第五节　多元复合函数的求导法则

一、多元复合函数的求导法则

在一元函数微分学中，复合函数的求导法则起着重要的作用.

现在我们把它推广到多元复合函数的情形.

下面按照多元复合函数不同的复合情形，分三种情况进行讨论.

1. 复合函数的中间变量均为一元函数的情形

定理 1 如果函数 及 都在点 t 可导，函数 在对应点 具有连续偏导数，则复合函数 在对应点 t 可导，且其导数可用下列公式计算：

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}$$

证明 设 t 获得增量 Δt,

则 $\Delta u=\phi(t+\Delta t)-\phi(t)$, $\Delta v=\psi(t+\Delta t)-\psi(t)$;

由于函数 在点 有连续偏导数 z $=$

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+\varepsilon_1\Delta u+\varepsilon_2\Delta v,$$

当 Λ , u $\longrightarrow$ 时, $\varepsilon_1 \rightarrow 0$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta t}+\varepsilon_1\frac{\Delta u}{\Delta t}+\varepsilon_2\frac{\Delta v}{\Delta t}$$

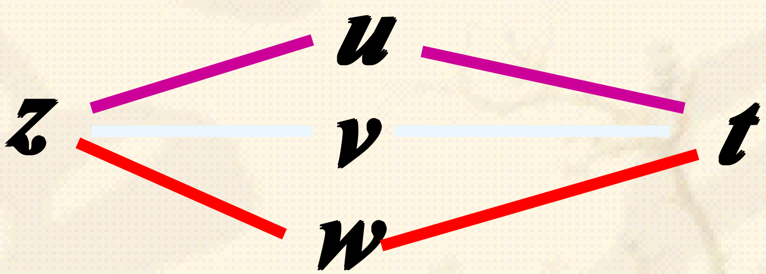
当 时 Λ t , $\longrightarrow$

$$\frac{\Delta u}{\Delta t}\rightarrow\frac{\mathrm{d}u}{\mathrm{d}t},\quad \frac{\Delta v}{\Delta t}\rightarrow\frac{\mathrm{d}v}{\mathrm{d}t},$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}.$$

上述定理的结论可推广到中间变量多于两个的情况.

如 $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t}$$



以上公式中的导数 $\frac{\mathrm{d}z}{\mathrm{d}t}$ 称为全导数.

例 1 设 ，而 ， ，

求全导数 .

解 $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial t}$$

$$=ve^{t}-u\sin t+\cos t$$

$$=e^{t}\cos t-e^{t}\sin t+\cos t$$

$$=e^{t}(\cos t-\sin t)+\cos t.$$

例2 若可微函数 $f(x,y)$ 对任意正实数 λ 满足

$$f(\lambda x,\lambda y)=\lambda^k f(x,y),$$

则称 $f(x,y)$ 为 **k 次齐次函数**. 证明 k 次齐次函数满足方程

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=kf(x,y).$$

证 设 $u=\lambda x, v=\lambda y,$ 则由已知条件有等式

$$f(u,v)=\lambda^k f(x,y)$$

上述等式左边可以看作以 u,v 为中间变量 λ 为自变量的函数. 等式两端对 λ 求导数，得

$$\frac{\partial f}{\partial u}\cdot\frac{du}{d\lambda}+\frac{\partial f}{\partial v}\cdot\frac{dv}{d\lambda}=k\lambda^{k-1}f(x,y),$$

即

$$x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}=k\lambda^{k-1}f(x,y),$$

上式对任意正实数 λ 都成立，特别取 $\lambda=1$，即得证所证等式

$$x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}=kf(x,y).$$

2. 复合函数的中间变量均为多元函数的情形

定理 2 如果 及 都在点

具有对 和 的偏导数，且函数

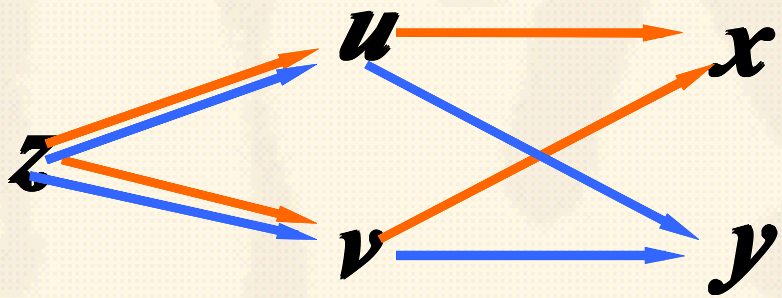
在对应点 具有连续偏导数，则复合函数

在对应点 的两个

偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

链式法则如图示



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

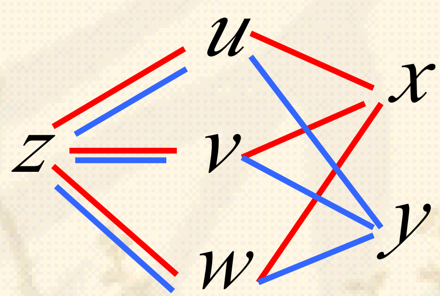
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

定理 2 推广，设 、 、 都在点 具有对 和 的偏导数，复合函数 在对应点 的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial x}$$

$$+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$

，



．

例 3 设 x ∂y ，而 ， ，

求 和 .

解 $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=e^{u}\sin v\cdot y+e^{u}\cos v\cdot 1=e^{u}(y\sin v+\cos v),$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=e^{u}\sin v\cdot x+e^{u}\cos v\cdot 1=e^{u}(x\sin v+\cos v).$$

例 4 设 $\frac{\quad}{\partial x}$ ，$\frac{\quad}{\partial x \partial z}$ 具有二阶连续偏导数，求 和 .

解 令 $u = x + y + z,\quad v = xyz;$

记 $f_1' = \dfrac{\partial f(u,v)}{\partial u},\qquad f_{12}'' = \dfrac{\partial^2 f(u,v)}{\partial u \partial v},$

同理有 $f_2',\quad f_{11}'',\quad f_{22}''.$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1' + yzf_2';$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}(f_1' + yzf_2') = \frac{\partial f_1'}{\partial z} + yf_2' + yz\frac{\partial f_2'}{\partial z};$$

$$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{11}'' + xyf_{12}'';$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{21}'' + xyf_{22}'';$$

于是 $$\frac{\partial^2 w}{\partial x \partial z} = f_{11}'' + xyf_{12}'' + yf_2' + yz(f_{21}'' + xyf_{22}'')$$

$$= f_{11}'' + y(x+z)f_{12}'' + xy^2 zf_{22}'' + yf_2'.$$

3. 复合函数的中间变量既有一元函数又有多元函数的情形

定理 3 如果　　　　　　在点　　　　　具有对　和

的偏导数，函数　　　　　　在点　可导，且函数

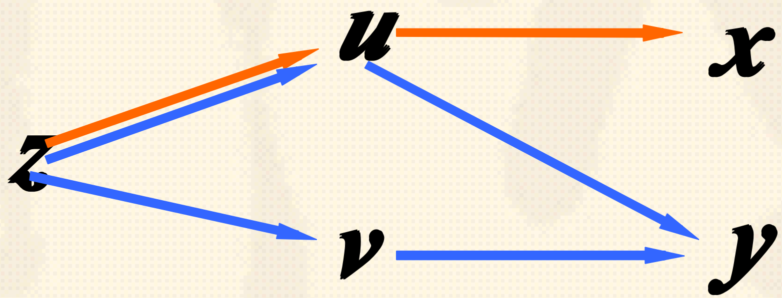
　　　　　　在对应点　　　　具有连续偏导数，则

复合函数　　　　　　　　　　　　在点　　　　的两个偏

导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}y}$$

链式法则如图示



$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}y}.$$

特殊地 $z=f(u,x,y)$ 其中 $u=\phi(x,y)$

即 $z=f[\phi(x,y),x,y]$, 令 $v=x$, $w=y$,

$$\frac{\partial v}{\partial x}=1,\quad \frac{\partial w}{\partial x}=0,\quad \frac{\partial v}{\partial y}=0,\quad \frac{\partial w}{\partial y}=1.$$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}.$$

区别类似

两者的区别

把复合函数

中的 看作不变而对 的偏导数

把

中的 及 作不

变而对 的 数

4.全微分形式不变性

设函数 具有连续偏导数，则有全微分

;当 、

时，有 .

全微分形式不变性的实质：

无论 z是自变量u、v的函数或中间变量u、v的函数，它的全微分形式是一样的.

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

$$=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y\right)$$

$$=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v.$$

例 5 已知，求和.

解 $\because \mathrm{d}(e^{-xy}-2z+e^{z})=0,$

$\therefore e^{-xy}\mathrm{d}(-xy)-2\mathrm{d}z+e^{z}\mathrm{d}z=0,$

$$(e^{z}-2)\mathrm{d}z=e^{-xy}(x\mathrm{d}y+y\mathrm{d}x)$$

$$\mathrm{d}z=\frac{ye^{-xy}}{(e^{z}-2)}\mathrm{d}x+\frac{xe^{-xy}}{(e^{z}-2)}\mathrm{d}y$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$