Lecture 10 Continuity of functions (I)

§ 1 The concept of continuity

Definitions 1.1 Suppose $y = f(x)$ is defined in a neighbourhood $O(x_0, r)(r > 0)$ of x_0. If $\lim_{x \to x_0} f(x) = f(x_0)$, then $y = f(x)$ is called continuous at x_0. That is, for any $\varepsilon > 0$, there is some $\delta > 0$ such that for any $x \in O(x_0, \delta)$,

$$|f(x) - f(x_0)| < \varepsilon.$$

If $f(x)$ is continuous at any $x \in X$, then $f(x)$ is called continuous in X.

Definitions 1.2 $f(x)$ is called right-sided continuous if $\lim\limits_{x \to x_0+0} f(x) = f(x_0)$ and left-sided continuous at x_0 if $\lim\limits_{x \to x_0-0} f(x) = f(x_0)$.

Theorem 1.1 $f(x)$ is continuous at x_0 if and only if $f(x)$ is not only right-sided continous, but also left-sided continous at x_0.

The proof easily follows from that of the limit case.

Example 1.1 Suppose $f(x)=\begin{cases}\sin x, x\geq 0\\ e^x, x<0\end{cases}$. Then $f(x)$ is right-sided continuous, but not left-sided continuous at $x=0$.

Example 1.2 Show that the functions $\sin x$, $\cos x$, x^2 are continous in R.

§2 Four arithmetic operations and properties of continuous functions

Theorem 2.1 If both $f(x)$ and $g(x)$ are continuous at x_0, then all functions $f(x)\pm g(x)$, $f(x)g(x)$ and

$\frac{f(x)}{g(x)}$ $(g(x_0) \neq 0)$ are continuous at x_0.

The proofs easily follow from similar arguments in those of corresponding limit cases.

Example 2.1 All functions $\tan x$, $\cot x$, $\sec x$ and $\csc x$ are continuous in their domains.

Theorem 2.2 If $f(x)$ is strictly increasing (resp. decreasing) and continuous in [a,b], $f(a)=\alpha$ and $f(b)=\beta$, then the inverse $x=f^{-1}(y)$ of $y=f(x)$ exists in $[\alpha,\beta]$ (resp. $[\beta,\alpha]$) and is also continuous and strictly increasing (resp. decreasing) in $[\alpha,\beta]$ (resp. $[\beta,\alpha]$).

We still need to prove the following two facts (we assume that $f(x)$ is strictly increasing):

(1) The domain of $f^{-1}(y)$ is $[\alpha,\beta]$;

(2) $x=f^{-1}(y)$ is continuous in $[\alpha,\beta]$.

We will prove these two facts later.

Example 2.2 Both functions $\arctan x$ and $\arcsin x$ are continuous in their domains.

Theorem 2.3 If $u=g(x)$ is continuous at x_0, $g(x_0)=u_0$, and $y=f(u)$ is continuous at u_0, then $y=f(g(x))$ is continuous at x_0.

Proof The hypothesis $\lim\limits_{u \to u_0} f(u) = f(u_0)$ shows that for any $\varepsilon > 0$, there is some $\eta > 0$ such that for all $x \in O(u_0, \eta)$,

$$|f(u) - f(u_0)| < \varepsilon.$$

The hypothesis $\lim\limits_{x \to x_0} g(x) = g(x_0)$ shows that for the given $\eta > 0$, we can find some $\delta > 0$ such that for all $x \in O(u_0, \delta)$,

$$|g(x) - g(x_0)| < \eta.$$

The discussions as above show that

$$|f(g(x)) - f(g(x_0))| < \varepsilon.$$

§ 3 The continuity of elementary functions

3.1 Trigonometric functions and their inverse functions

Proposition 3.1 All trigonometric functions and their inverse functions are continuous on their domains.

3.2 Exponent and logarithm functions

Proposition 3.2 a^x and $\log_a x$ are continuous.

Proof It suffices to prove the continuity of the function a^x. We divide our discussions into two cases.

Case1 $a > 1$

Claim 1 $\lim_{x \to 0} a^x = 1$.

If $x > 0$, let $x = \frac{1}{t}$. Then $x \le \frac{1}{[t]}$ and $1 \le a^x \le a^{\frac{1}{[t]}}$.

This yields that

$$\lim_{x \to 0+} a^x = 1 = a^0.$$

If $x < 0$, let $s = -x$. Then $s > 0$ and

$$\lim_{x \to 0-} a^x = \lim_{s \to 0+} a^{-s} = \frac{1}{\lim_{s \to 0+} a^s} = 1.$$

Claim 2 $\lim_{x \to x_0} a^x = a^{x_0}$.

Since $\lim_{x \to x_0} a^{x-x_0} = 1$, Claim 1 implies that

$$\lim_{x \to x_0}(a^x - a^{x_0}) = \lim_{x \to x_0} a^{x_0}(a^{x-x_0} - 1) = 0.$$

Hence $\lim_{x \to x_0} a^x = a^{x_0}$.

Case 2 $a < 1$

It follows from Case 1 that

$$\lim_{x \to x_0} a^x = \frac{1}{\lim_{x \to x_0}(\frac{1}{a})^x} = \frac{1}{(\frac{1}{a})^{x_0}} = a^{x_0}.$$

This completes the proof.

3.3 Power functions

Proposition 3.3 x^a is continuous on $(0,+\infty)$.

3.4 Hyperbolic functions

Proposition3.4 All hyperbolic functions are continuous.

§4 Classifications of the points of discontinuity

4.1 The points of discontinuity of first class

Definition 4.1 A point of discontinuity of $f(x)$ is said to be of first class if both $f(x_0+0)$ and $f(x_0-0)$ exist, but $f(x_0+0) \neq f(x_0-0)$.

Example 4.1 Classify the points of continuity of $y = x[x]$.

Hint:

$$f(x)=\begin{cases}-nx, & x\in[-n,-n+1)\\ \vdots & \\ -x, & x\in[-1,0)\\ \vdots & \\ 0, & x\in[0,1)\\ x, & x\in[1,2)\\ \vdots & \\ nx, & x\in[n,n+1)\end{cases}.$$

4.2 The points of discontinuity of second class

Definition 4.2 A point of discontinuity of $f(x)$ is said to be of second class if at least one of $f(x_0+0)$ and $f(x_0-0)$ does not exist.

Example 4.2.1 Classify the points of discontinuity of functions

$$y=\sin\frac{1}{x} \text{ and } y=e^{\frac{1}{x}}.$$

4.3 Removable discontinuities

Definition 4.3 A point of discontinuity of $f(x)$ is said to be removable if both $f(x_0+0)$ and $f(x_0-0)$ exist, and $f(x_0+0)=f(x_0-0)\neq f(x_0)$.

Example 4.3 Let $y=\begin{cases} x\sin\dfrac{1}{x}, & x\neq 0 \\ 2, & x=0 \end{cases}$.

Find and classify the points of discontinuity of y.

Example 4.4 Suppose that both $f(x)$ and $g(x)$ is continuous on (a,b). Define

$$h(x)=\max\{f(x),g(x)\}$$

Show that $h(x)$ is continuous on (a,b).

Hint: :

$$h(x)=\max\{f(x),g(x)\}=\frac{1}{2}\{f(x)+g(x)+|f(x)-g(x)$$

Example 4.5 Classify the point of discontinuity of $f(x)$ where

$$f(x)=\begin{cases}\dfrac{\log(1+x)}{x}, & x>0\\ 0 & x=0\\ \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}, & x<0\end{cases}.$$

Hint: Find the limits

$$\lim_{x\to 0+0}\frac{\log(1+x)}{x} \quad \text{and} \quad \lim_{x\to 0-0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

Homework: Page93: 1(3);
Page94: 7; 9; 12; 14(1, 3, 5, 7); 15(1)