Lecture 4 Properties and four arithmetic operations of limits of sequences

§ 1 Properties

Theorem 1.1 Let $\{x_n\}$ and $\{y_n\}$ be two sequences.

If

$$\lim_{n\to\infty} x_n = A = \lim_{n\to\infty} y_n$$

and for all n, $x_n \le z_n \le y_n$,

then

$$\lim_{n\to\infty} z_n = A.$$

Proof For any $\varepsilon > 0$, the assumption $\lim_{n\to\infty} x_n = A$ implies that there is some $N_1 > 0$ such that for all $n > N_1$,

$$A - \varepsilon < x_n < A + \varepsilon.$$

And the assumption $\lim_{n\to\infty} y_n = A$ implies that there is some $N_2 > 0$ such that for all $n > N_2$, $A - \varepsilon < y_n < A + \varepsilon$.

Let $N = \max\{N_1, N_2\}$. Then for all $n > N$,

$$A - \varepsilon < x_n \le z_n \le y_n < A + \varepsilon.$$

This shows that $\lim_{n\to\infty} z_n = A$.

Corollay 1.1 If there is some N such that for any $n > N$. $a \le z_n \le y_n$ and $\lim_{n\to\infty} y_n = a$.

Then
$$\lim_{n\to\infty} z_n = a.$$

Corollay 1.2 If there is some N such that for any $n > N$. $x_n \le z_n \le a$ and $\lim_{n\to\infty} x_n = a$.

Then
$$\lim_{n\to\infty} z_n = a.$$

Examples 1.1

(1) Find $\lim_{n\to\infty} \frac{n+1}{n^2}$;

(2) Suppose $a_i > 0$ $(i = 1, 2, \cdots, k)$. Prove that

$$\lim_{n\to\infty} \sqrt[n]{a_1^n + a_2^n + \cdots + a_k^n} = \max\{a_1^n, a_2^n, \cdots, a_k^n\}.$$

(3) Suppose $0<\alpha<1$. Prove that $\lim_{n\to\infty}[(n+1)^\alpha - n^\alpha]=0$.

Hint : The proof of (3) follows from the following estimates:

$$\begin{aligned} 0<(n+1)^\alpha - n^\alpha &= n^\alpha[(1+\frac{1}{n})^\alpha - 1] \\ &< n^\alpha[(1+\frac{1}{n}) - 1] \\ &= \frac{1}{n^{1-\alpha}}. \end{aligned}$$

Definition 1.1 A sequence $\{x_n\}$ is called bounded if there is a constant $M>0$ such that for all n,

$$|x_n| \le M.$$

Or we call $\{x_n\}$ bounded if there are two constants A and B such that $A \le x_n \le B$, where A is called a lower bound and B a upper bound.

Theorem 1.2 If $\lim_{n\to\infty} x_n$ exists, then $\{x_n\}$ is bounded.

Proof Assume that $\lim_{n\to\infty} x_n = a$ and let $\varepsilon_0 = 1$.

Then the hypothesis $\lim_{n\to\infty} x_n = a$ implies that there is some $N > 0$ such that for all $n > N$,

$$1 - a < x_n < a + 1.$$

Let $M = \max\{|x_1|, |x_2|, \cdots, |x_N|, |1+a|, |1-a|\}$. Then for all n,

$$|x_n| \le M .$$

This shows that $\{x_n\}$ is bounded.

Examples 1.2 (1) Find $\lim\limits_{n\to\infty}(1+\frac{1}{n^2})(1+\frac{2}{n^2})\cdots(1+\frac{n}{n^2})$;

Hint: (1) For $x>0$, $\frac{x}{1+x}<\log(1+x)<x$;

(2) $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$;

(3) $\frac{1}{n^2+i} \ge \frac{1}{n^2+n}$.

Examples 1.3 Prove $\lim\limits_{n\to\infty}(2n+1)=0$ does not hold.

Hint: $\lim_{n\to\infty} x_n \neq a$ if and only if there is some $\varepsilon > 0$ such that for any $N > 0$, there must exist some $n > N$ such that

$$|x_n - a| \geq \varepsilon.$$

Proof Let $\varepsilon = 1$ and $|2n+1| \geq 1$. Then $n \geq 0$. This implies that for each n,

$$|2n+1| \geq \varepsilon.$$

Hence $\lim_{n\to\infty}(2n+1) = 0$ does not hold.

Homework: Page 55: 6; 7; 8 (2); 9(1, 3, 5).
Page 56: 11.