Lecture 5 Four arithmetic operations, and monotonic and bounded sequences

§ 1 Four arithmetic operations

Theorem 1.1 If $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$,

then $$\lim_{n\to\infty}(x_n \pm y_n) = a \pm b.$$

The proof directly follows from the definition.

Theorem 1.2 If $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} x_n = b$,

then $$\lim_{n\to\infty}(x_n y_n) = ab.$$

The proof of Theorem 1.2 follows from the following two statements:

(1) The definition of the limit; and

(2) If $\lim_{n\to\infty} x_n = a$, then $\{x_n\}$ is bounded.

Remark 1.1 In general, the inverses of Theorem 1.1 and 2.2 are not valid. We can take $x_n = n,\ \ y_n = -n$, and $x_n = n,\ \ y_n = \frac{1}{n}$ as counterexamples.

Theorem 1.3 If $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b \neq 0$, then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}.$$

Proof For any $\varepsilon > 0$, there is some $N_1 > 0$ such that for all $n > N_1$,

$$a-\varepsilon < x_n < a+\varepsilon .$$

And there is some $N_2 > 0$ such that for all $n > N_2$,

$$b-\varepsilon < y_n < b+\varepsilon .$$

It follows from $by_n \to b^2 > \frac{b^2}{2}$ that there is some $N_3 > 0$,

Such that for all $n > N_3$,

$$|by_n| > \frac{b^2}{2}.$$

Since

$$\left|\frac{x_n}{y_n} - \frac{a}{b}\right| = \frac{|bx_n - ay_n|}{|by|} \le \frac{|b|\,|x_n - a| + |a|\,|y_n - b|}{|by|},$$

By Let $N=\max\{N_1,N_2,N_3\}$, we see that for all $n>N$

$$\left|\frac{x_n}{y_n}-\frac{a}{b}\right|\le\frac{2}{b^2}\,(|a|+|b|)\varepsilon .$$

This shows that

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\frac{a}{b}.$$

Examples 1.1 Find the following limits.

(1) $\lim\limits_{n\to\infty}\dfrac{n^2+2n+5}{n^2+1}$;

(2) $\lim\limits_{n\to\infty}\dfrac{1}{n}\left[(x+\dfrac{1}{n}a)+(x+\dfrac{2}{n}a)+\cdots+(x+\dfrac{n-1}{n}a)\right]$;

(3) $\lim\limits_{n\to\infty}\sin^2\left(\pi\sqrt{n^2+n}\right)$..

Hint of (3) ① $\sin^2 x=\dfrac{1-\cos 2x}{2}$;

② $\cos x=\cos(2n\pi-x)$.

Examples 1.2 Find the error in the following inference.

$$1=\lim_{n\to\infty}(n\frac{1}{n})=\lim_{n\to\infty}n\cdot\lim_{n\to\infty}\frac{1}{n}=0.$$

Theorem 1.4 $\lim\limits_{n\to\infty}x_n=A$ if and only if there is some sequence $\{\varepsilon_n\}$ such that $x_n=A+\varepsilon_n$ with $\lim\limits_{n\to\infty}\varepsilon_n=0$.

By letting $\varepsilon_n=x_n-A$, the proof easily follows.

§ 2 Monotonic and bounded sequences

We give the following result as an axiom.

Theorem 2.1 If $\{x_n\}$ is monotonic and bounded,

then $\lim_{n\to\infty} x_n$ exists.

Example 2.1 Suppose $a > 0$ is a constant.

Let $y_1 = \sqrt{a}$, $y_2 = \sqrt{a+\sqrt{a}}$, $\cdots$, $y_n = \underbrace{\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}_{n}$, $\cdots$.

First prove that $\lim_{n\to\infty} y_n$ exists and then find this limit.

Proof ① Obviously, $y_{n+1} > y_n$;

② $y_{n+1}^2 = a + y_n < a + y_{n+1}$, $y_{n+1} < \sqrt{a} + 1$.

These show that $\{y_n\}$ is decreasing and bounded.

Theorem 2.1 implies that $\lim\limits_{n\to\infty} y_n$ exists.

Assume that $l = \lim\limits_{n\to\infty} y_n$. It follows from $l = \sqrt{a+1}$ that

$$l^2 - l - a = 0.$$

We see that

$$l = \frac{1+\sqrt{1+4a}}{2}.$$

Example 2.2 Let $\{y_n = (1+\frac{1}{n})^n\}$. Prove $\lim\limits_{n\to\infty} y_n$ exists.

Proof ① Since

$$y_n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n!} + \cdots + \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n}).$$

and

$$y_{n+1} = 1 + 1 + \frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1}) + \cdots$$

$$+ \frac{1}{n!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1})$$

$$+ \frac{1}{(n+1)!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n}{n+1}),$$

We see that

$$y_n < y_{n+1}.$$

It follows that $\{y_n\}$ is increasing.

②
$$0 < y_n < 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}$$
$$=1+1+\frac{1}{1.2}+\cdots+\frac{1}{(n-1)n}$$
$$=1+1+1-\frac{1}{n}<3.$$

Hence $\{y_n\}$ is increasing and bounded. Theorem 2.1 implies that $\lim_{n\to\infty} y_n$ exists, which is denoted by

$$e = 2.71828182845926\cdots, \text{ i.e.,}$$

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e.$$

Example 2.3 Let $x_n = \frac{n^k}{a^n}$, where $a > 1$ and $k > 0$ are constant. Prove $\lim_{n\to\infty} x_n$ exists.

Hint: Since

$$\lim_{n\to\infty}(1+\frac{1}{n})^k = 1$$

and $a > 1$, we easily know that there is some $N > 0$ such that for all $n > N$,

$$(1+\frac{1}{n})^k < a.$$

Example 2.4 Let $x \in R$ and $y_n = \underbrace{\sin \sin \cdots \sin}_{n} x$.

Prove $\lim_{n\to\infty} y_n(x)$ exists.

Proof ① without loss of generality, we may assume that $\sin x > 0$. It follows that

$$y_n(x) < y_{n-1}(x).$$

This shows that $\{y_n(x)\}$ is decreasing.

② Obviously, $0 < y_n(x) < \sin x$.

The above discussions show that $\{y_n(x)\}$ is decreasing and bounded. Theorem 2.1 implies that $\lim_{n\to\infty} y_n$ exists, which is denoted by l. Then $l = \sin l$. Hence $l = 0$.

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