Lecture 7 Limits of functions

§ 1 Definition of the limit of a function at x_0

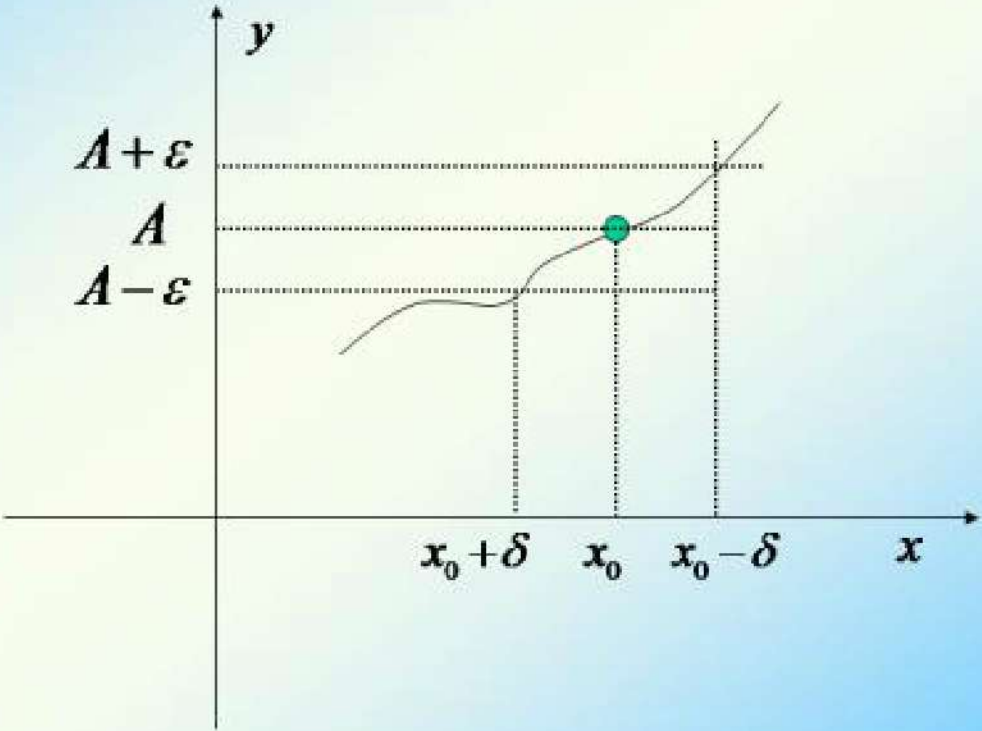
Definition 1.1 Let $f(x)$ be well defined on a neighbourhood $O(x_0,\delta)$ of x_0 with x_0 deleted (which is denoted by $O(\hat{x}_0,\delta)$ in the following) and A a constant.

If for any $\varepsilon > 0$, there is some $\delta > 0$ such that for all x:

$$0 < |x - x_0| < \delta, \quad |f(x) - A| < \varepsilon,$$

then we call A the limit of $f(x)$ at x_0, denoted by

$$\lim_{x \to x_0} f(x) = A.$$



Examples 1.1 (1) Prove $\lim_{x\to 0}(x\sin\frac{1}{x})=0$;

(2) $\lim_{x\to 1}\frac{(x-2)(x-1)}{x-3}=0$;

(3) $\lim_{x\to 2}\frac{x^2+1}{2x+1}=1$.

Theorem 1.1 If $\lim_{x\to x_0} f(x)=A$ then $\lim_{x\to x_0}|f(x)|=|A|$.

The proof easily follows from the definition and the following estimate:

$$\big||f(x)|-|A|\big|\le|f(x)-A|.$$

The function sgn(x) shows that the converse of Theorem 1.1 does not hold.

Theorem 1.2 $\lim_{x\to x_0} f(x)=0$ if and only if $\lim_{x\to x_0}|f(x)|=0$.

The proof is obvious.

§2 Operations

Theorem 2.1 If $\lim_{x\to x_0} f(x)=A$, and $\lim_{x\to x_0} g(x)=B$, then

$\lim_{x\to x_0}(f(x)\pm g(x))=A\pm B$ and $\lim_{x\to x_0}(g(x)f(x))=AB$.

Theorem 2.2 If $\lim_{x\to x_0} f(x)=A$, and $\lim_{x\to x_0} g(x)=B\neq 0$, then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{A}{B}.$$

Proof It follows from $\lim\limits_{x \to x_0} f(x) = A$ and $\lim\limits_{x \to x_0} g(x) = B$, that for any $\varepsilon > 0$, there is some $\delta_1 > 0$ such that for all $x \in O(\hat{x}_0, \delta_1)$,

$$|f(x) - A| < \varepsilon ;$$

and there is some $\delta_2 > 0$ such that for all $x \in O(\hat{x}_0, \delta_2)$,

$$|g(x) - B| < \varepsilon .$$

Since $\lim\limits_{x \to x_0} Bg(x) = |B^2|$, we see that there is $\delta_3 > 0$ such that for all $x \in O(\hat{x}_0, \delta_3)$,

$$\frac{1}{2} B^2 < Bg(x) < \frac{3}{3} B^2 .$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\delta > 0$ and for all $x \in O(\hat{x}_0, \delta)$,

$$\left|\frac{f(x)}{g(x)} - \frac{A}{B}\right| < \left|\frac{Bf(x) - Ag(x)}{Bg(x)}\right| < \frac{2(|A|+|B|)}{B^2}\varepsilon.$$

This shows that

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{A}{B}.$$

Example 2.1 Show that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

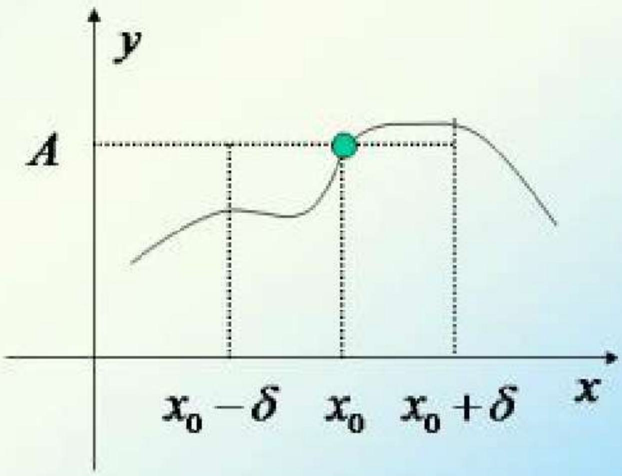
Theorem 2.3 $\lim_{x\to x_0} f(x) = A$ if and only if there is some sequence $\{\varepsilon(x)\}$ such that $f(x) = A + \varepsilon(x)$ with

$$\lim_{x\to x_0}\varepsilon(x) = 0.$$

By letting $\varepsilon(x) = f(x) - A$, the proof easily follows.

§3 Properties (I)

Theorem 3.1 If $\lim_{x \to x_0} f(x) = A > 0$, then there is some $\delta > 0$ such that for all $x \in O(\hat{x}_0, \delta)$, $f(x) > 0$.



Proof Let $\varepsilon=\dfrac{A}{2}$. Then there is some $\delta>0$ satisfying

there exists some neighbourhood $O(x_0,\delta)$ of x_0 such that

for all $x\in O(\hat{x}_0,\delta)$,

$$|f(x)-A|<\frac{A}{2}.$$

This implies that

$$0<\frac{A}{2}<f(x)<\frac{3}{2}A.$$

Corollary 3.1 If $\lim\limits_{x\to x_0} f(x)=A$, $\lim\limits_{x\to x_0} g(x)=B$ and $A>B$,

then there exists some $\delta>0$ such that for all $x\in O(x_0,\delta)$,

$$f(x)>g(x).$$

By letting $F(x) = f(x) - g(x)$, we easily know that the proof follows from Theorem 3.1.

Corollary 3.2 If $f(x) \geq 0$ and $\lim_{x \to x_0} f(x)$ exists, then $\lim_{x \to x_0} f(x) \geq 0$.

Proof Suppose $\lim_{x \to x_0} f(x) < 0$. Then by Theorem 3.1, it is impossible.

Corollary 3.3 If $\lim_{x \to x_0} f(x) = A > B$, then there is some $\delta > 0$ such that for all $x \in O(\hat{x}_0, \delta)$, $f(x) > B$.

Corollary 3.4 (Uniqueness) If $\lim_{x\to x_0} f(x) = A$ and $\lim_{x\to x_0} f(x) = B$, then $A = B$.

Theorem 3.2 If there is some $\delta > 0$ such that for all $x \in O(\hat{x}_0, \delta)$,

$$f(x) \le g(x) \le h(x)$$

and , $\lim_{x\to x_0} f(x) = A = \lim_{x\to x_0} h(x)$, then $\lim_{x\to x_0} g(x) = A$.

Proof For any $\varepsilon > 0$, there is some $\delta_1 > 0$ such that for all $x \in O(\hat{x}_0, \delta_1)$,

$$A - \varepsilon < f(x) < A + \varepsilon .$$

Also there is some $\delta_2 > 0$ such that for all $x \in O(\hat{x}_0, \delta_2)$,

$$A-\varepsilon < h(x) < A+\varepsilon .$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x \in O(\hat{x}_0, \delta)$,

$$A-\varepsilon < g(x) < A+\varepsilon .$$

This shows that

$$|g(x)-A| < \varepsilon ,$$

which implies that

$$\lim_{x \to x_0} g(x) = A.$$

Homework: Page76: 1 (1,3); 2(1)