrm{d}x = n\int_0^T f(x)\mathrm{d}x \quad (n\in\mathbf{N})$, 并由此计算

$$\int_0^{n\pi}\sqrt{1+\sin 2x}\,\mathrm{d}x$$

(1) $\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x$

(2) $$\int_a^{a+nT} f(x)\mathrm{d}x = \sum_{k=0}^{n-1}\int_{a+kT}^{a+kT+T} f(x)\mathrm{d}x$$

将 $a+kT$ 看作(1)中的 a, 则有

$$\int_{a+kT}^{a+kT+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x$$

$$= n\int_0^T f(x)\mathrm{d}x \quad (n\in\mathbf{N})$$

$$\int_0^{n\pi}\sqrt{1+\sin 2x}\,\mathrm{d}x$$

$$= n\int_0^{\pi}\sqrt{1+\sin 2x}\,\mathrm{d}x$$

$\sqrt{1+\sin 2x}$ 是以 π 为周期的周期函数

$$\int_0^{n\pi}\sqrt{1+\sin 2x}\,\mathrm{d}x = n\int_0^{\pi}\sqrt{1+\sin 2x}\,\mathrm{d}x$$

$$= n\int_0^{\pi}\sqrt{(\cos x+\sin x)^2}\,\mathrm{d}x$$

$$= n\int_0^{\pi}\left|\cos x+\sin x\right|\mathrm{d}x$$

$$= n\sqrt{2}\int_0^{\pi}\left|\sin(x+\tfrac{\pi}{4})\right|\mathrm{d}x$$

令 $t = x+\frac{\pi}{4}$

$$= n\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\left|\sin t\right|\mathrm{d}t$$

$$= n\sqrt{2}\int_0^{\pi}\left|\sin t\right|\mathrm{d}t$$

$$= n\sqrt{2}\int_0^{\pi}\sin t\,\mathrm{d}t = 2\sqrt{2}\,n$$

(1) $\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^{T} f(x)\,\mathrm{d}x$

二、定积分的分部积分法

定理2. 设 $u(x),\, v(x) \in C^1[a,\, b]$, 则

$$\int_a^b u(x)\, v'(x)\,\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b u'(x)\, v(x)\,\mathrm{d}x$$

证: $\because\ [u(x)\, v(x)]' = u'(x)v(x) + u(x)v'(x)$

$\big\downarrow$ 两端在 $[a,b]$ 上积分

$$u(x)\, v(x)\Big|_a^b = \int_a^b u'(x)v(x)\,\mathrm{d}x + \underline{\int_a^b u(x)v'(x)\,\mathrm{d}x}$$

$$\therefore\quad \int_a^b u(x)\, v'(x)\,\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b u'(x)\, v(x)\,\mathrm{d}x$$

例5. 计算 $\int_0^{\frac{1}{2}} \arcsin x \, dx.$

解：原式 $= x\arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} dx$

$$= \frac{\pi}{12} + \frac{1}{2}\int_0^{\frac{1}{2}} (1-x^2)^{\frac{-1}{2}} d(1-x^2)$$

$$= \frac{\pi}{12} + (1-x^2)^{\frac{1}{2}} \Big|_0^{\frac{1}{2}}$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

例6. 证明 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \mathrm{d}x$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ 为偶数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3}, & n \text{ 为奇数} \end{cases}$$

证: 令 $t = \frac{\pi}{2} - x$, 则

$$\int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x = -\int_{\frac{\pi}{2}}^0 \sin^n (\tfrac{\pi}{2} - t) \mathrm{d}t = \int_0^{\frac{\pi}{2}} \cos^n x \mathrm{d}x$$

令 $u = \sin^{n-1} x$, $v' = \sin x$, 则 $u' = (n-1)\sin^{n-2} x \cos x$,

$v = -\cos x$

$$\therefore \quad I_n = \left[-\cos x \cdot \sin^{n-1} x\right]\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \mathrm{d}x$$

（其中 $\left[-\cos x \cdot \sin^{n-1} x\right]\Big|_0^{\frac{\pi}{2}} = 0$）

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$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \,\mathrm{d}x$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1-\sin^2 x)\,\mathrm{d}x$$

$$= (n-1)\, I_{n-2} - (n-1)\, I_n$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x$$

由此得递推公式 $I_n = \frac{n-1}{n} I_{n-2}$

于是

$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \cdots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0$$

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \cdots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$$

而

$$I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}, \qquad I_1 = \int_0^{\frac{\pi}{2}} \sin x \,\mathrm{d}x = 1$$

故所证结论成立 .