Lecture 34 Special class for exercises in chapter 4

Example 1 (page 150-12) Suppose the intersection point of x−axis and the tangent line of the curve $y = x^n$ at $(1,1)$ is $(\xi, 0)$. Find the limit $\lim_{n\to\infty} y(\xi)$.

Solution It follows from $y' = nx^{n-1}$ that the equation of the tangent line of the curve $y = x^n$ at $(1,1)$ is

$$y = nx - (n-1).$$

This yields that the intersection point with x−axis is $(\frac{n-1}{n}, 0)$. Hence

$$\lim_{n\to\infty} y(\xi) = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n = e^{-1}.$$

Example 2 (page 150-13) Suppose the equation of the parabola is $y = x^2 + ax + b$. Find the condition for the point (x_0, y_0) to satisfy that through which there are two tangent lines, or only one tangent line or none of the parabola.

Solution It follows from $y' = 2x + a$ that the equation of the tangent line of the parabola at the point (x_1, y_1) is

$$y = (2x_1 + a)(x - x_1) + y_1.$$

Then
$$\begin{cases} y_0 = (2x_1 + a)(x_0 - x_1) + y_1 \\ y_1 = x_1^2 + ax_1 + b \end{cases},$$

which implies that

$$x_1^2 - 2x_0x_1 + y_0 - ax_0 - b = 0.$$

Hence if $y_0 < x_0^2 + ax_0 + b$, then there are two tangent lines; if $y_0 = x_0^2 + ax_0 + b$, then there is only one, and if $y_0 > x_0^2 + ax_0 + b$, there is none.

Example 3 (page 150-14) What is the constant a to satisfy that the line $y = x$ is tangent to the curve $y = \log_a x$? What are the coordinates of the tangent point?

Solution It follows from $y' = \dfrac{1}{x \log a}$ that the equation of the tangent line at (x_0, y_0) is

$$y = \frac{1}{x_0 \log a}(x - x_0) + y_0 .$$

The assumption shows that

$$\begin{cases} \frac{1}{x_0 \log a} = 1 \\ y_0 = x_0 \\ y_0 = \log_a x_0 \end{cases} .$$

Hence

$$a = e^{\frac{1}{e}}$$

and

$$x_0 = y_0 = e .$$

Example 4 (page 172-13) Find the one-sided derivatives of the following functions at which they are not derivable.

(1) $y=\left|\log|x|\right|$; (2) $y=|\tan x|$.

Hint (1)

$$y=\begin{cases}\log(-x), & x\le -1\\ -\log(-x), & -1<x<0\\ -\log x, & 0<x<1\\ \log x, & x\ge 1\end{cases}$$

Hint (2)

$$y=\begin{cases}-\tan x, & n\pi-\dfrac{\pi}{2}<x<n\pi\\ \tan x, & n\pi\le x\le n\pi+\dfrac{\pi}{2}\end{cases}.$$

Example 5 (Page 182-6) Suppose

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0 \\ 0, & x=0 \end{cases}.$$

Show that $f^{(n)}(0)=0$.

Proof Claim I $\lim_{t\to\infty}\frac{P_k(t)}{e^{t^2}}=0$, where $P_k(t)$ denotes a polynomial with degree k. The proof is obvious.

Claim II $\left(e^{-\frac{1}{x^2}}\right)^{(n)}=P_k(\frac{1}{x})e^{-\frac{1}{x^2}}$ for some polynomial, $P_k(t)$ with degree k. This can be proved by induction. By letting $t=\frac{1}{x}$, the proof follows from Claim I and II.

Example 6 (Page 182-11) Suppose

$$\varphi(x)=\frac{f(x)-f(a)}{f'(a)}\left[1+\frac{f(x)-f(a)}{f'^2(a)}\left(f'(a)-\frac{1}{2}f''(a)\right)\right].$$

Find $\varphi'(a)$ and $\varphi''(a)$.

Solution Since

$$\varphi'(x)=\frac{f'(x)}{f'(a)}\left[1+\frac{f(x)-f(a)}{f'^2(a)}\left(f'(a)-\frac{1}{2}f''(a)\right)\right]$$

$$+\frac{f(x)-f(a)}{f'^3(a)}\left(f'(a)-\frac{1}{2}f''(a)\right)f'(x),$$

we see that $\varphi'(a)=1$ and

$$\varphi''(a)=\lim_{x\to a}\frac{\varphi'(x)-\varphi'(a)}{x-a}=2.$$

Example 7 (Page 183-17) Suppose $y=e^x$. Under the following conditions, find d^2y, respectively.

(1) x is an independent variable;

(2) x is a dependent variable.

Solution (1) $d^2y=e^x dx^2$.

(2) Let $x=x(t)$. Then $dy=e^{x(t)}x'(t)\,dt$ and

$$d^2y=e^{x(t)}\left(x'(t)\right)^2 dt^2+e^{x(t)}x''(t)dt^2=e^{x(t)}\left(\left(dx(t)\right)^2+d^2x(t)\right).$$

Hence

$$d^2y=e^x\left((dx)^2+d^2x\right).$$