Chapter 2

Lecture 3 The limits of sequences and their properties

§ 1 The concept of limits of sequences

1.1 Sequences

$$\{x_n\}: x_1, x_2, \cdots, x_n, \cdots.$$

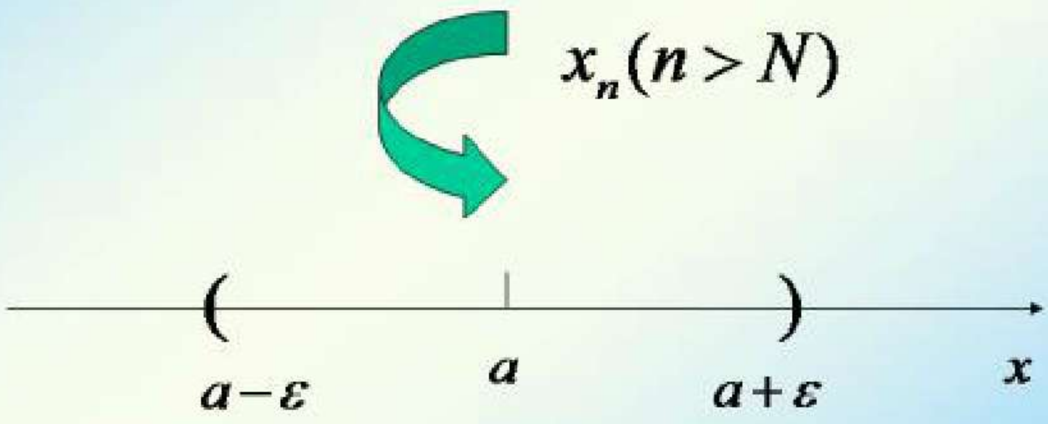
1.2 Definition of limits of sequences

Suppose $\{x_n\}$ is a sequence and a is a constant.

If for any $\varepsilon > 0$, there is some $N > 0$ such that for all

$n > N$, $|x_n - a| < \varepsilon$, then we call that the limit of $\{x_n\}$ exists, which is a. Further, we denote it by

$$\lim_{n \to \infty} x_n = a.$$



Example 1.2 (1) Prove $\lim_{n\to\infty}\frac{1}{n}=0$;

(2) Prove $\lim_{n\to\infty} q^n = 0$ if $|q|<1$;

(3) Prove $\lim_{n\to\infty}\dfrac{n^2-n+\frac{1}{4}}{4n^2+2n+1}=\frac{1}{4}$.

Theorem 1.2.1 If $\lim_{n\to\infty} x_n = A$ exists, then $\lim_{n\to\infty}|x_n|=|A|$.

The proof easily follows from the following inequality:

$$||x_n|-|A|| \le |x_n - A|.$$

By letting $x_n=(-1)^n$, we easily know that the converse of Theorem 1.2.1 does not hold.

Theorem 1.2.2 $\lim\limits_{n\to\infty} x_n = 0$ if and only if $\lim\limits_{n\to\infty} |x_n| = 0$.

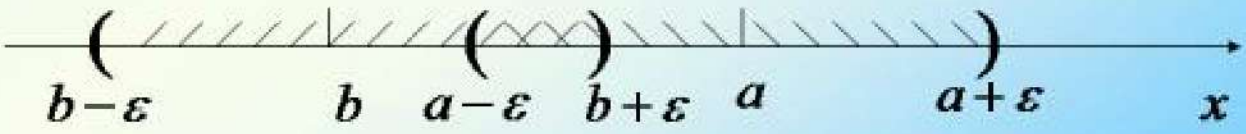
The proof is obvious.

§ 2 Properties of limits of sequences

Theorem 2.1 If $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} x_n = b$ and $a > b$, then there is some $N > 0$ such that for all $n > N$,

$$x_n > y_n.$$

Hint:



Proof Let $\varepsilon_0 = \frac{b-a}{2}$.

Then it follows from $\lim_{n\to\infty} x_n = a$ that there is some $N_1 > 0$ such that for all $n > N_1$,

$$|x_n - a| < \frac{b-a}{2}.$$

This show that for all $n > N_1$,

$$x_n < \frac{a+b}{2}.$$

It follows from $\lim_{n\to\infty} x_n = b$ that there is some $N_2 > 0$ such that for all $n > N_2$,

$$|y_n - a| < \frac{b-a}{2}.$$

This show that for all $n > N_2$,

$$y_n > \frac{a+b}{2}.$$

Let $N > \max\{N_1, N_2\}$, then for all $n > N$,

$$x_n < \frac{a+b}{2} < y_n.$$

The proof is finished.

Similar arguments can show the following.

Theorem 2.2 If $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} x_n = b$ and $a<b$, then there is some $N>0$ such that for all $n>N$,

$$x_n < y_n.$$

Corollary 2.1 If $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} x_n = b$ and there is some $N>0$ such that for all $n>N$, $x_n \le y_n$, then

$$a \le b.$$

In particular, if for all $n>N$, $x_n \le b$, and $\lim\limits_{n\to\infty} x_n = a$ then $a \le b$.

Corollary 2.2 If $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} x_n = b$ and there is some $N>0$ such that for all $n>N$, $x_n \ge y_n$, then

$$a \ge b.$$

In particular, if for all $n > N$, $x_n \le b$, and $\lim_{n\to\infty} x_n = a$ then $a \le b$.

The proofs of Corollaries 2.1 and 2.2 easily follow from Theorems 2.1 and 2.2 by contradiction.

Remark 2.1 Although the condition $x_n > y_n$ is satisfied, in general, we can not always have $a > b$.

For example, $x_n = 1 + \frac{2}{n}$, $y_n = 1 + \frac{1}{n}$.

Corollary 2.3 If $\lim_{n\to\infty} x_n = a$ and $a < b$, then there is some $N > 0$ such that for all $n > N$, $x_n < b$.

Corollary 2.4 If $\lim_{n\to\infty} x_n = a$ and $a > b$, then there is some $N > 0$ such that for all $n > N$, $x_n > b$.

Corollary 2.5 If $\lim_{n\to\infty} x_n = a$ and $a < c$, then there is some $N > 0$ such that for all $n > N$, $x_n < c$.

Corollary 2.6 If $\lim_{n\to\infty} x_n = a$ and $a > b$, then there is $N > 0$ such that for all $n > N$, $x_n > c$.

Remark 2.2 In Corollaries 2.3 ~2.6, if $a = b$ or $a = c$, then the conclusions do not always hold.

For example, $x_n = 1 + \frac{(-1)^n}{n}$. Obviously, $b = c = 1$.

Theorem 2.3 If $\lim_{n \to \infty} x_n$ exists, then $\lim_{n \to \infty} x_n$ is unique.

Proof We prove this result by contradiction.

Suppose there are $A \neq B$ (without loss of generality, we may assume that $A > B$) such that

$$\lim_{n \to \infty} x_n = A \text{ and } \lim_{n \to \infty} x_n = B.$$

Let $\varepsilon_0 = \frac{A-B}{2}$. Then there is some $N_1 > 0$ such that for all $n > N_1$,

$$|x_n - A| < \frac{A-B}{2}.$$

This implies that for all $n > N_1$,

$$x_n > \frac{A+B}{2}.$$

Also there is some $N_2 > 0$ such that for all $n > N_2$,

$$|x_n - B| < \frac{A-B}{2}.$$

This shows that for all $n > N_2$,

$$x_n < \frac{A+B}{2}.$$

Let $N > \max\{N_1, N_2\}$. Then for all $n > N$,

$$\frac{A+B}{2} < x_n < \frac{A+B}{2}.$$

This contradiction completes the proof.

Remark 2.3 In fact, the proof of Theoroem 2.3 easily follows from Theorem 2.1.

Example 2.1 Prove $\lim_{n\to\infty}\frac{n^2-n+2}{3n^2+2n-4}=\frac{1}{3}$.

Proof Since

$$\left|\frac{n^2-n+2}{3n^2+2n-4}-\frac{1}{3}\right|=\left|\frac{5n-10}{3n^2+6(n^2+n-2)}\right|,$$

We see if $n\geq 2$,then

$$\left|\frac{n^2-n+2}{3n^2+2n-4}-\frac{1}{3}\right|\leq\frac{n-2}{3n^2+6(n^2+n-2)}<\frac{5}{3n}.$$

This shows that for all $\varepsilon>0$, if

$$\frac{5}{3n} < \varepsilon,$$

then $$n > \frac{5}{3\varepsilon}.$$

Let $N = \max\left\{\left[\frac{5}{3\varepsilon}\right]+1, 2\right\}$. Then for all $n > N$,

$$\left|\frac{n^2 - n + 2}{3n^2 + 2n - 4} - \frac{1}{3}\right| < \varepsilon.$$

Hence $$\lim_{n\to\infty}\frac{n^2 - n + 2}{3n^2 + 2n - 4} = \frac{1}{3}.$$

Homework Page54: 2 (1, 5, 7) ; 4 (1); 5