Lecture 6 Infinitesimals and infinity

§ 1 Infinitesimals

Definition 1.1 If $\lim_{n\to\infty} x_n = 0$, then $\{x_n\}$ is called an infinitesimal.

For example, when $n \to \infty$, both sequences $\{\frac{1}{n}\}$ and $\{0\}$ are infinitesimals.

Theorem 1.1 If $\{x_n\}$ is bounded and $\{y_n\}$ is an infinitesimal, then $\{x_n y_n\}$ is still an infinitesimal.

Proof since $\{x_n\}$ is bounded, we see that there is some $M>0$ such that for all $n>0$,

$$|x_n| \le M.$$

It follows from $\{y_n\}$ being an infinitesimal that for any $\varepsilon>0$, there is some $N>0$ such that for all $n>N$,

$$|y_n| \le \varepsilon.$$

Hence for all $n>N$,

$$|x_n y_n| \le M\varepsilon.$$

This shows that

$$\lim_{n\to\infty} x_n y_n = 0.$$

Example 1.1 Find the limits.

(1) $\lim\limits_{n\to\infty}\frac{\sin n^2}{\sqrt{n}}$; (2) $\lim\limits_{n\to\infty}(2+\frac{1}{n})^2\sin\frac{\sqrt[3]{n}}{\sqrt{n}}$.

§ 2 Infinity

Definition2.1 If $\lim\limits_{n\to\infty} x_n=\infty$, then $\{x_n\}$ is called an infinity.

This is equivalent to the following:

For any $G>0$, there is some $N>0$ such that for all $n>N$,

$$|x_n|>G.$$

There are two cases:

Case I $\lim_{n\to\infty} x_n = +\infty$ if and only if for any $G>0$, there is some $N>0$ such that for all $n>N, x_n > G$.

Case II $\lim_{n\to\infty} x_n = -\infty$ if and only if for any $G>0$, there is some $N>0$ such that for all $n>N, x_n < -G$.

Example 2.1 Prove $\{2^n\}$ is an infinity.

Proof For any $G>0$, let $2^n > G$.

Take $N = [\log_2^G]+1$. Then for all $n>N$,

$$|2^n| > G.$$

The proof is finished.

§ 3 Their relationships

Theorem 3.1 If $\{x_n\}$ is an infinity, then $\{\frac{1}{x_n}\}$ is an infinitesimal, and the converse also holds if for each n, $x_n \neq 0$.

Proof For any $\varepsilon > 0$, let $G = \frac{1}{\varepsilon}$. Then the hypothesis $\{x_n\}$ being an infinity implies that there is some $N > 0$ such that for all $n > N$,

$$|x_n| > \frac{1}{\varepsilon}.$$

Hence $\left|\frac{1}{x_n}\right| < \varepsilon$. It follows that $\lim_{n\to\infty}\frac{1}{x_n} = 0$.

The proof for the converse is the same.

Theorem 3.2 If both $\{x_n\}$ and $\{y_n\}$ are positive (resp. negative) infinities, then $\{x_n + y_n\}$ is still an infinity.

The proof is obvious.

Theorem 3.3 If $\{x_n\}$ is an infinity and $\{y_n\}$ is bounded, Then $\{x_n + y_n\}$ is still an infinity.

The proof is obvious.

Theorem 3.4 If $\{x_n\}$ is an infinity and $\{y_n\}$ satisfies that there is some $N>0$ and $q>0$ such that for all $n>N$, $|y_n|\geq q$, then $\{x_n y_n\}$ is an infinity.

Proof Since $\{x_n\}$ is an infinity, we see that there is $G>0$ and $N>0$ such that for all $n>N$,

$$|x_n|>G \text{ and } |x_n y_n|>Gq.$$

These show that $\{x_n y_n\}$ is an infinity.

The following is a direct consequence of Theorem 3.4.

Corollary 3.1 If $\{x_n\}$ is an infinity and $\lim_{n\to\infty} y_n = a \neq 0$, then $\{x_n y_n\}$ is an infinity.

Remark 3.1 The condition $a \neq 0$ is necessary. For example, we take $x_n = n$ and $y_n = \frac{1}{n}$.

Example 3.1 Let $x_n = \dfrac{a_0 n^k + a_1 n^{k-1} + \cdots + a_k}{b_0 n^l + b_1 n^{l-1} + \cdots + b_l}$, where $a_0 \neq 0$, $b_0 \neq 0$. Discuss the existence of $\lim_{n\to\infty} x_n$.

Example 3.2 Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof Let $\sqrt[n]{n} = 1 + x_n$. Then

$$\frac{1}{2}n(n-1)x_n^2 < n.$$

This implies that for all $n \geq 2$,

$$0 < x_n < \sqrt{\frac{2}{n-1}}.$$

This yields that

$$\lim_{n\to\infty} x_n = 0.$$

Hence

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Remark 3.2 As an exercise, please show that for any $a > 0$, $\lim_{n\to\infty} \sqrt[n]{a} = 1$.

Example 3.3 Find $\lim_{n\to\infty} \frac{3n+\sin n}{n^2+\cos n+1}[\arctan n+\sin^2 n]$.

Example3.4 If $a_n > 0$ and $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{\frac{1}{n}} = a$

Proof We divide our discussions into two cases.

Case 1 $a = 0$.

It follows that $\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = 0$.

By using the fact

$$0 < \sqrt[n]{a_1 a_2 \cdots a_n} < \frac{a_1+a_2+\cdots+a_n}{n},$$

We see that $\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{\frac{1}{n}} = 0.$

Case 2 $a > 0$

It follows that $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

This implies that

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a, \quad \lim_{n\to\infty} \frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n} = \frac{1}{a}$$

and

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} = a.$$

It follows from the following double inequalities

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}} < \sqrt[n]{a_1 a_2 \cdots a_n} < \frac{a_1+a_2+\cdots+a_n}{n}$$

that

$$\lim_{n\to\infty}(a_1 a_2 \cdots a_n)^{\frac{1}{n}} = a.$$

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