Optimization of LPN Solving Algorithms

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Abstract. In this article we focus on constructing an algorithm that automatizes the generation of LPN solving algorithms from the considered parameters. When searching for an algorithm to solve an LPN instance, we make use of the existing techniques and optimize their use. We formalize an LPN algorithm as a path in a graph G and our algorithm is searching for the optimal paths in this graph. The results bring improvements over the existing work by a factor from 2^8 to 2^{10}, i.e. we improve the results of the covering code from ASIACRYPT'14. Furthermore, we propose concrete practical codes and a method to find good codes.

1 Introduction

The Learning Parity with Noise (LPN) problem can be seen as a noisy system of linear equations in the binary domain. More specifically, we have a secret s and an adversary that has access to an LPN oracle which provides him tuples of uniformly distributed binary vectors v_i and the inner product between s and v_i to which some noise was added. The noise is represented by a Bernoulli variable with a probability τ to be 1. The goal of the adversary is to recover the secret s. The LPN problem is a particular case of the well-known Learning with Errors (LWE) [34] problem where instead of working in $\mathbb{Z}_2$ we extend the work to a ring $\mathbb{Z}_q$.

The LPN problem is attractive as it is believed to be resistant to quantum computers. Thus, it can be a good candidate for replacing the number-theoretic problems such as factorization and discrete logarithm (which can be easily broken by a quantum algorithm). Also, given its structure it can be implemented in lightweight devices. The LPN problem is used in the design of the *HB*-family of authentication protocols [10,20,24,25,27,31] and several cryptosystems base their security on its hardness [1,15,16,17,21,26].

Previous Work. LPN is believed to be hard. So far, there is no reduction from hard lattice problems to certify the hardness (like in the case of LWE). Thus, the best way to assess its hardness is by trying to design and improve algorithms that solve it. Over the years, the LPN problem was analyzed and there exist several solving algorithms. The first algorithm to target LPN is the BKW algorithm [6].

This algorithm can be described as a Gaussian elimination on blocks of bits (instead on single bits) where the secret is recovered bit by bit. Improvements of it were presented in [19,29]. One idea that improves the algorithm is the use of the fast Walsh-Hadamard transform as we can recover several bits of the secret at once. In their work [29], Levieil and Fouque provide an analysis with the level of security achieved by different LPN instances and propose secure parameters. Using BKW as a black-box, Lyubashevsky [30] presents an LPN solving algorithm useful for the case when the number of queries is restricted to an adversary. The best algorithm to solve LPN was presented at ASIACRYPT'14 [23] and it introduces the use of the covering codes to improve the performance. An analysis on the existing LPN solving algorithms is presented in [7,8].

For the case when the secret is sparse, i.e. its Hamming weight is small, the classical Gaussian elimination proves to give better results [7,8,9,11].

The LPN algorithms consist of two parts: one in which the size of the secret is reduced and one in which part of the secret is recovered. Once a part of the secret is recovered, the queries are updated and the algorithm restarts to recover the rest of the secret. When trying to recover a secret s of k bits, it is assumed that k can be written as $a \cdot b$, for $a, b \in \mathbb{N}$ (i.e. secret s can be seen as a blocks of b bits). Usually all the reduction steps reduce the size by b bits and the solving algorithm recovers b bits. While the use of the same parameter, i.e. b, for all the operations may be convenient for the implementation, we search for an algorithm that may use different values for each reduction step. We discover that small variations from the fixed b can bring important improvements in the time complexity of the whole algorithm.

Our Contribution. In this work we first *analyze the existing* LPN *algorithms and study the operations that are used in order to reduce the size of the secret. We adjust the expressions of the complexities of each step* (as in some works they were underestimated in the literature). For instance, the results from [23] are displayed on the second column of Table 1. As discussed in [7,8] and in the ASIACRYPT presentation, the authors of [23] used a too optimistic approximation for the bias introduced by their new reduction method, the covering codes. Some complexity terms are further missing (as discussed in Section 2.2) or are not in bit operations. Adjusting the computation with a correct bias for a concrete code and the data complexity to make their algorithm work, results in a worse time complexity, illustrated on the third column of Table 1 (details for this computation are provided as an additional material for this paper).

Second, we *improve the theory behind the covering code reduction, show the link with perfect and quasi-perfect codes and propose an algorithm to find good codes* (in [23], only a hypothetical code was assumed to be close to a

(k,τ)	[23]	[23] corrected	our results
$(512, 0.125)$	$2^{79.9}$ (article) $2^{79.7}$ (presentation) [1]	$2^{89.04}$ $2^{82.27}$	$2^{79.37}$
$(532, 0.125)$	$2^{81.82}$	$2^{90.43}$	$2^{81.64}$
$(592, 0.125)$	$2^{88.07}$	$2^{97.87}$	$2^{88.25}$

Table 1: Time complexity to solve LPN (in bit operations)

perfect code; the second column of Table 1 is based on this favorable code but other columns use real codes that we built).

Third, we *optimize the order and the parameters used by the operations that reduce the size of the secret such that we minimize the time complexity required*. We *design an algorithm that combines the reduction steps and finds the optimal strategy to solve* LPN. We *automize the process of finding* LPN *solving algorithms, i.e. given a random* LPN *instance, our algorithm provides the description of the steps that optimize the time complexity*. In our formalization we call such algorithms "optimal chains". We perform a security analysis of LPN based on the results obtained by our algorithm and compare our results with the existing ones. We discover that we improve the complexity compared with the results from [7,8,29] and [23]. Applying our algorithm that improves the bias from the covering code and optimizes the reduction steps, gives a much better performance, illustrated on the fourth column of Table 1.

Preliminaries & Notations. Given a domain $\mathcal{D}$, we denote by $x \xleftarrow{U} \mathcal{D}$ the fact that x is drawn uniformly at random from $\mathcal{D}$. By Ber_τ we denote the Bernoulli distribution with parameter τ. By Ber_τ^k we denote the binomial distribution with parameters k and τ. Let $\langle \cdot, \cdot \rangle$ denote the inner product, $\mathbb{Z}_2 = \{0,1\}$ and $\oplus$ denote the bitwise XOR. The Hamming weight of a vector v is denoted by $HW(v)$.

Organization. In Section 2 we formally define the LPN problem and describe the main tools used to solve it. We carefully analyze the complexity of each step and show in footnote where it differs from the existing literature. Section 3 introduces the bias computation for perfect and quasi-perfect codes. We provide an algorithm to find good codes. The algorithm that searches the optimal strategy to solve LPN is presented in Sections 4 and 5. We illustrate and compare our results in Section 6 and conclude in Section 7. We put in additional material details of our results: the complete list of the chains we obtain (for Table 3 and Table 4), an example of complete solving algorithm, the random codes that we use for the covering code reduction, and an analysis of the results from [23] to obtain Table 1.

[1] `http://des.cse.nsysu.edu.tw/asiacrypt2014/doc/1-1_SolvingLPNUsingCoveringCodes.pdf`

2 LPN

2.1 LPN Definition

The LPN problem can be seen as a noisy system of equations in $\mathbb{Z}_2$ where one is asked to recover the unknown variables. Below, we present the formal definition.

Definition 1 (LPN oracle). *Let $s \xleftarrow{U} \mathbb{Z}_2^k$, let $\tau \in]0, \frac{1}{2}[$ be a constant noise parameter and let Ber_τ be the Bernoulli distribution with parameter τ. Denote by $D_{s,\tau}$ the distribution defined as*

$$\{(v,c) \mid v \xleftarrow{U} \mathbb{Z}_2^k, c = \langle v,s\rangle \oplus d, d \leftarrow \mathsf{Ber}_\tau\} \in \mathbb{Z}_2^{k+1}.$$

An LPN oracle $O_{s,\tau}^{\mathsf{LPN}}$ is an oracle which outputs independent random samples according to $D_{s,\tau}$.

Definition 2 (Search LPN problem). *Given access to an LPN oracle $O_{s,\tau}^{\mathsf{LPN}}$, find the vector s. We denote by $\mathsf{LPN}_{k,\tau}$ the LPN instance where the secret has size k and the noise parameter is τ. Let $k' \leq k$. We say that an algorithm $\mathcal{M}$ (n,t,m,θ,k')-solves the search $\mathsf{LPN}_{k,\tau}$ problem if*

$$\Pr[\mathcal{M}^{O_{s,\tau}^{\mathsf{LPN}}}(1^k) = (s_1 \dots s_{k'}) \mid s \xleftarrow{U} \mathbb{Z}_2^k] \geq \theta,$$

and $\mathcal{M}$ runs in time t, uses memory m and asks at most n queries from the LPN oracle.

Remark that we consider here the problem of recovering only a part of the secret. Throughout the literature this is how the LPN problem is formulated. The reason for doing so is that the recovery of the first k' bits dominates the overall complexity. Once we recover part of the secret, the new problem of recovering a shorter secret of $k-k'$ bits is easier.

The LPN problem has a decisional form where one has to distinguish between random vectors of size $k+1$ and the samples from the LPN oracle. In this paper we are interested only in finding algorithms for the search version.

We define $\delta = 1-2\tau$. We call δ the bias of the error bit d. We have $\delta = E((-1)^d)$, with $E(\cdot)$ the expected value. We denote the bias of the secret bits by δ_s. As s is a uniformly distributed random vector, at the beginning we have $\delta_s = 0$.

2.2 Reduction and Solving Techniques

Depending on how many queries are given from the LPN oracle, the LPN solving algorithms are split in 3 categories. With a *linear number of queries*,

the best algorithms are exponential, i.e. with $n = \Theta(k)$ the secret is recovered in $2^{\Theta(k)}$ time [32,36]. Given a *polynomial number of queries* $n = k^{1+\eta}$, with $\eta > 0$, one can solve LPN with a sub-exponential time complexity of $2^{O(\frac{k}{\log\log k})}$ [30]. When $\tau = \frac{1}{\sqrt{k}}$ we can improve the result of [30] and have a complexity of $e^{\frac{1}{2}\sqrt{k}(\ln k)^2 + O(\sqrt{k}\ln k)}$ [9]. The complexity improves but remains in the sub-exponential range with a *sub-exponential number of queries*. For this category, we have the BKW [6], LF1, LF2 [29], FMICM [19] and the covering code algorithm [23]. All these algorithms solve LPN with a time complexity of $2^{O(\frac{k}{\log k})}$ and require $2^{O(\frac{k}{\log k})}$ queries. In the special case when the noise is sparse, a simple Gaussian elimination can be used for the recovery of the secret [7,11]. LF2, covering code or the Gaussian elimination prove to be the best one, depending on the noise level [7].

All these algorithms have a common structure: given an $\mathsf{LPN}_{k,\tau}$ instance with a secret s, they reduce the original LPN problem to a new LPN problem where the secret s' is of size $k' \leq k$ by applying several *reduction techniques*. Then, they recover s' using a *solving method*. The queries are updated and the process is repeated until the whole secret s is recovered. We present here the list of reduction and solving techniques used in the existing LPN solving algorithms. In the next section, we combine the reduction techniques such that we find the optimal reduction phases for solving different LPN instances.

We assume for all the reduction steps that we start with n queries, that the size of the secret is k, the bias of the secret bits is δ_s and the bias of the noise bits is δ. After applying a reduction step, we will end up with n' queries, size k' and biases δ' and δ'_s. Note that δ_s averages over all secrets although the algorithm runs with one target secret. In our analysis we make the following heuristic assumption:

> **Stochastic equivalence approximation**: The average probability of success of the solving algorithm over the distribution of the key is not affected by considering the average bias.

We will see that complexities only depend on k, n, and the parameters of the steps. Actually, only the probability of success is concerned with this heuristic.

We have the following reduction steps:

- *sparse-secret* changes the secret distribution. In the formal definition of LPN, we take the secret s to be a random row vector of size k. When other reduction steps or the solving phase depends on the distribution of s, one can transform an LPN instance with a random s to a new one where s has the same distribution as the initial noise, i.e. $s \leftarrow \mathsf{Ber}_\tau^k$. The reduction performs the following steps: from the n queries select k of them:

$(v_{i_1}, c_{i_1}), \ldots, (v_{i_k}, c_{i_k})$ where the row vectors v_{i_j}, with $1 \leq j \leq k$, are linearly independent. Construct the matrix M as $M = [v_{i_1}^T \cdots v_{i_k}^T]$ and rewrite the k queries as $sM + d' = c'$, where $d' = (d_{i_1}, \ldots, d_{i_k})$. With the rest of $n-k$ queries we do the following:

$$c'_j = \langle v_j (M^T)^{-1}, c' \rangle \oplus c_j = \langle v_j (M^T)^{-1}, d' \rangle \oplus d_j = \langle v'_j, d' \rangle \oplus d_j$$

We have $n-k$ new queries (v'_j, c'_j) where the secret is now d'. In [23], the authors use an algorithm which is unappropriately called "the four Russians algorithm" [2]. This way, the complexity should be of $O\left(\min_{\chi \in \mathbb{N}} \left(kn' \lceil \frac{k}{\chi} \rceil + k^3 + k\chi 2^\chi\right)\right)$.[1] Instead, we use the Bernstein algorithm [4]. Thus, we have:

sparse-secret : $k' = k$; $n' = n-k$; $\delta' = \delta$; $\delta'_s = \delta$ Complexity: $O\left(\frac{n'k^2}{\log_2 k - \log_2 \log_2 k} + k^2\right)$

- *partition-reduce*(b) is used by the BKW algorithm. Recall that the queries received from the oracle are of the form $(v, \langle v, s \rangle \oplus d)$. In this reduction, the v vectors are sorted in equivalence classes according to their values on a block of b bits. These b positions are chosen randomly. Two queries in the same equivalence class have the same values on the b positions. In each equivalence class, we choose a representative vector and xor it with the rest of the queries. This will give vectors with only 0 on this window of b bits. Afterwards the representative vector is dropped. This operation reduces the secret to $k-b$ effective bits (since b bits of the secret are always xored with zero). The new bias is δ^2 as the new queries are xor of two initial ones and the number of queries is reduced by 2^b (as there are 2^b equivalence classes).

partition-reduce(b) : $k' = k-b$; $n' = n-2^b$; $\delta' = \delta^2$; $\delta'_s = \delta_s$ Complexity: $O(kn)$

  The next reduction, *xor-reduce*(b), is always better than *partition-reduce*(b). Nevertheless, we keep this operation in our analysis for backward compatibility with existing algorithms (e.g. to fill our Table 1 with the algorithms from [23]).
- *xor-reduce*(b) was first used by the LF2 algorithm. Similar to *partition-reduce*, the queries are grouped in equivalence classes according to the values on b random positions. In each equivalence class, we perform the xoring of every pair of queries. The size of the secret is reduced by b bits and the new bias is δ^2. The expected new number of queries is $E(\sum_{i<j} 1_{v_i \text{ matches } v_j \text{ on the } b\text{-bit block}}) = \frac{n(n-1)}{2^{b+1}}$ [2]. When $n \approx 1 + 2^{b+1}$, the number of queries are maintained. For $n > 1 + 2^{b+1}$, the number of queries will increase.

[1] but the $k^3 + k\chi 2^\chi$ terms is missing in [23].

[2] In [8], the number of queries was approximated to $\frac{\frac{n}{2^b}\left(\frac{n}{2^b}-1\right)}{2}$ which is less favourable.

$\boxed{\begin{array}{l} \textit{xor-reduce}(b) : k' = k-b;\ n' = \frac{n(n-1)}{2^{b+1}};\ \delta' = \delta^2;\ \delta'_s = \delta_s \\ \text{Complexity: } O(k \cdot \max(n, n')) \end{array}}$

– *drop-reduce*(b) is a reduction used only by the BKW algorithm. It consists in dropping all the queries that are not 0 on a window of b bits. Again, these b positions are chosen randomly. In average, we expect that half of the queries are 0 on a given position. For b bits, we expect to have $\frac{n}{2^b}$ queries that are 0 on this window. The bias is unaffected and the secret is reduced by b bits.

$\boxed{\begin{array}{l} \textit{drop-reduce}(b) : k' = k-b;\ n' = \frac{n}{2^b};\ \delta' = \delta;\ \delta'_s = \delta_s \\ \text{Complexity: } O(n(1+\frac{1}{2}+\ldots+\frac{1}{2^{b-1}})) \end{array}}$

The complexity of $n(1+\frac{1}{2}+\ldots+\frac{1}{2^{b-1}}) = O(n)$ comes from the fact that we don't need to check all the b bits: once we find a 1 we don't need to continue and just drop the corresponding query.

– *code-reduce*(k, k', params) is a method used by the covering code algorithm presented in ASIACRYPT'14. In order to reduce the size of the secret, one uses a linear code $[k, k']$ (which is defined by params) and approximates the v_i vectors to the nearest codeword g_i. We assume that decoding is done in linear time for the code considered. (For the considered codes, decoding is indeed based on table look-ups.) The noisy inner product becomes:

$$\begin{aligned} \langle v_i, s\rangle \oplus d_i &= \langle g'_i G, s\rangle \oplus \langle v_i - g_i, s\rangle \oplus d_i \\ &= \langle g'_i, sG^T\rangle \oplus \langle v_i - g_i, s\rangle \oplus d_i \\ &= \langle g_i, s'\rangle \oplus d'_i, \end{aligned}$$

where G is the generator matrix of the code, $g_i = g'_i G$, $s' = sG^T \in \{0,1\}^{k'}$ and $d'_i = \langle v_i - g_i, s\rangle \oplus d_i$. We denote $\mathsf{bc} = E((-1)^{\langle v_i - g_i, s\rangle})$ the bias of $\langle v_i - g_i, s\rangle$. We will see in Section 3 how to construct a $[k, k']$ linear code making bc as large as possible.

Here, bc averages the bias over the secret although s is fixed by *sparse-secret*. It gives the correct average bias δ over the distribution of the key. The Stochastic equivalence approximation justifies this analysis.

By this transform, no query is lost.

$\boxed{\begin{array}{l} \textit{code-reduce}(k, k', \mathsf{params}) : k';\ n' = n;\ \delta' = \delta \cdot \mathsf{bc};\ \delta'_s \text{ depends on } \delta_s \text{ and } G \\ \text{Complexity: } O(kn) \end{array}}$

The way δ'_s is computed is a bit more complicated than for the other types of reductions. However, δ_s only plays a role in the *code-reduce* reduction, and we will not consider algorithms that use more than one *code-reduce* reduction.

It is easy to notice that with each reduction operation the number of queries decreases or the bias is getting smaller. In general, for solving LPN, one tries to

lose as few queries as possible while maintaining a large bias. We will study in the next section what is a good combination of using these reductions.

After applying the reduction steps, we assume we are left with an $\mathsf{LPN}_{k',\delta'}$ instance where we have n' queries. The two main solving techniques encountered in the LPN algorithms are:

- *majority* is used by the BKW algorithm. For this solving algorithm we have queries of the form $(1, s_i \oplus d_j)$, where s_i is the i^{th} bit of the secret and $d_j \leftarrow \mathsf{Ber}_{(1-\delta')/2}$. For this, from the n' queries, we chose only those that have Hamming weight 1. Given that the probability for a noise bit to be set on 1 is smaller than $\frac{1}{2}$, most of the queries will be $(1, s_i)$. The majority of the $\frac{n'}{2^{k'}}$ queries decide what is the value of s_i. The number of queries needed to make the correct guess is given by the Chernoff bounds. According to [7], the *majority* needs $n' = 2\ln\left(\frac{k'}{\theta}\right) \cdot \delta'^{-2} 2^{k'}$ queries in order to bound the failure probability of guessing wrong any of k' bits by θ, with $0 < \theta < 1$. The complexity of *majority* is $O(n')$. As the next solving method is always better than *majority*, we do not detail this further.
- Walsh Hadamard Transform (WHT) is used by most of the LPN algorithms. This method recovers a block of the secret by computing the fast Walsh Hadamard transform on the function $f(x) = \sum_i 1_{v_i = x} (-1)^{\langle v_i, s \rangle \oplus d_i}$. The Walsh-Hadamard transform is

$$\hat{f}(\nu) = \sum_x (-1)^{\langle \nu, x \rangle} f(x) = \sum_i (-1)^{\langle v_i, s+\nu \rangle \oplus d_i}$$

  For $\nu = s$, we have $\hat{f}(s) = \sum_i (-1)^{d_i}$. We know that most of the noise bits are set to 0. So, $\hat{f}(s)$ is large and we suppose it is the largest value in the table of $\hat{f}$. Using again the Chernoff bounds, we need to have $n' = 8\ln(\frac{2^{k'}}{\theta})\delta'^{-2}$ [7] queries in order to bound the probability of guessing wrongly the k'-bit secret by θ. We can improve further by applying directly the Central Limit Theorem and obtain a heuristic bound $\varphi(-\sqrt{\frac{n'}{2\delta'^{-2}-1}}) \leq 1 - (1-\theta)^{\frac{1}{2^{k'}-1}}$, where $\varphi(x) = \frac{1}{2} + \frac{1}{2}\mathsf{erf}(\frac{x}{\sqrt{2}})$ and erf is the Gauss error function. We obtain that

$$\sqrt{n'} \geq -\sqrt{2\delta'^{-2} - 1} \cdot \varphi^{-1}\left(1 - (1-\theta)^{\frac{1}{2^{k'}-1}}\right). \tag{1}$$

We can derive the approximation of Selçuk [35] that $n' \geq 4\ln(\frac{2^{k'}}{\theta})\delta'^{-2}$. We give the details of our results in Appendix A. Complexity of the $\mathsf{WHT}(k')$ is $O(k'2^{k'} + k'n')$ as we use the fast Walsh Hadamard Transform[3] [4].

$\mathsf{WHT}(k')$;
Requires $\sqrt{n'} \geq -\sqrt{2\delta'^{-2} - 1} \cdot \varphi^{-1}\left(1 - (1-\theta)^{\frac{1}{2^{k'}-1}}\right)$
Complexity: $O(k'2^{k'}\frac{\log_2 n'+1}{2} + k'n')$

Given the reduction and solving techniques, an $\mathsf{LPN}_{k,\tau}$ solving algorithm runs like this: we start with a k-bit secret and with n queries from the LPN oracle. We reduce the size of the secret by applying several reduction steps and we end up with n' queries where the secret has size k'. We use one solving method, e.g. the WHT, and recover the k'-bit secret with a probability of failure bounded by θ. We chose $\theta = \frac{1}{3}$. We have recovered a part of the secret. To fully recover the whole secret, we update the queries and start another chain to recover more bits, and so on until the remaining $k - k'$ bits are found. For the second part of the secret we will require for the failure probability to be θ^2 and for the i^{th} part it will be θ^i. Thus, if we recover the whole secret in i iterations, the total failure probability will be bounded by $\theta + \theta^2 + \cdots + \theta^i$. Given that we take $\theta = \frac{1}{3}$, we recover the whole secret with a success probability larger than 50%. Experience shows that the time complexity for the first iteration dominates the total complexity.

As we can see in the formulas of each possible step, the computations of k', n', and of the complexity do not depend on the secret weight. Furthermore, the computation of biases is always linear. So, the correct average bias (over the distribution of the key made by the *sparse-secret* transform) is computed. Only the computation of the success probability is non-linear but the Stochastic equivalence approximation is invoked to solve this issue. As is only matters in WHT, we will see in Appendix A that the approximation is justified.

3 Bias of the Code Reduction

In this section we present how to compute the bias introduced by a *code-reduce*. Recall that the reduction *code-reduce*(k,k') introduces a new noise:

$$\langle v_i, s\rangle \oplus d_i = \langle g'_i, s'\rangle \oplus \langle v_i - g_i, s\rangle \oplus d_i,$$

[3] The second term $k'n'$ illustrates the cost of constructing the function f. In cases where $n' > 2^{k'}$ this is the dominant term and it should not be ignored. This was missing in [23,8]. For the instance $\mathsf{LPN}_{592,0.125}$ from [23] this makes a big difference as $k' = 64$ and $n' = 2^{69}$; the complexity of WHT with the second term is 2^{75} vs 2^{70} from [23]. Given that is must be repeated 2^{13} (as 35 bits of the secret are guessed), the cost of WHT is 2^{88}.

[4] Normally, the values $\hat{f}(v)$ have an order of magnitude of $\sqrt{n'}$ so we have $\frac{1}{2}\log_2 n'$ bits.

where $g_i = g_i' G$ is the nearest codeword of v_i and $s' = sG^T$. Then the noise bc can be computed by the following formula:

$$\begin{aligned}\mathsf{bc} &= E((-1)^{\langle v_i - g_i, s\rangle}) = \sum_{e\in\{0,1\}^k} \Pr[v_i - g_i = e] E((-1)^{\langle e,s\rangle}) \\ &= \sum_{w=0}^{k} \sum_{\substack{e\in\{0,1\}^k,\\ \mathsf{HW}(e)=w}} \Pr[v_i - g_i = e]\delta_s^w = E\left(\delta_s^{\mathsf{HW}(v_i-g_i)}\right)\end{aligned}$$

for a δ_s-sparse secret. (We recall that the *sparse-secret* reduction step randomizes the secret and that we make the Stochastic equivalence approximation.)

From this formula, we can see that the decoding algorithm $v_i \to g_i$ making $\mathsf{HW}(v_i - g_i)$ minimal makes bc maximal. In this case, we obtain

$$\mathsf{bc} = E\left(\delta_s^{d(v_i,C)}\right), \tag{2}$$

where C is the code and $d(v_i, C)$ denotes the Hamming distance of v_i from C.

For a code C, the *covering radius* is $\rho = \max_v d(v,C)$. The *packing radius* is the largest radius R such that the balls of this radius centered on all codewords are non-overlapping. So, the packing radius is $R = \left\lfloor \frac{D-1}{2} \right\rfloor$ where D is the minimal distance. We further have $\rho \geq \left\lfloor \frac{D-1}{2} \right\rfloor$. A *perfect code* is characterized by $\rho = \left\lfloor \frac{D-1}{2} \right\rfloor$. A *quasi-perfect code* is characterized by $\rho = \left\lfloor \frac{D-1}{2} \right\rfloor + 1$.

Theorem 1. *We consider a $[k, k', D]$ linear code C, where k is the length, k' is the dimension, and D is the minimal distance. For any integer r and any positive bias δ_s, we have*

$$\mathsf{bc} \leq 2^{k'-k} \sum_{w=0}^{r} \binom{k}{w} (\delta_s^w - \delta_s^{r+1}) + \delta_s^{r+1}$$

where bc is a function of δ_s defined by (2). Equality for any $\delta_s \in]0,1[$ implies that C is perfect or quasi-perfect. In that case, we do have equality when taking the packing radius $r = R = \left\lfloor \frac{D-1}{2} \right\rfloor$.

By taking r as the largest integer such that $\sum_{w=0}^{r} \binom{k}{w} \leq 2^{k-k'}$ (which is the packing radius $R = \left\lfloor \frac{D-1}{2} \right\rfloor$ for perfect and quasi-perfect codes), we can see that if a perfect $[k, k']$ code exists, it makes bc maximal. Otherwise, if a quasi-perfect $[k, k']$ code exists, it makes bc maximal.

Proof. Let decode be an optimal deterministic decoding algorithm. The formula gives us that

$$\mathsf{bc} = 2^{-k} \sum_{g\in C} \sum_{v\in \mathsf{decode}^{-1}(g)} \delta_s^{\mathsf{HW}(v-g)}$$

We define $\mathsf{decode}_w^{-1}(g) = \{v \in \mathsf{decode}^{-1}(g); \mathsf{HW}(v-g) = w\}$ and $\mathsf{decode}_{>r}^{-1}(g)$ the union of all $\mathsf{decode}_w^{-1}(g)$ for $w > r$. For all r, we have

$$\begin{aligned}
&\sum_{v \in \mathsf{decode}^{-1}(g)} \delta_s^{\mathsf{HW}(v-g)} \\
&= \sum_{w=0}^{r} \binom{k}{w} \delta_s^w + \sum_{w=0}^{r} \left(\#\mathsf{decode}_w^{-1}(g) - \binom{k}{w} \right) \delta_s^w + \sum_{w>r} \delta_s^w \#\mathsf{decode}_w^{-1}(g) \\
&\leq \sum_{w=0}^{r} \binom{k}{w} \delta_s^w + \sum_{w=0}^{r} \left(\#\mathsf{decode}_w^{-1}(g) - \binom{k}{w} \right) \delta_s^w + \delta_s^{r+1} \#\mathsf{decode}_{>r}^{-1}(g) \\
&\leq \sum_{w=0}^{r} \binom{k}{w} \delta_s^w + \delta_s^{r+1} \left(\#\mathsf{decode}^{-1}(g) - \sum_{w=0}^{r} \binom{k}{w} \right)
\end{aligned}$$

where we used $\delta_s^w \leq \delta_s^{r+1}$ for $w > r$, $\#\mathsf{decode}_w^{-1}(g) \leq \binom{k}{w}$ and $\delta_s^w \geq \delta_s^{r+1}$ for $w \leq r$. We further have equality if and only if the ball centered on g of radius r is included in $\mathsf{decode}^{-1}(g)$ and the ball of radius $r+1$ contains $\mathsf{decode}^{-1}(g)$. By summing over all $g \in C$, we obtain the result.

So, the equality case implies that the packing radius is at least r and the covering radius is at most $r+1$. Hence, the code is perfect or quasi-perfect. Conversely, if the code is perfect or quasi-perfect and r is the packing radius, we do have equality. □

So, for quasi-perfect codes, we can compute

$$\mathsf{bc} = 2^{k'-k} \sum_{w=0}^{R} \binom{k}{w} (\delta_s^w - \delta_s^{R+1}) + \delta_s^{R+1} \tag{3}$$

with $R = \left\lfloor \frac{D-1}{2} \right\rfloor$. For perfect codes, the formula simplifies to

$$\mathsf{bc} = 2^{k'-k} \sum_{w=0}^{R} \binom{k}{w} \delta_s^w \tag{4}$$

3.1 Bias of a Repetition Code

Given a $[k,1]$ repetition code, the optimal decoding algorithm is the majority decoding. We have $D = k$, $k' = 1$, $R = \left\lfloor \frac{k-1}{2} \right\rfloor$. For k odd, the code is perfect so $\rho = R$. For k even, the code is quasi-perfect so $\rho = R+1$. Using (3) we obtain

$$\mathsf{bc} = \begin{cases} \sum_{w=0}^{\frac{k-1}{2}} \frac{1}{2^{k-1}} \binom{k}{w} \delta_s^w & \text{if } k \text{ is odd} \\ \sum_{w=0}^{\frac{k}{2}-1} \frac{1}{2^{k-1}} \binom{k}{w} \delta_s^w + \frac{1}{2^k} \binom{k}{k/2} \delta_s^{\frac{k}{2}} & \text{if } k \text{ is even} \end{cases}$$

We give below the biases obtained for some $[k,1]$ repetition codes.

$[k,1]$	bias
$[2,1]$	$\frac{1}{2}\delta_s + \frac{1}{2}$
$[3,1]$	$\frac{3}{4}\delta_s + \frac{1}{4}$
$[4,1]$	$\frac{3}{8}\delta_s^2 + \frac{1}{2}\delta_s + \frac{1}{8}$
$[5,1]$	$\frac{5}{8}\delta_s^2 + \frac{5}{16}\delta_s + \frac{1}{16}$
$[6,1]$	$\frac{5}{16}\delta_s^3 + \frac{15}{32}\delta_s^2 + \frac{3}{16}\delta_s + \frac{1}{32}$
$[7,1]$	$\frac{35}{64}\delta_s^3 + \frac{21}{64}\delta_s^2 + \frac{7}{64}\delta_s + \frac{1}{64}$
$[8,1]$	$\frac{35}{128}\delta_s^4 + \frac{7}{16}\delta_s^3 + \frac{7}{32}\delta_s^2 + \frac{1}{16}\delta_s + \frac{1}{128}$
$[9,1]$	$\frac{63}{128}\delta_s^4 + \frac{21}{64}\delta_s^3 + \frac{9}{64}\delta_s^2 + \frac{9}{256}\delta_s + \frac{1}{256}$
$[10,1]$	$\frac{63}{256}\delta_s^5 + \frac{105}{256}\delta_s^4 + \frac{15}{64}\delta_s^3 + \frac{45}{512}\delta_s^2 + \frac{5}{256}\delta_s + \frac{1}{512}$

3.2 Bias of a Perfect Code

In [23], the authors assume a perfect code. In this case, $\sum_{w=0}^{R}\binom{k}{w} = 2^{k-k'}$ and we can use (4) to compute bc. There are not so many binary linear codes which are perfect. Except the repetition codes with odd length, the only ones are the trivial codes $[k,k,1]$ with $R = \rho = 0$ and $\mathsf{bc} = 1$, the Hamming codes $[2^\ell - 1, 2^\ell - \ell - 1, 3]$ for $\ell \geq 2$ with $R = \rho = 1$, and the Golay code $[23,12,7]$ with $R = \rho = 3$.

For the Hamming codes, we have

$$\mathsf{bc} = 2^{-\ell}\sum_{w=0}^{1}\binom{2^\ell - 1}{w}\delta_s^w = \frac{1 + (2^\ell - 1)\delta_s}{2^\ell}$$

For the Golay code, we obtain

$$\mathsf{bc} = 2^{-11}\sum_{w=0}^{3}\binom{23}{w}\delta_s^w = \frac{1 + 23\delta_s + 253\delta_s^2 + 1771\delta_s^3}{2^{11}}$$

Formulae (2), (4), (3) for bc are new. In [23,8], bc was approximated to

$$\mathsf{bc}(w) = 1 - 2\frac{1}{S(k,\rho)}\sum_{\substack{i\leq\rho,\\ i \text{ odd}}}\binom{w}{i}S(k-w,\rho-i),$$

where w is the Hamming weight of the k-bit secret and $S(k',\rho)$ is the number of k'-bit strings with weight at most ρ. Intuitively the formula counts the number of $v_i - g_i$ that produce an odd number of xor with the 1's of the secret. (See [7,8,23].) So, [23] assumes a fixed value for the weight w of the secret and considers the probability that w is not correct. If w is lower, the actual bias is larger but if w is larger, the computed bias is overestimated and the algorithm fails. The advantage of our formula is that we average over w and do not have a probability of failure.

For instance, with a $[3,1]$ repetition code, the correct bias is $\mathsf{bc} = \frac{3}{4}\delta_s + \frac{1}{4}$ following our formula. With a fixed w, it is of $\mathsf{bc}(w) = 1 - \frac{w}{2}$ following [7,23]. The probability of w to be correct is $\binom{k}{w}\tau^w(1-\tau)^{k-w}$. We take the example of $\tau = \frac{1}{3}$ so that $\delta_s = \frac{1}{3}$ and $\delta = \frac{1}{2}$.

w	$\mathsf{bc}(w)$	$\Pr[w]$	$\Pr[w], \tau = \frac{1}{3}$
0	1	$(1-\tau)^3$	0.2963
1	$\frac{1}{2}$	$3\tau(1-\tau)^2$	0.4444
2	0	$3\tau^2(1-\tau)$	0.2222
3	$-\frac{1}{2}$	τ^3	0.0370

So, by taking $w = 1$, we have $\delta = \mathsf{bc}(w)$ but the probability of failure is about $\frac{1}{4}$. Our approach uses the same bias with no failure.

To solve $\mathsf{LPN}_{512,0.125}$, [23] uses a $[124,64]$ perfect code with $\rho = 14$ (note that such a code does not exist) and $w = 16$. Our formula gives $\mathsf{bc} \approx 2^{-5.74}$ but $\mathsf{bc}(w) \approx 2^{-7.05}$ and $\Pr[\leq w] \approx 0.6189$. So, our analysis predicts better performances (a larger bias on average instead of a smaller one in 62% of cases and a failure otherwise). In the presentation of [23] at the conference, the authors rather used a $[180,60]$ perfect code ($\rho = 33$) with w such that $\mathsf{bc}(w) \geq \varepsilon_{\mathsf{set}} = 2^{-14.78}$. We have $\mathsf{bc}(w) \approx 2^{-13.56}$ and $\Pr[\text{success}] \approx 0.17$.

3.3 Using Quasi-Perfect Codes

If C' is a $[k-1,k',D]$ perfect code with $k' > 1$ and if there exists some codewords of odd length, we can extend C', i.e., add a parity bit and obtain a $[k,k']$ code C. Clearly, the packing radius of C is at least $\left\lfloor \frac{D-1}{2} \right\rfloor$ and the covering radius is at most $\left\lfloor \frac{D-1}{2} \right\rfloor + 1$. For $k' > 1$, there is up to one possible length for making a perfect code of dimension k'. So, C is a quasi-perfect, its packing radius is $\left\lfloor \frac{D-1}{2} \right\rfloor$ and its covering radius is $\left\lfloor \frac{D-1}{2} \right\rfloor + 1$.

If C' is a $[k+1,k',D]$ perfect code with $k' > 1$, we can puncture it, i.e., remove one coordinate by removing one column from the generating matrix. If we chose to remove a column which does not modify the rank k', we obtain a $[k,k']$ code C. Clearly, the packing radius of C is at least $\left\lfloor \frac{D-1}{2} \right\rfloor - 1$ and the covering radius is at most $\left\lfloor \frac{D-1}{2} \right\rfloor$. For $k' > 1$, there is up to one possible length for making a perfect code of dimension k'. So, C is a quasi-perfect, its packing radius is $\left\lfloor \frac{D-1}{2} \right\rfloor - 1$ and its covering radius is $\left\lfloor \frac{D-1}{2} \right\rfloor$.

Hence, we can use extended Hamming codes $[2^\ell, 2^\ell - \ell - 1]$ with packing radius 1 for $\ell \geq 3$, punctured Hamming codes $[2^\ell - 2, 2^\ell - \ell - 1]$ with packing radius 0 for $\ell \geq 3$, the extended Golay code $[24,12]$ with packing radius 3, and the punctured Golay code $[22,12]$ with packing radius 2,

There actually exist many constructions for quasi-perfect linear binary codes. We list a few in Table 2. We took codes listed in [14, Table 1], [33, p. 122], [22,

p. 47], [18, Table 1], [13, p. 313], and [3, Table I]. In Table 2, k, k', D, and R denote the length, the dimension, the minimal distance, and the packing radius, respectively.

Table 2: Perfect and Quasi-Perfect Binary Linear Codes

name	type	$[k,k',D]$	R	comment	ref.
	P	$[k,k,1]$, $k \geq 1$	0	$[*,\ldots,*]$	
r	P	$[k,1,k]$, k odd	$\frac{k-1}{2}$	repetition code	
H	P	$[2^\ell-1,2^\ell-\ell-1,3]$, $\ell \geq 3$,	1	Hamming code	
G	P	$[23,12,7]$	3	Golay code	
	QP	$[k,k-1,1]$	0	$[*,\ldots,*,0]$	
r	QP	$[k,1,k]$, k even	$\frac{k}{2}-1$	repetition code	
eG	QP	$[24,12,8]$	3	extended Golay code	
pG	QP	$[22,12,6]$	2	punctured Golay code	
eH	QP	$[2^\ell,2^\ell-\ell-1,4]$, $\ell \geq 2$	1	extended Hamming code	
	QP	$[2^\ell-1,2^\ell-\ell,1]$, $\ell \geq 2$,	0	Hamming with an extra word	
pH	QP	$[2^\ell-2,2^\ell-\ell-1,2]$, $\ell \geq 2$	0	punctured Hamming	
HxH	QP	$[2*(2^\ell-1),2*(2^\ell-\ell-1)]$, $\ell \geq 2$	1	Hamming $\times$ Hamming	[14]
upack	QP	$[2^\ell-2,2^\ell-\ell-2,3]$, $\ell \geq 3$	1	uniformly packed	[14]
2BCH	QP	$[2^\ell-1,(2^\ell-1)-(2*\ell)]$, $\ell \geq 3$	2	2-e.c. BCH	[14]
Z	QP	$[2^\ell+1,(2^\ell+1)-(2*\ell)]$, $\ell > 3$ even	2	Zetterberg	[14]
rGop	QP	$[2^\ell-2,(2^\ell-2)-(2*\ell)]$, $\ell > 3$ even	2	red. Goppa	[14]
iGop	QP	$[2^\ell,(2^\ell)-(2*\ell)]$, $\ell > 2$ odd	2	irred. Goppa	[14]
Mclas	QP	$[2^\ell-1,(2^\ell-1)-2*\ell]$, $\ell > 2$ odd	2	Mclas	[14]
S	QP	$[5,2]$, $[9,5]$, $[10,5]$, $[11,6]$	1	Slepian	[33]
S	QP	$[11,4]$	2	Slepian	[33]
FP	QP	$[15,9]$, $[21,14]$, $[22,15]$, $[23,16]$	1	Fontaine-Peterson	[33]
W	QP	$[19,10]$, $[20,11]$, $[20,13]$, $[23,14]$	2	Wagner	[33]
P	QP	$[21,12]$	2	Prange	[33]
FP	QP	$[25,12]$	3	Fontaine-Peterson	[33]
W	QP	$[25,15]$, $[26,16]$, $[27,17]$, $[28,18]$, $[29,19]$, $[30,20]$, $[31,20]$	1	Wagner	[33]
GS	QP	$[13,7]$, $[19,12]$	1	GS85	[22]
BBD	QP	$[7,3,3]$, $[9,4,4]$, $[10,6,3]$, $[11,7,3]$, $[12,7,3]$, $[12,8,3]$, $[13,8,3]$, $[13,9,3]$, $[14,9,3]$, $[15,10,3]$, $[16,10,3]$, $[17,11,4]$, $[17,12,3]$, $[18,12,4]$, $[18,13,3]$, $[19,13,3]$, $[19,14,3]$, $[20,14,4]$	1	BBD08	[3]
BBD	QP	$[22,13,5]$	2	BBD08	[3]

3.4 Finding the Optimal Concatenated Code

The linear code $[k,k']$ is typically instantiated by a concatenation of elementary codes for practical purposes. They are indeed easy to implement and to decode. For $[k,k']$ we have the concatenation of $[k_1,k'_1],\ldots,[k_m,k'_m]$ codes, where $k_1+\cdots+k_m=k$ and $k'_1+\cdots+k'_m=k'$. Let v_{ij},g_{ij},s'_j denote the j^{th} part of v_i,g_i,s'

respectively, corresponding to the concatenated $[k_j, k'_j]$ code. The bias of $\langle v_{ij} - g_{ij}, s_j \rangle$ in the code $[k_j, k'_j]$ is denoted by bc_j. As $\langle v_i - g_i, s \rangle$ is the xor of all $\langle v_{ij} - g_{ij}, s_j \rangle$, the total bias introduced by this operation is computed as $\mathsf{bc} = \prod_{j=1}^{k'} \mathsf{bc}_j$ and the combination $\mathsf{params} = ([k_1, k'_1], \ldots, [k_m, k'_m])$ is chosen such that it gives the highest bias.

The way these params are computed is the following: we start by computing the biases for all elementary codes. I.e. we compute the biases for all codes from Table 2. We may add random codes that we found interesting.[5] Next, for each $[i, j]$ code we check to see if there is a combination of $[i-n, j-m]$,$[n, m]$ codes that give a better bias, where $[n, m]$ is either a repetition code, a Golay code or a Hamming code. We illustrate below the algorithm to find the optimal concatenated code.

Algorithm 1 Finding the optimal params and bias

Input: k
Output: table for the optimal bias for each $[i, j]$ code, $1 \leq j < i \leq k$

initialize all $\mathsf{bias}(i, j) = 0$
initialize $\mathsf{bias}(1, 1) = 1$
initialize the bias for all elementary codes
for all $j : 2$ to k **do**
    for all $i : j+1$ to k **do**
        for all elementary code $[n, m]$ **do**
            if $|\mathsf{bias}(i-n, j-m) \cdot \mathsf{bias}(n, m)| > |\mathsf{bias}(i, j)|$ **then**
                $\mathsf{bias}(i, j) = \mathsf{bias}(i-n, j-m) \cdot \mathsf{bias}(n, m)$
                $\mathsf{params}(i, j) = \mathsf{params}(i-n, j-m) \cup \mathsf{params}(n, m)$

Using $O(k)$ elementary codes, this procedure takes $O(k^3)$ time and we can store all params for any combination $[i, j]$, $1 \leq j < i \leq k$ with $O(k^2)$ memory.

4 The Graph of Reduction Steps

Having in mind the reduction methods described in Section 2, we formalize an LPN solving algorithm in terms of finding the best chain in a graph. The intuition is the following: in an LPN solving algorithm we can see each reduction step as an edge from a $(k, \log_2 n)$ instance to a new instance $(k', \log_2 n')$ where the secret is smaller, $k' \leq k$, we have more or less number of queries and the noise has a different bias. For example, a *partition-reduce*(b) reduction turns an $(k, \log_2 n)$ instance with bias δ into $(k', \log_2 n')$ with bias δ' where $k' = k - b$,

[5] The random codes that we used are provided as an additional material to this paper.

$n' = n - 2^b$ and $\delta' = \delta^2$. By this representation, the reduction phase represents a chain in which each edge is a reduction type moving from LPN with parameters (k,n) to LPN with parameters (k',n') and that ends with an instance (k_i,n_i) used to recover the k_i-bit length secret by a solving method. We choose the fast Walsh-Hadamard transform as the solving method as it was proven to be more efficient than the majority phase used by the BKW algorithm.

As described before, we formalize the reduction phase as a chain of reduction steps in a graph $G = (V,E)$. The set of vertices V is composed of $V = \{1,\ldots,k\} \times L$ where L is a set of real numbers. For instance, we could take $L = \mathbb{R}$ or $L = \mathbb{N}$. For efficiency reasons, we could even take $L = \{0,\ldots,\eta\}$ for some bound η. Every vertex saves the size of the secret and the logarithmic number of queries; i.e. a vertex $(k,\log_2 n)$ means that we are in an instance where the size of the secret is k and the number of queries available is n. An edge from one vertex to another is given by a reduction step. An edge from $(k,\log_2 n)$ to a $(k',\log_2 n')$ has a label indicating the type of reduction and its parameters (e.g. *partition-reduce*$(k-k')$ or *code-reduce*(k,k',params)). This reduction defines some α and β coefficients such that the bias δ' after reduction is obtained from the bias δ before the reduction by

$$\log_2 \delta'^2 = \alpha \log_2 \delta^2 + \beta$$

where $\alpha,\beta \in \mathbb{R}$.

We denote by $\lceil\lambda\rceil_L$ the smallest element of L which is at least equal to λ and by $\lfloor\lambda\rfloor_L$ the largest element of L which is not larger than λ. In general, we could use a rounding function $\mathsf{Round}_L(\lambda)$ such that $\mathsf{Round}_L(\lambda)$ is in L and approximates λ.

The reduction steps described in Subsection 2.2 can be formalized as follows:

- *sparse-secret*: $(k,\log_2 n) \to (k,\mathsf{Round}_L(\log_2(n-k)))$ and $\alpha = 0, \beta = 0$
- *partition-reduce*(b): $(k,\log_2 n) \to (k-b,\mathsf{Round}_L(\log_2(n-2^b))$ and $\alpha = 2, \beta = 0$
- *xor-reduce*(b): $(k,\log_2 n) \to (k-b,\mathsf{Round}_L(\log_2\left(\frac{n(n-1)}{2^{b+1}}\right)))$ and $\alpha = 2, \beta = 0$
- *drop-reduce*(b): $(k,\log_2 n) \to (k-b,\mathsf{Round}_L(\log_2(\frac{n}{2^b})))$ and $\alpha = 1, \beta = 0$
- *code-reduce*(k,k',params): $(k,\log_2 n) \to (k',\log_2 n)$ and $\alpha = 1, \beta = \log_2 \mathsf{bc}^2$, where bc is the bias introduced by the covering code reduction using a $[k,k']$ linear code defined by params.

Below, we give the formal definition of a reduction chain.

Definition 3 (Reduction chain). *Let* $\mathcal{R} = \{$*sparse-secret*, *partition-reduce*(b), *xor-reduce*(b), *drop-reduce*(b), *code-reduce*$(k,k',\mathsf{params})\}$ *for* $k,k',b \in \mathbb{N}$. *A* ***reduction chain*** *is*

a sequence

$$(k_0, \log_2 n_0) \xrightarrow{e_1} (k_1, \log_2 n_1) \xrightarrow{e_2} \ldots \xrightarrow{e_i} (k_i, \log_2 n_i),$$

where the change $(k_{j-1}, \log_2 n_{j-1}) \to (k_j, \log_2 n_j)$ *is performed by one reduction from* $\mathcal{R}$*, for all* $0 < j \leq i$*.*

A chain is ***simple*** *if it is accepted by the automaton from Figure 1.*

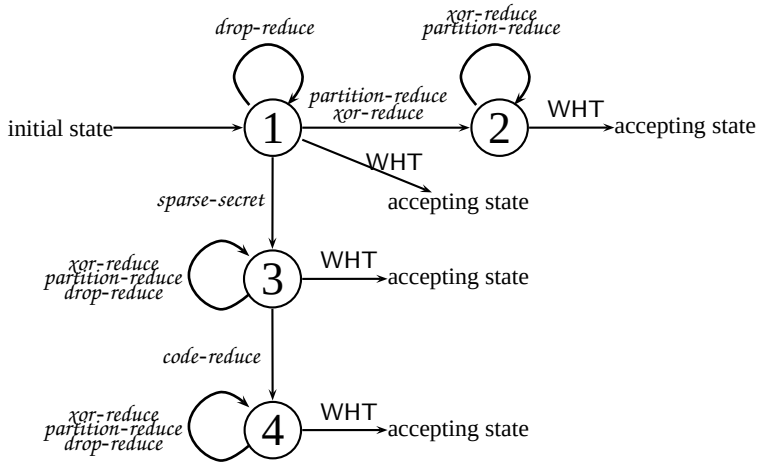

Fig. 1: Automaton accepting simple chains

Definition 4 (Exact chain). *An* ***exact chain*** *is a simple reduction chain for* $L = \mathbb{R}$*. I.e.* Round_L *is the identity function.*

A chain which is not exact is called **rounded**.

Remark: Restrictions for simple chains are modelled by the automaton in Figure 1. We restrict to simple chains as they are easier to analyze. Indeed, *sparse-secret* is only used to raise δ_s to make *code-reduce* more effective. And, so far, it is hard to analyze sequences of *code-reduce* steps as the first one may destroy the uniform and high δ_s for the next ones. This is why we exclude multiple *code-reduce* reductions in a simple chain. So, we use up to one *sparse-secret* reduction, always one before *code-reduce*. And *sparse-secret* occurs before δ decreases. For convenience, we will add a state of the automaton to the vertex in V.

For solving LPN we are interested in those chains that end with a vertex $(k_i, \log_2 n_i)$ which allows to call a WHT solving algorithm to recover the k_i-bit secret. We call these chains valid chains and we define them below.

Definition 5 (Valid reduction chain). *Let*

$$(k_0, \log_2 n_0) \xrightarrow{e_1} (k_1, \log_2 n_1) \xrightarrow{e_2} \ldots \xrightarrow{e_i} (k_i, \log_2 n_i)$$

be a reduction chain. Let $\delta_i = 1 - 2\tau_i$ *be the bias corresponding to the edge* $(k_i, \log_2 n_i)$. *We say the chain is a* θ-***valid reduction chain*** *if* n_i *satisfies (1) for solving an* $\mathsf{LPN}_{k_i,\tau_i}$ *instance.*

The *time complexity* of a chain $(e_1, \ldots, e_i)$ is simply the sum of the complexity of each reduction step $e_1, e_2, \ldots, e_i$ and WHT. We further define the **max-complexity** of a chain which is the maximum of the complexity of each reduction step and WHT. The max-complexity is a good approximation of the complexity. Our goal is to find a chain with optimal complexity. What we achieve is that, *given a set L*, we find a *rounded* chain with optimal *max-complexity* up to some given precision.

4.1 Towards Finding the Best LPN Reduction Chain

In this section we present the algorithm that helps finding the optimal valid chains for solving LPN. As aforementioned, we try to find the valid chain with optimal max-complexity for solving an $\mathsf{LPN}_{k,\tau}$ instance in our graph G.

The first step of the algorithm is to construct the directed graph $G = (V, E)$. We take the set of vertices $V = \{1, \ldots, k\} \times L \times \{1, 2, 3, 4\}$ which indicate the size of the secret, the logarithmic number of queries and the state in the automaton in Figure 1. Each edge $e \in E$ represents a reduction step and is labelled with the following information: $(k_1, \log_2 n_1, st) \xrightarrow{\alpha,\beta,t} (k_2, \log_2 n_2, st')$ where t is one of the reduction steps and α and β save information about how the bias is affected by this reduction step.

The graph has $O(k \cdot |L|)$ vertices and each vertex has $O(k)$ edges. So, the size of the graph is $O(k^2 \cdot |L|)$.

Thus, we construct the graph G with all possible reduction steps and from it we try to see what is the optimal simple rounded chain in terms of max-complexity. We present in Algorithm 2 the procedure to construct the graph G that contains all possible reduction steps with a time complexity bounded by 2^η (As explained below, Algorithm 2 is not really used).

The procedure of finding the optimal valid chain is illustrated in Algorithm 3. The procedure of finding a chain with upper bounded max-complexity is illustrated in Algorithm 4.

Algorithm 4 receives as input the parameters k and τ for the LPN instance, the parameter θ which represents the bound on the failure probability in recovering the secret. Parameter η represents an upper bound for the logarithmic

Algorithm 2 Construction of graph G

1: **Input**: k, τ, L, η
2: **Output**: graph $G = (V, E)$ containing all the reduction steps that have a complexity smaller than 2^η

3: $V = \{1, \ldots, k\} \times L \times \{1, \ldots, 4\}$
4: E is the set of all $((i, \eta_1, st), (j, \eta_2, st'))$ labelled by (α, β, t) such that there is a $st \xrightarrow{t} st'$ transition in the automaton and for
5: $t =$ *sparse-secret*:
6: **for all** $\eta : 1$ such that $\mathsf{lcomp} \leq \eta$ **do** set the edge
7: where $i = k$, $(j, \eta_2) = (i, \mathsf{Round}_L(\log_2(2^{\eta_1} - i)))$, $\alpha = 1$, $\beta = 0$, $\mathsf{lcomp} = \log_2(\frac{(2^{\eta_1} - i)i^2}{log_2 i - log_2 log_2 i} + i^2)$
8: $t =$ *partition-reduce*:
9: **for all** (i, η_1, b) such that $b \geq 1$ and $\mathsf{lcomp} \leq \eta$ **do** set the edge
10: where $(j, \eta_2) = (i - b, \mathsf{Round}_L(\log_2(\eta_1 - 2^b)))$, $\alpha = 2$, $\beta = 0$, $\mathsf{lcomp} = \log_2 i + \eta_1$
11: $t =$ *xor-reduce*:
12: **for all** (i, η_1, b) such that $b \geq 1$ and $\mathsf{lcomp} \leq \eta$ **do** set the edge
13: where $(j, \eta_2) = (i - b, \mathsf{Round}_L(\eta_1 - 1 + \log_2(\frac{2^{\eta_1}}{2^b} - 1)))$, $\alpha = 2$, $\beta = 0$, $\mathsf{lcomp} = \log_2 i + \max(\eta_1, \eta_2)$
14: $t =$ *drop-reduce*:
15: **for all** (i, η_1, b) such that $b \geq 1$ and $\mathsf{lcomp} \leq \eta$ **do** set the edge
16: where $(j, \eta_2) = (i - b, \mathsf{Round}_L(\eta_1 - b))$, $\alpha = 1$, $\beta = 0$, $\mathsf{lcomp} = \log_2 b + \eta_1$
17: $t =$ *code-reduce*:
18: **for all** (i, η_1, j) such that $j < i$ and $\mathsf{lcomp} \leq \eta$ **do** set the edge
19: where $\eta_2 = \eta_1$, $\alpha = 1$, $\beta = \log_2 \mathsf{bc}^2$, $\mathsf{lcomp} = \log_2 i + \eta_1$, bc is the bias from the optimal $[i, j]$ code

complexity of each reduction step. Given η, we build the graph G which contains all possible reductions with time complexity smaller than 2^η (Step 4). Note that we don't really call Algorithm 2. Indeed, we don't need to store the edges of the graph. We rather keep a way to enumerate all edges going to a given vertex (in Step 11) by using the rules described in Algorithm 2.

For each vertex, we iteratively define Δ^{st} and Best^{st}, the best reduction step to reach a vertex and the value of the corresponding error bias. The best reduction step is the one that maximizes the bias. We define these values iteratively until we reach a vertex from which the WHT solving algorithm succeeds with complexity bounded by 2^η. Once we have reached this vertex, we construct the chain by going backwards, following the Best pointers.

We easily prove what follows by induction.

Lemma 1. *At the end of the iteration of Algorithm 4 for* (j, η_2, st'), $\Delta^{st'}_{j,\eta_2}$ *is the maximum of* $\log_2 \delta^2$, *where* δ *is the bias obtained by an* Round_L*-rounded simple*

Algorithm 3 Search for a rounded chain with optimal max-complexity

Input: $k, \tau, \theta,$ precision
Output: a valid simple rounded chain in which rounding uses a given precision

set found $=$ `bruteforce` ▷ found is the best found algorithm
set increment $= k$
set $\eta = k$ ▷ 2^η is a bound on the max-complexity
repeat
    set increment $\leftarrow \frac{1}{2}$increment
    define $L = \{0, \text{precision}, 2 \times \text{precision}, \ldots\} \cap [0, \eta - \text{increment}]$
    run $(\text{out}, \text{success}) = \text{Search}(k, \tau, \theta, L, \eta - \text{increment})$ with Algorithm 4
    if success **then**
        set found $=$ out
        set $\eta = \eta -$ increment
until increment $\leq$ precision
output found

chain from a vertex of form $(k, \eta_1, 0)$ *to* (j, η_2, st') *with max-complexity bounded by* 2^η *(*$\Delta^{st'}_{j,\eta_2} = -\infty$ *if there is no such chain).*

Lemma 2. *If there exists a simple* Round_L*-rounded chain* c *ending on state* (k_j, η_j, st_j) *and max-complexity bounded by* 2^η*, there exists one* c' *such that* $\Delta^{st_i}_{i,\eta_i} = \log_2 \delta_i^2$ *at each step.*

Proof. Let c'' be a simple chain ending on (k_j, η_j, st_j) with $\Delta^{st_j}_{j\eta_j} = \log_2 \delta_j^2$. Let $(k_{j-1}, \eta_{j-1}, st_{j-1})$ be the preceding vertex in c''. We apply Lemma 2 on this vertex by induction to obtain a chain c'''. Since the complexity of the last edge does not depend on the bias and $\alpha \geq 0$ in the last edge, we construct the chain c', by concatenating c''' with the last edge of c''. □

Theorem 2. *Algorithm 4 finds a* θ*-valid simple* Round_L*-rounded chain for* $\text{LPN}_{k,\tau}$ *with max-complexity bounded by* 2^η *if there exists one.*

Proof. We use Lemma 2 and the fact that increasing δ^2 keeps constraint (1) valid. □

If we used $L = \mathbb{R}$, Algorithm 4 would always find a valid simple chain with bounded max-complexity when it exists. Instead, we use rounded chains and hope that rounding still makes us find the optimal chain.

So, we build Algorithm 3. In this algorithm, we look for the minimal η for which Algorithm 4 returns something by a divide and conquer algorithm. First, we set η as being in the interval $[0, k]$ where the solution for $\eta = k$ corresponds to a brute-force search. Then, we cut the interval in two pieces and see if the

Algorithm 4 Search for a best LPN reduction chain with max-complexity bounded to η

Input: k, τ, θ, L, η
Output: a valid simple rounded chain with max-complexity bounded to η

$\delta = 1 - 2\tau$
Construct the graph G using Algorithm 2 with parameters k, τ, L, η
for all $\eta_1 \in L$ **do**
    set $\Delta^0_{k,\eta_1} = \log_2 \delta^2$, $\mathsf{Best}^0_{k,\eta_1} = \perp$
    set $\Delta^{st}_{k,\eta_1} = -\infty$, $\mathsf{Best}^{st}_{k,\eta_1} = \perp$ ▷ Δ^{st} stores the best bias for a vertex (k, η_1, st) in a chain, and Best^{st} is the edge ending to this vertex in this chain
for $j : k$ downto 1 **do** ▷ Search for the optimal chain
    for $\eta_2 \in L$ in decreasing order **do**
        set $\Delta^{st}_{j,\eta_2} = 0$, $\mathsf{Best}^{st} = \perp$ for all st
        foreach st' and each edge e to (j, η_2, st')
            set (i, η_1, st) to the origin of e and α and β as defined by e
            if $\alpha\Delta^{st}_{i,\eta_1} + \beta \geq \Delta^{st'}_{j,\eta_2}$ **then** set $\Delta^{st'}_{j,\eta_2} = \alpha\Delta^{st}_{i,\eta_1} + \beta$, $\mathsf{Best}^{st} = e$
        if $\eta_2 > 1 - \Delta^{st'}_{j,\eta_2} + 2\log_2\left(-\varphi^{-1}(1-(1-\theta)^{\frac{1}{2^{j-1}}})\right)$ and $j + \log_2 j \leq \eta$ **then**
            Construct the chain c ending by $\mathsf{Best}^{st'}_{j,\eta_2}$ and output (c, true)
output $(\perp, \mathsf{false})$

lower interval has a solution. If it does, we iterate in this interval. Otherwise, we iterate in the other interval. We stop once the amplitude of the interval is lower than the requested precision. The complexity of Algorithm 3 is of $\log_2 \frac{k}{\mathsf{precision}}$ calls to Algorithm 4.

Theorem 3. *Algorithm 3 finds a* θ*-valid simple* Round_L*-rounded chain for* $\mathsf{LPN}_{k,\tau}$ *with parameter* precision*, with optimal rounded max-complexity, where the rounding function approximates* $\log_2$ *up to* precision *if there exists one.*

Proof. Algorithm 3 is a divide-and-conquer algorithm to find the smallest η such that Algorithm 4 finds a valid simple Round_L-rounded chain of max-complexity bounded by 2^η. □

We can see that the complexity of Algorithm 4 is of $O\left(k^2 \cdot |L|\right)$ iterations as vertices have k possible values for the secret length and $|L|$ possible values for the logarithmic number of equations. So, it is linear in the size of the graph. Furthermore, each type of edge to a fixed vertex has $O(k)$ possible origins. The memory complexity is $O(k \cdot |L|)$, mainly to store the $\Delta_{k,\eta}$ and $\mathsf{Best}_{k,\eta}$ tables. We also use Algorithm 1 which has a complexity $O(k^3)$ but we run it only once during precomputation. Algorithm 3 sets $|L| \sim \frac{k}{\mathsf{precision}}$. So, the complexity of Algorithm 3 is $O\left(k^3 + \frac{k^3}{\mathsf{precision}} \times \log \frac{k}{\mathsf{precision}}\right)$.

5 Chains with a Guessing Step

In order to further improve our valid chain we introduce two new reduction steps to our algorithm. As it is done in [23,5], we guess part of the bits of the secret. More precisely, we assume that b bits of the secret have a Hamming weight smaller or equal to w. The influence on the whole algorithm is more complicated: it requires to iterate the WHT step $\sum_{i=0}^{w}\binom{w}{i}$ times. The overall complexity must further be divided by $\sum_{i=0}^{w}\binom{w}{i}\left(\frac{1-\delta_s}{2}\right)^i\left(\frac{1+\delta_s}{2}\right)^{w-i}$. Note that this generalized *guess-secret* step was used in [23].

We formalize this step as following:

– *guess-secret*(b,w) guesses that b bits of the secret have a Hamming weight smaller or equal to w. The b positions are chosen randomly. The number of queries remains the same, the noise is the same and the size of the secret is reduced by b bits. Thus, for this step we have

> *guess-secret*$(b,w): k' = k-b; n' = n; \delta' = \delta; \delta'_s = \delta$
> Complexity: $O(nb)$ (included in *sparse-secret*) and
> the Walsh transform has to be iterated $\sum_{i=0}^{w}\binom{w}{i}$ times and
> the complexity of the whole algorithm is divided by
> $\sum_{i=0}^{w}\binom{w}{i}\left(\frac{1-\delta_s}{2}\right)^i\left(\frac{1+\delta_s}{2}\right)^{w-i}$

This step may be useful for a sparse secret, i.e. τ is small, as when we reduce the size of the secret with a very small cost. In order to accommodate this new step we would have to add a transition from state 3 to state 3 in the automaton that accepts the simple chains (See Figure 1).

To find the optimal chain using *guess-secret*(b,w), we have to make a loop over all possible b and all possible w. We run the full search $O(k^2)$ times. The total complexity is thus $O\left(\frac{k^5}{\text{precision}}\times\log\frac{k}{\text{precision}}\right)$.

The second reduction technique is denoted by *trunc-reduce*. Given the queries $(v,\langle v,s\rangle\oplus d)$, one can just pick a random position, j, and reduce the secret by 1 bit if the j^{th} bit of v is 0. If the j^{th} bit is 1, then we can still reduce the size by 1 by adding s_j to the error term; i.e. the updated query is $(v',\langle v',s'\rangle\oplus d\oplus s_j)$, where $s'(v')$ is $s($ respectively $v)$ vector without the j^{th} bit. When the bit of v is 0 this change does not affect the original query. When it is 1, then the new bias introduced is δ_s, the bias of s_j. We can take $\delta_s = 1-2\tau$. Thus, the expected bias is $\frac{1+\delta_s}{2}$. We can generalize this reduction and zero b bits of the secret by introducing an error term that has a bias of $(\frac{1+\delta_s}{2})^b$. One point is that we must make the secret sparse (thus use *sparse-secret*) and also randomize the b bits which are truncated so that they do have a bias of δ_s and are not constant. We do

this by a special treatment during the *sparse-secret* step: for each output equation we use b new input equations and use these errors bits as the randomized b secret bits. The Stochastic equivalence approximation helps again.

We have the following:

– *trunc-reduce*(b) introduces b bits of the secret in the error term. We can easily see that, by this operation, the number of queries remains the same, the size of the secret is reduced by b bits and the new bias is $\delta \cdot (\frac{1+\delta_s}{2})^b$.

> *trunc-reduce*(b) : $k' = k - b$; $n' = n$; $\delta' = \delta \cdot (\frac{1+\delta_s}{2})^b$; $\delta'_s = \delta_s$
> Complexity: $O(1)$

As we do not follow the evolution of δ_s beyond *code-reduce*, the *trunc-reduce* step can only be done in between *sparse-secret* and *code-reduce* in a chain. When considering *trunc-reduce*(b) outside this case, we under-estimate δ_s to 0 and have $\delta' = \delta \cdot 2^{-b}$. As for *guess-secret*, the *trunc-reduce* step requires a transition from state 3 to itself in the automaton.

6 Results

We illustrate in this section the results obtained by running Algorithm 4 for different LPN instances taken from [7,8]. They vary from taking $k = 32$ to $k = 768$, with the noise levels: $0.05, 0.1, 0.125, 0.2$ and 0.25. In Table 3 we display the logarithmic time complexity we found for solving LPN without using *guess-secret*.[6]

τ	k						
	32	48	64	100	256	512	768
0.05	$13.89^{11.26}_{0.1}$	$14.51^{12.94}_{0.1ct}$	$16.04^{14.43}_{0.1c}$	$20.47^{18.46}_{0.1c}$	$36.85^{34.68}_{0.1c}$	$58.00^{55.27}_{0.1c}$	$77.07^{74.24}_{0.1c}$
0.1	$15.04^{12.70}_{0.1}$	$18.58^{16.43}_{0.1}$	$21.58^{19.38}_{0.1c}$	$27.83^{25.44}_{0.1c}$	$47.13^{44.61}_{0.1c}$	$74.15^{71.33}_{0.1c}$	$99.48^{96.58}_{0.1c}$
0.125	$15.66^{13.52}_{0.1}$	$19.29^{17.00}_{0.1}$	$22.94^{20.50}_{0.1}$	$28.91^{26.30}_{0.1}$	$50.36^{47.90}_{0.1c}$	$79.46^{76.66}_{0.1c}$	$106.45^{103.48}_{0.1c}$
0.2	$17.01^{14.80}_{0.1}$	$21.25^{19.23}_{0.1}$	$24.42^{22.00}_{0.1}$	$32.06^{29.75}_{0.1}$	$56.86^{54.28}_{0.1c}$	$89.63^{86.86}_{0.1c}$	$121.20^{118.43}_{0.1}$
0.25	$18.42^{16.30}_{0.1}$	$22.34^{20.43}_{0.1}$	$26.86^{24.58}_{0.1}$	$32.94^{30.75}_{0.1}$	$59.47^{56.88}_{0.1}$	$94.98^{92.21}_{0.1}$	$128.32^{125.58}_{0.1ct}$

entry of form $a^b_{c...}$: $a = \log_2$ complexity, $b = \log_2$ max-complexity, $c =$ precision
subscript c means that a *code-reduce* is used
subscript t means that a *trunc-reduce* is used

Table 3: Logarithmic time complexity on solving LPN without *guess-secret*

[6] Complete results are provided as an additional material to this paper.

τ	k						
	32	48	64	100	256	512	768
0.05	$11.85^{10.90}_{0.1g13o}$	$13.01^{12.52}_{0.1g23o}$	$14.44^{13.74}_{0.1g38o}$	$17.20^{16.19}_{0.1g75o}$	$30.13^{28.02}_{0.1g178o}$	$49.56^{47.29}_{1g417o}$	$68.15^{65.98}_{1g682o}$
0.1	$12.41^{11.65}_{0.1g23o}$	$15.23^{14.25}_{0.1g37o}$	$17.71^{16.76}_{0.1g52o}$	$24.02^{22.14}_{0.1g77o}$	$46.50^{43.87}_{0.1g132o}$	$74.13^{71.47}_{1g1}$	$99.63^{96.79}_{1g2}$
0.125	$13.30^{12.40}_{0.1g26o}$	$16.49^{15.49}_{0.1g39o}$	$20.57^{18.61}_{0.1g36o}$	$27.26^{24.93}_{0.1g48o}$	$50.36^{47.90}_{0.1}$	$79.71^{77.01}_{1g1}$	$106.51^{103.68}_{1}$
0.2	$17.01^{14.80}_{0.1}$	$21.25^{19.23}_{0.1}$	$24.42^{22.00}_{0.1}$	$32.06^{29.75}_{0.1}$	$56.86^{54.28}_{0.1}$	$89.65^{87.04}_{1}$	$121.12^{118.57}_{1}$
0.25	$18.42^{16.30}_{0.1}$	$22.34^{20.43}_{0.1}$	$26.86^{24.58}_{0.1}$	$32.94^{30.75}_{0.1}$	$59.47^{56.88}_{0.1}$	$95.00^{92.53}_{1}$	$128.40^{125.77}_{1g1}$

entry of form $a^b_{c\ldots}$: $a = \log_2$ complexity, $b = \log_2$ max-complexity, $c =$ precision
subscript o means that a only 1 bit of the secret is found by WHT
subscript gb means that a *guess-secret*$(b,\cdot)$ is used

Table 4: Logarithmic time complexity on solving LPN with *guess-secret*

Sequence of chains. If we analyze in more details one of the chains that we obtained, e.g. the chain for $\mathsf{LPN}_{512,0.125}$, we can see that it first uses a *sparse-secret*. Afterwards, the secret is reduced by applying 5 times the *xor-reduce* and one *code-reduce* at the end of the chain. With a total complexity of $2^{79.46}$ and $\theta < 33\%$ it recovers 65 bits of the secret.

$$(512, 62.2) \xrightarrow{\textit{sparse-secret}} (512, 61.2) \xrightarrow{\textit{xor-reduce}(54)} (458, 67.4) \xrightarrow{\textit{xor-reduce}(66)}$$
$$(392, 67.8) \xrightarrow{\textit{xor-reduce}(67)} (325, 67.6) \xrightarrow{\textit{xor-reduce}(66)} (259, 68.2) \xrightarrow{\textit{xor-reduce}(67)}$$
$$(192, 68.4) \xrightarrow{\textit{code-reduce}} (65, 68.4) \xrightarrow{\mathsf{WHT}}$$

The code used is a $[192, 65]$ concatenation made of the $[25, 15]$ quasi-perfect Wagner code, some $[19, 4]$, $[16, 5]$, and $[18, 5]$ random codes that we found and 6 copies of a $[19, 6]$ random code that we found. By manually tuning the number of equations without rounding, we can obtain with $n = 2^{61.1915}$ a complexity of $2^{79.37}$. This is the value from Table 1.

On the guess-secret reduction. Our results show that the *guess-secret* step does not bring any significant improvement. If we compare Table 3 with Table 4 we can see that in few cases the guess step improves the total complexity. For $k \geq 512$,

some results are not better than Table 3. This is most likely due to the lower precision used in Table 4.

We can see several cases where, at the end of a chain with *guess-secret*, only one bit of the secret is recovered by WHT. If only 1 bit of the secret is recovered by non-bruteforce methods, the next chain for $\mathsf{LPN}_{k-1,\tau}$ will have to be run several times, given the *guess-secret* step used in the chain for $\mathsf{LPN}_{k,\tau}$. Thus, it might happen that the first chain does not dominate the total complexity. So, our strategy to use sequences of chains has to be revised, but most likely, the final result will not be better than sequences of chains without *guess-secret*. We want to avoid these chains ending with 1 bit recovery.

There is only one case where a *guess-secret* without a chain ending with 1 bit brings a little improvement: $\mathsf{LPN}_{512,0.1}$: $2^{74.15}$ vs. $2^{74.13}$ with $b = 1$. Most likely, this performance with *guess-secret* can be cancelled by finding better codes.

On the trunc-reduce reduction. We did not see much use of the *trunc-reduce* step. The only use is in the chain for $\mathsf{LPN}_{48,0.05}$ and $\mathsf{LPN}_{768,0.25}$ (see Table 3) where one bit is truncated. Again, a better code would most likely lead us to better chains with no *trunc-reduce*.

Comparing the results. For practical values we compare our results with the previous work. We take as reference the analysis done in [23,29] and [7,8].

In Tables 5 and 6 we display the analysis conducted in [29,7,8]. In [29], only the query complexity is given. We took the query complexity from [29] and computed what is the time complexity in order to compare with our results. Our results are better by a factor of 2^3 up to 2^8, depending on the parameters.

τ	k		
	256	512	768
0.05	50	79	102
0.125	56	88	125
0.25	64	100	142

Table 5: Security of LPN [29]

τ	k		
	256	512	768
0.05	42	65	87
0.125	52	82	109
0.25	64	99	139

Table 6: Security of LPN [7,8]

The comparison with [23] was shown in Table 1 in Introduction. From the work [23] from ASIACRYPT'14 we have that $\mathsf{LPN}_{512,0.125}$ can be solved in time complexity of $2^{79.7}$ (in fact, more as some complexities were underestimated). We do better, provide concrete codes and we even remove the *guess-secret* step

with an optimized use of a code. Thus, the results of Algorithm 4 improve all the existing results on solving LPN.

7 Conclusion

In this article we have proposed an algorithm for creating reduction chains with the optimal max-complexity. The results we obtain bring improvements to the existing work and to our knowledge we have the best algorithm for solving $\mathsf{LPN}_{512,0.125}$. For instance, we improve the results from ASIACRYPT'14 [23] by showing how to select the code and to optimize the sequence of reduction steps. Furthermore, for the covering codes, we propose concrete codes for different LPN instances. We believe that our algorithm could be further adapted and automatized if new reduction techniques would be introduced.

As future works, we could look at applications to the LWE problem. Kirchner and Fouque [28] improve the LWE solving algorithms by refining the modulus switching. We could also look at ways to keep track of biases of secret bits bitwise, in order to allow cascades of *code-reduce* steps and random use of *trunc-reduce*.

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A Approximating n by using Central Limit Theorem

In order to approximate the number of queries needed to solve the LPN instance we consider when the Walsh Hadamard Transform fails to give the correct secret. This scenario happens when for another $s' \neq s$, we have that $\hat{f}(s') > \hat{f}(s)$. Following the analysis from [7] this translates to $HW(y) \leq HW(d')$, for $y = A's'^T + c'^T$ and $d' = A's^T + c'^T$. For each s', we take y as a uniformly distributed random vector. I.e.

$$\Pr[\hat{f}(s') > \hat{f}(s)] = \Pr\left[\sum_{i=1}^{n'}(y_i - d'_i) \leq 0\right].$$

Let $X_1, \ldots, X_{n'}$ be random variable corresponding to $X_i = y_i - d'_i$. Since $E(y_i) = \frac{1}{2}$, $E(d'_i) = \frac{1}{2} - \frac{\delta'}{2}$ and y_i and d'_i are independent, we have that $E(X_i) = \frac{\delta'}{2}$ and $Var(X_i) = \frac{2-\delta'^2}{4}$. By using the Central Limit Theorem we obtain that

$$\Pr[X_1 + \ldots + X_{n'} \leq 0] \approx \varphi\left(\frac{-\sqrt{n'}\delta'}{\sqrt{2-\delta'^2}}\right),$$

and φ can be calculated by $\varphi(x) = \frac{1}{2} + \frac{1}{2}\mathsf{erf}(\frac{x}{\sqrt{2}})$, where erf is the Gauss error function. Applying the reasoning for any $s' \neq s$ we obtain that the failure probability is bounded by θ if

$$(1 - \Pr[X_1 + \ldots + X_{n'} \leq 0])^{2^{k'}-1} \geq 1 - \theta$$

From this inequality and the previous approximation we deduce the following

$$\begin{aligned}
&(1-\Pr[X_1+\ldots+X_{n'}\leq 0])^{2^{k'}-1}\geq 1-\theta \Leftrightarrow\\
&\varphi\left(\frac{-\sqrt{n'}\delta'}{\sqrt{2-\delta'^2}}\right)\leq 1-(1-\theta)^{\frac{1}{2^{k'}-1}} \Leftrightarrow\\
&\sqrt{n'}\geq -\sqrt{2\delta'^{-2}-1}\varphi^{-1}\left(1-(1-\theta)^{\frac{1}{2^{k'}-1}}\right)
\end{aligned}$$

Thus, we require to have $\sqrt{n'}\approx -\sqrt{2\delta'^{-2}-1}\varphi^{-1}\left(1-(1-\theta)^{\frac{1}{2^{k'}-1}}\right)$. If we use the approximation $\varphi\left(\frac{-\sqrt{n'}\delta'}{\sqrt{2-\delta'^2}}\right)\approx \frac{1}{\sqrt{n'}}\frac{\sqrt{2-\delta'^2}}{\delta'}\frac{e^{-\frac{n'\delta'^2}{2(2-\delta'^2)}}}{\sqrt{2\pi}}$, we obtain that $n'\geq 2(2\delta'^{-2}-1)\ln(\frac{2^{k'}-1}{\theta})$. This brings an improvement of factor two over the Hoeffding bound that requires $n'\geq 8\ln(\frac{2^{k'}}{\theta})\delta'^{-2}$.

On the validity of the Stochastic independence approximation. The above computation makes sense when we can use the Stochastic independence approximation. We now look at how to avoid using it. So, the secret d generated by the *sparse-secret* reduction is randomly sampled once. The bias $\delta'(d)$ now depends on this secret d. We only know $E(\delta'(d))=\delta'$. The above computation is only making sure that

$$1-(1-\varphi(E(Z)))^{2^{k'}-1}\leq\theta$$

where $Z=\frac{-\sqrt{n'}\delta'(d)}{\sqrt{2-\delta'(d)^2}}$ which depends on d. We now want to study the average probability of failure

$$p=E\left(1-(1-\varphi(Z))^{2^{k'}-1}\right)$$

Typically, $\delta'(d)$ is concentrated around an average value depending on $\mathsf{HW}(d)$. So, if $E(Z)\ll 0$, values of Z close to 0 occur with rare probability.

The value of k' in WHT is typically not too small. Since we want the probability of failure to be below 33%, we need $\varphi(Z)$ to be very small. In this region of the curve, of the φ function, we can thus approximate $E(\varphi(Z))\approx\varphi(E(Z))$. The $t\mapsto 1-(1-t)^{2^{k'}-1}$ is concave. So, thanks to the Jensen inequality, the average probability of failure is

$$p\leq 1-(1-E(\varphi(Z)))^{2^{k'}-1}\approx 1-(1-\varphi(E(Z)))^{2^{k'}-1}\leq\theta$$

So, for k' not too small, what we obtain with the Stochastic independence approximation is a safe condition for having an average probability of failure below θ.