Lecture 2 Compound functions, the inverse of functions and elementary functions

§ 1 Compound functions

1.1 An example

Let

$$E = \frac{1}{2} m v^2, \quad v = gt.$$

Then $E = \frac{1}{2} m g^2 t^2$. This is a compound function.

1.2 Definition

We are given two functions: $y = f(u)$, $u = g(x)$.

If $R_g \subset D_f$, then $y = f(g(x))$ is called a compound function of functions f and g, denoted by $y = f \circ g$, where R_g denotes the range of g, D_f the domain of f.

Remark 1.2.1 The hypothesis $R_g \subset D_f$ is important.

Example 1.2.1 Suppose $f(x)=x^2+2$, $g(x)=\sin x$.

Find the following compound functions: $f(g(x))$, $g(f(x))$ and $f(f(x))$.

Example 1.2.2 Consider the composition of two functions $y=\sqrt{1+u}$ and $u=x^2-5$.

§ 2 Inverse of a function

2.1 Definition

We are given a function $y=f(x)$ with the domain D

and the range R. If for any $y \in R$, there is a unique preimage $x \in D$ which corresponds to y, then we call this function the inverse of the function $y = f(x)$, denoted by $x = f^{-1}(y)$ or $y = f^{-1}(x)$.

For example ,the inverse of function $y = x^3$ is $x = y^{\frac{1}{3}}$, it is always denoted by $y = x^{\frac{1}{3}}$.

2.2 A criterion

Theorem 2.2.1 If $y = f(x)$ is strictly increasing (resp. decreasing) in its domain D with the rang R, then its inverse $y=f^{-1}(x)$ exists and $y=f^{-1}(x)$ is also strictly increasing (resp. decreasing) in R.

Proof (1) Existence

Suppose that there are $y \in R$ and $x_1 \neq x_2 \in D$ such

That $x_1=f^{-1}(y)$ and $x_2=f^{-1}(y)$, then $f(x_1)=y=f(x_2)$. By the monotonicity of $y=f(x)$, we see that $x_1=x_2$.

This contradiction proves the existence.

(2) Monotonicity

It suffices to consider the case that $y=f(x)$ is strictly increasing. The proof for the case that $y = f(x)$ is strictly increasing easily follows from a similar discussion.

For any $y_1 < y_2$, let $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$.

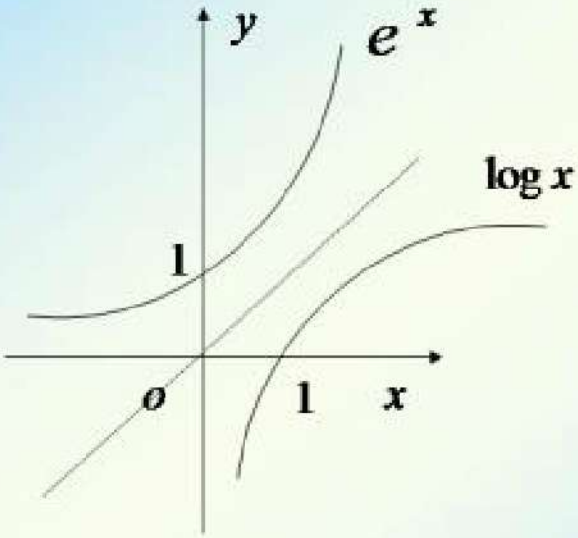
Suppose $x_1 \geq x_2$. Then $f(x_1) \geq f(x_2)$ i.e $y_1 \geq y_2$.

This is the desired contradiction.

The proof is complete.

Theorem 2.2.2 If the inverse of $y = f(x)$ exists , then the graph of $f(x)$ is symmetric to that of $f^{-1}(x)$ with the symmetric axis $y = x$.

For example , $y = e^x$ and $y = \log x$.

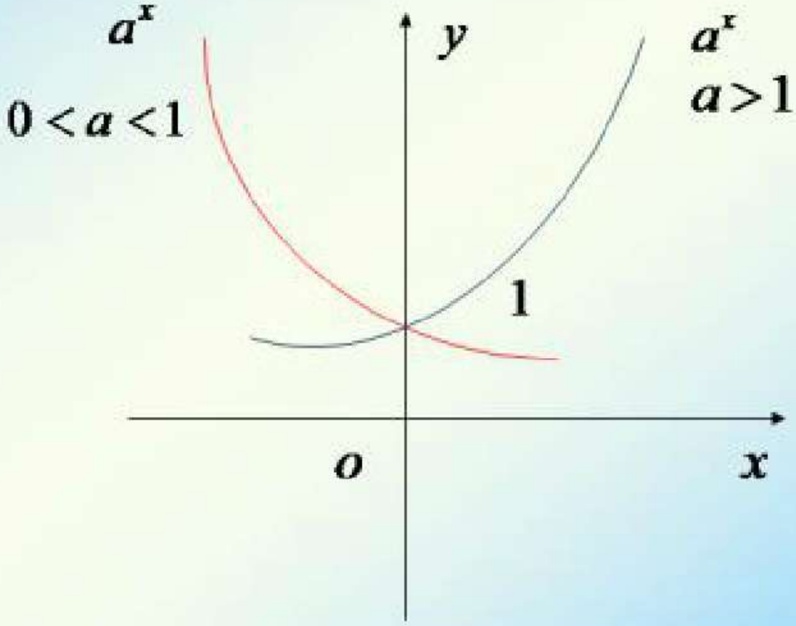


§ 3 Elementary functions

3.1 Basic elementary functions

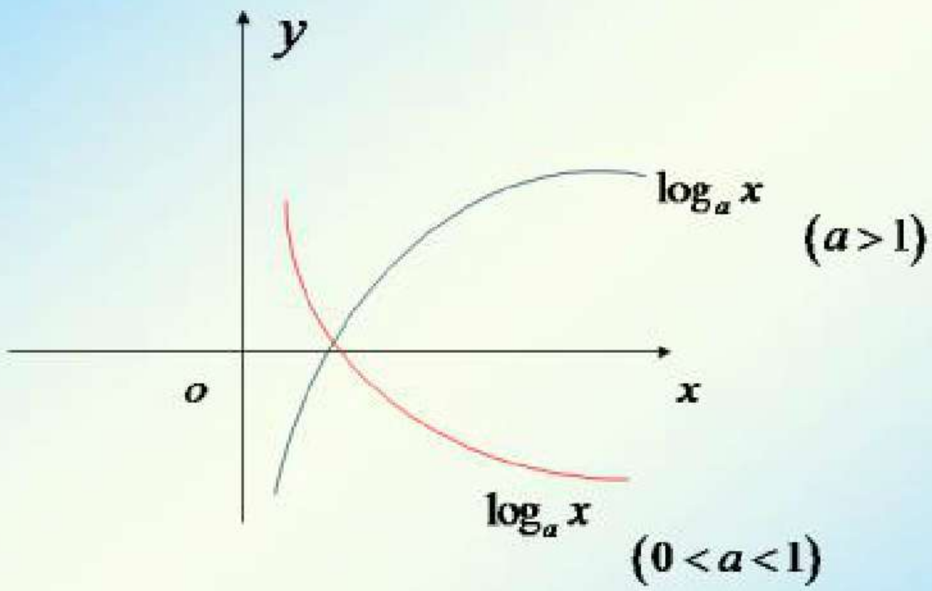
3.1.1 Exponent functions

$$y = a^x \quad (a > 0,\ a \neq 1) \ (y = e^x).$$



3.1.2 Logarithm functions

$$y = \log_a^x \quad (a > 0, a \neq 1) \quad (y = \log x)$$



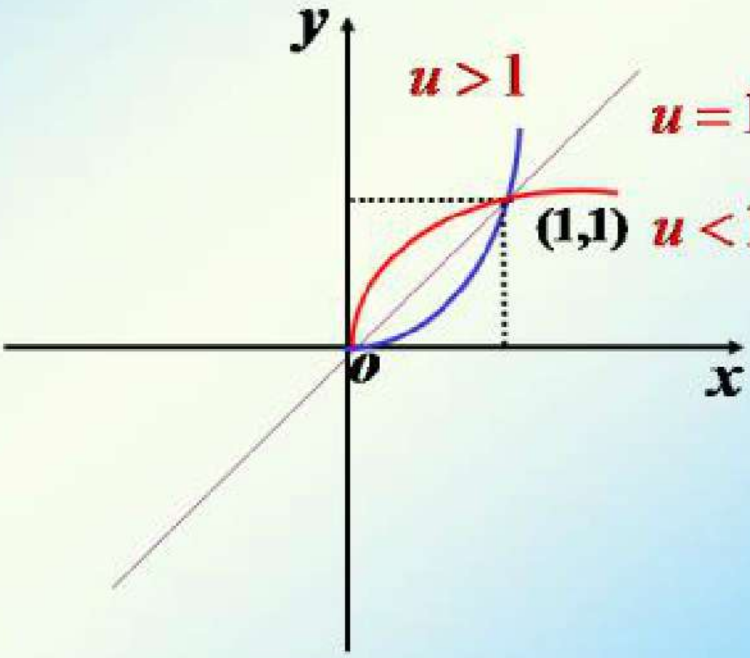
Lemma 3.1.2.1 For all $x > 0$, $\frac{x}{1+x} < \log(1+x) < x$.

This proof will be given later.

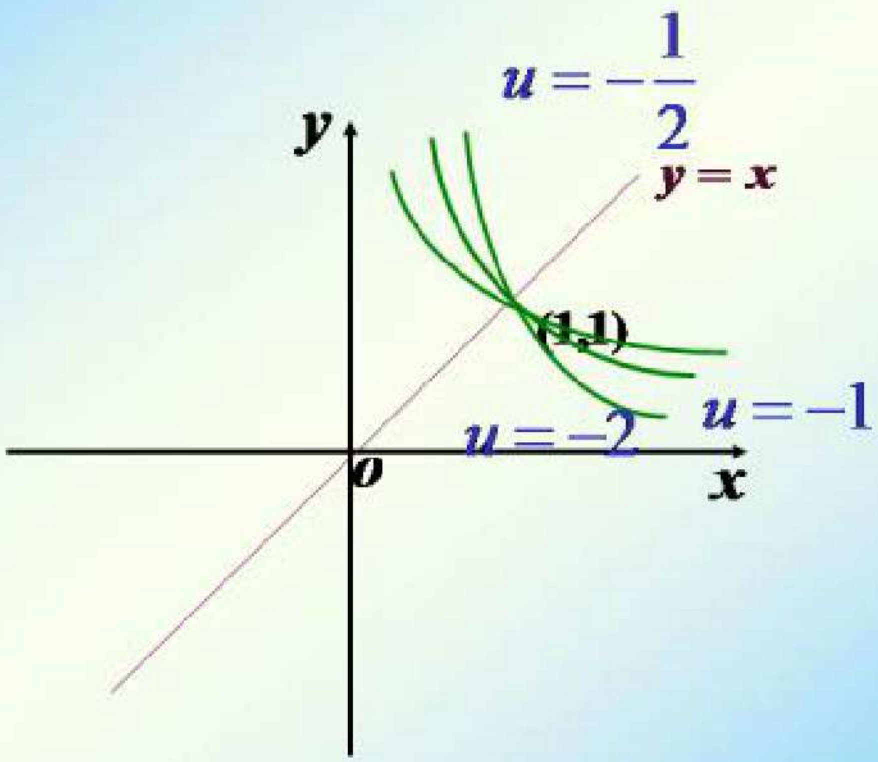
3.1.3 Power functions

$$y = x^u \ (u \neq 0).$$

① If $u > 0$, then $y = x^u \ (x > 0)$ is increasing;



② If $u<0$, then $y=x^u\ (x>0)$ is decreasing.



3.1.4 Trigonometric functions

1 $y = \sin x$; 2 $y = \cos x$;

3 $y = \tan x$; 4 $y = \cot x$;

5 $y = \sec x$; 6 $y = \csc x$.

3.1.5 Anti-trigonometric functions

1 $y = \arcsin x$; 2 $y = \arccos x$;

3 $y = \arctan x$; 4 $y = arc\cot x$;

5 $y = arc\sec x$; 6 $y = arc\csc x$.

3.1.6 Hyperbolic functions

1 $shx=\dfrac{e^x-e^{-x}}{2}$; 2 $chx=\dfrac{e^x+e^{-x}}{2}$;

3 $thx=\dfrac{shx}{chx}$; 4 $cthx=\dfrac{chx}{shx}$.

Propositions (1) $ch^2x=1+sh^2x$;

(2) $sh2x=2shxchx$, $ch2x=sh^2x+ch^2x$;

(3) $ch(x\pm y)=chxchy\pm shxshy$;

(4) $sh(x\pm y)=shxchy\pm chxshy$.

3.2 Elementary functions

An elementary function is a function which is obtained from the above six classes of basic elementary functions by using finitely many times of four arithmetic operations and composition operations.

Homework Page 22:3 (1,4); 5; 6; 7(3, 4).

Page 28: 3(1). Page29: 5; 9; 10.