Lecture 11 Continuity of functions (II) and the concept of the orders of infinitesimals and infinities

§ 1 Properties of continuous functions on closed intervals

1.1 The Weierstrass boundedness theorem

Theorem 1.1 Suppose $f(x)$ is continuous in $[a, b]$. Then $f(x)$ is bounded in $[a, b]$.

1.2 The Weierstrass maximal-value theorem

Theorem 1.2 Suppose $f(x)$ is continuous in $[a, b]$.

Then $f(x)$ attains its maximum value and its minimum value in $[a, b]$.

1.3 The root existence theorem

Theorem 1.3 Suppose $f(x)$ is continuous in $[a, b]$ and $f(a)f(b) < 0$. Then there is some $\zeta \in [a,b]$ such that

$$f(\zeta) = 0.$$

1.4 The Bolzano-Cauchy intermediate-value theorem

Theorem 1.4 Suppose $f(x)$ is continuous in $[a, b]$. Let $M = \max_{x\in[a,b]} \{f(x)\}$ and $m = \min_{x\in[a,b]} \{f(x)\}$.

Then for any $q \in [m, M]$, there is some $\xi \in [a,b]$ such that

$$f(\xi) = q.$$

Hint Consider $g(x) = f(x) - q$. The proof easily follows from Theorem 1.3.

1.5 The cantor-Heine theorem on uniformly continuity

Definition 1.5 $f(x)$ is called uniformly continuous in X if for any $\varepsilon > 0$, there is some $\delta > 0$ which depends only on ε such that for any pair x_1, $x_2 \in X$, if $|x_1 - x_2| < \delta$, then

$$|f(x_1) - f(x_2)| < \varepsilon .$$

Example 1.5 Prove that $\sin\frac{1}{x}$ is uniformly continuous on $(c,1)$ $(c > 0)$, but not on $(0, 1)$.

Hint (1) $\left|\sin\frac{1}{x_1} - \sin\frac{1}{x_2}\right| \le \frac{|x_1 - x_2|}{2|x_1 x_2|}$;

(2) Let $x_{n1} = \frac{1}{2n\pi}$, and $x_{2n} = \frac{1}{2n\pi + \frac{\pi}{2}}$.

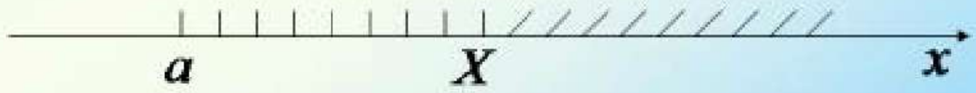
Theorem 1.5 Suppose $f(x)$ is continuous on $[a, b]$. Then $f(x)$ is uniformly continuous on $[a, b]$.

§ 2 Added exercises

Example 2.1 If $f(x)$ is continuous in $[a,+\infty)$ and $\lim_{x\to+\infty} f(x)$ exists, then $f(x)$ is bounded on $[a,+\infty)$.

Hint By the existence of $\lim_{x\to+\infty} f(x)$, we divide $[a,+\infty)$ into the union of two parts, $[a,+\infty)=[a,X]\cup(X,+\infty)$.

Then $f(x)$ is bounded in $[a,+\infty)$, we divide into $[a,X]$ and $(X,+\infty)$, respectively.

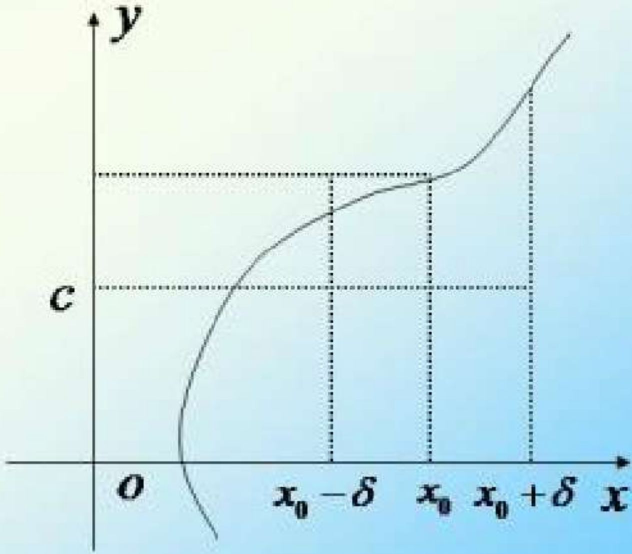


Example 2.2 If $f(x)$ is continuous at x_0 and $f(x_0)>0$, then there is some neighborhood $O(x_0,\delta)$ of x_0 such that for all

$x \in O(x_0,\delta)$, $f(x) \ge c$, where c is a positive constant.

Hint In the definition of $\lim_{x\to x_0} f(x) = f(x_0)$, we take

$$\varepsilon = \frac{f(x)}{2}.$$



Example 2.3 If $f \in C[a,b]$ and $a < x_1 < \cdots < x_n < b$, then there is some $\xi \in [x_1, x_n]$ such that

$$f(\xi) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Hint By using the weierstrass-Cauchy maximal-value theorem and the intermediate value theorem in $[x_1, x_n]$.

§ 3 Infinitesimals

Suppose $\lim_{x \to x_0} f(x) = 0 = \lim_{x \to x_0} g(x)$.

3.1 The comparison of infinitesimals

Definition 3.1 (1) $f(x)$ is higher order than $g(x)$ if

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=0;$$

(2) $f(x)$ and $g(x)$ are of the same order if

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=A\neq 0;$$

(3) $f(x)$ is equivalent to $g(x)$ if $\lim_{x\to x_0}\frac{f(x)}{g(x)}=1$,

which is denoted by

$$f(x)\sim g(x).$$

Example 3.1 Show that $\sin x\sim x$, $\tan x\sim x$ and $\ln(x+1)\sim x$.

Example 3.2 Find the following limits.

① $\lim_{x\to\infty}(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x})$; ② $\lim_{x\to 0}(1+\sin x)^{\frac{1}{2x}}$;

③ $\lim_{x\to\infty}\frac{\log(x^2-x+1)}{\log(x^{10}+x+1)}$; ④ $\lim_{x\to\infty}(\sin\frac{1}{x}+\cos\frac{1}{x})^x$.

Hint of ③ $\frac{\log(x^2-x+1)}{\log(x^{10}+x+1)}=\frac{2\log|x|+\log\frac{1-x}{x^2}}{10\log|x|+\frac{x+1}{x^{10}}}$.

Homework Page 95: 17(1);

Page 97: 1(1, 3, 5, 7);

Page 98: 2(1, 3, 5).