Lecture 35 Special class for exercises in chapter 5

Example 1 (page 190-8) Suppose for any $x \geq 0$,

$$\sqrt{x+1}-\sqrt{x}=\frac{1}{2\sqrt{x+\theta(x)}},\ \frac{1}{4}\leq\theta(x)<\frac{1}{2}.$$

Then

$$\lim_{x\to 0+0}\theta(x)=\frac{1}{4},\lim_{x\to+\infty}\theta(x)=\frac{1}{2}.$$

Proof Obviously, $\theta(x)=\dfrac{2\sqrt{x^2+x}+(1-2x)}{4}$.

The result easily follows.

Example 2 (page 205-13) Find α and β such that

$$\lim_{x\to+\infty}(\sqrt[4]{16x^4-8x^3+10x-7}-\alpha x-\beta)=0.$$

Solution Since

$$\lim_{x\to+\infty}(\sqrt[4]{16x^4-8x^3+10x-7}-\alpha x-\beta)$$

$$=\lim_{x\to+\infty}2x\left(\sqrt[4]{1-\frac{1}{2x}+\frac{5}{6x^3}-\frac{7}{16x^4}}-\frac{\alpha}{2}-\frac{\beta}{2x}\right),$$

we see that

$$\lim_{x\to+\infty}\left(\sqrt[4]{1-\frac{1}{2x}+\frac{5}{6x^3}-\frac{7}{16x^4}}-\frac{\alpha}{2}-\frac{\beta}{2x}\right)=0,$$

which shows that

$$\alpha=2.$$

It follows from

$$\sqrt[4]{1-\frac{1}{2x}+\frac{5}{6x^3}-\frac{7}{16x^4}}-\frac{\alpha}{2}-\frac{\beta}{2x}=1-\frac{1}{8x}+o(\frac{1}{x})$$

that

$$=\lim_{x\to+\infty}2x\left(\sqrt[4]{1-\frac{1}{2x}+\frac{5}{6x^3}-\frac{7}{16x^4}}-\frac{\alpha}{2}-\frac{\beta}{2x}\right)$$

$$=\lim_{x\to+\infty}\left(2x-\frac{1}{4}-2x-\beta+o(\frac{1}{x})\right)$$

$$=-\beta-\frac{1}{4}=0.$$

Hence

$$\beta=-\frac{1}{4}.$$

Example 3 (Page 228-23) Suppose that $f(x)$ is downward convex and $f'(x_0)$.

Show that

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0).$$

Proof For any $x > x_0$, let $x_1 = \frac{x_0 + x}{2}$. Then

$$\frac{f(x) - f(x_0)}{x - x_0} \geq \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Let $x_2 = \frac{x_0 + x_1}{2}$. Then

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \geq \frac{f(x_2) - f(x_0)}{x_2 - x_0}.$$

By repeating this procedure, we get a sequence $\{x_n\}$ such that

$\lim_{n\to\infty}\{x_n\} = x_0$ and

$$\frac{f(x)-f(x_0)}{x-x_0} \geq \frac{f(x_n)-f(x_0)}{x_n-x_0}.$$

Hence

$$\frac{f(x)-f(x_0)}{x-x_0} \geq \lim_{n\to\infty}\frac{f(x_n)-f(x_0)}{x_n-x_0} = f'(x_0).$$

A similar argument as above shows that for any $x < x_0$,

$$\frac{f(x)-f(x_0)}{x-x_0} \leq \lim_{n\to\infty}\frac{f(x_n)-f(x_0)}{x_n-x_0} = f'(x_0).$$

The proof is complete.

Example 4 (Page 245-3) Suppose $f''(x)$ exists. Show that $\lim\limits_{h\to 0}\dfrac{f(x+h)+f(x-h)-2f(x)}{h^2}=f''(x)$.

Proof Obviously,

$$\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}=\lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

$$=\frac{1}{2}\lim_{h\to 0}\left(\frac{f'(x+h)-f'(x)}{h}+\frac{f'(x-h)-f'(x)}{-h}\right)$$

$$=f''(x).$$

The proof is complete.