Lecture 19 Properties of definite integrals

§ 1 Properties

Proposition 1.1 If $f \in R[a,b]$, then for any $k \in R$, $kf \in R[a,b]$ and

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$

The proof is easy.

Proposition 1.2 If both $f, g \in R[a,b]$, then $f \pm g \in R[a,b]$ and

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$$

The proof follows from the definition.

Proposition 1.3 If both $f, g \in R[a,b]$, then $f \cdot g \in R[a,b]$.

Proof It follows from $f, g \in R[a,b]$ that there is some $M > 0$ such that

$$|f(x)| \leq M \text{ and } |g(x)| \leq M.$$

For any $\varepsilon > 0$, there is some $\delta > 0$ such that for any partition $P = \{x_i\}_{i=0}^{n}$ with $\lambda(p) < \delta$,

$$\sum_{i=1}^{n} \omega_i(f)\,\Delta x_i < \varepsilon \text{ and } \sum_{i=1}^{n} \omega_i(g)\,\Delta x_i < \varepsilon.$$

This yields that

$$\sum_{i=1}^{n} \omega_i(fg)\,\Delta x_i < M\sum_{i=1}^{n} \omega_i(f)\,\Delta x_i + M\sum_{i=1}^{n} \omega_i(g)\,\Delta x_i$$
$$< 2M\varepsilon.$$

Proposition 1.4 If $f(x) \geq 0$ and $f \in R[a,b]$, then $\int_a^b f(x)dx \geq 0.$

The proof directly follows from the definition.

Corollary 1.5 If $f, g \in R[a,b]$ and $f(x) \leq g(x)$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Proposition 1.6 If $f \in R[a,b]$, then $|f| \in R[a,b]$, and

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof For any $\varepsilon > 0$, there is some $\delta_1 > 0$ such that for any partition $P = \{x_i'\}_{i=0}^{n}$ with $\lambda(P_1) < \delta_1$,

$$\sum_{[a,c]} \omega_i(f)\,\Delta x_i' < \varepsilon .$$

It follows from

$$\omega_i(|f|) \le \omega_i(f)$$

that

$$\sum_{i=1}^{n} \omega_i(|f|)\,\Delta x_i < \varepsilon .$$

Hence

$$\left|\int_a^b f(x)dx\right| = \left|\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i\right| \le \lim_{\lambda\to 0}\sum_{i=1}^{n} |f(\xi_i)|\,\Delta x_i = \int_a^b |f(x)|\,dx .$$

The following example shows that the converse of Proposition 1.6 is not true.

Example 1.1 Let $D(x)=\begin{cases} 1, & \text{x is rational} \\ -1, & \text{x is irrational} \end{cases}$. Discuss the integrability of $D(x)$ and $|D(x)|$ on $[a,b]$.

Solution Obviously, for any partition $P=\{x_i\}_{i=1}^{n}$ of $[a,b]$ and each i, the oscillation ω_i of $f(x)$ on $[x_{i-1},x_i]$ satisfies that $\omega_i=2$. This shows that $D(x)$ is not integrable.

Since $|D(x)|$ is a constant which is continuous, we know that it is integrable.

Proposition 1.7 Suppose $a<c<b$ and suppose $f\in R[a,c]$ and $f\in R[c,b]$. Then $f\in R[a,b]$ and

$$\int_a^b f(x)dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

Proof For any $\varepsilon>0$, it follows from the hypotheses $f\in R[a,c]$ and $f\in R[c,b]$ that there are partitions $P'=\{x_i\}_{i=0}^{t}$ of $[a,c]$ and $P''=\{x_i\}_{i=t}^{n}$ of $[c,b]$ such that

$$\sum_{[a,c]}\omega_i(f)\,\Delta x_i<\frac{1}{2}\varepsilon \text{ and } \sum_{[c,b]}\omega_i(f)\,\Delta x_i<\frac{1}{2}\varepsilon.$$

Let $P=P'\cup P''=\{x_i\}_{i=0}^{n}$. Then P is a partition of $[a,b]$ and

$$\sum_{i=1}^{n}\omega_i(f)\Delta x_i=\sum_{i=1}^{t}\omega_i(f)\Delta x_i+\sum_{i=t}^{n}\omega_i(f)\Delta x_i<\varepsilon.$$

The integrability of $f(x)$ on $[a,b]$ follows from the first criterion.

It follows that

$$\int_a^b f(x)dx=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i=\lim_{\lambda\to 0}\sum_{i=1}^{t}f(\xi_i)\Delta x_i+\lim_{\lambda\to 0}\sum_{i=t+1}^{n}f(\xi_i)\Delta x_i$$

$$=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

Corollary 1.8 For any a, b, c, if $f \in R[a,c]$ and $f \in R[c,b]$, then $f \in R[a,b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proposition 1.9 (The first mean-value theorem) Suppose

(1) $f \in C[a,b]$;

(2) g is nonnegative (or nonpositive) and g is integrable.

Then there is some $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

Proof Without loss of generality, we may assume that $g(x) \geq 0$. Let $M = \max_{x \in [a,b]} \{f(x)\}$ and $m = \min_{x \in [a,b]} \{f(x)\}$.

Then

$$mg(x) \leq f(x)g(x) \leq Mg(x).$$

It follows that

$$m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx.$$

(1) If $\int_a^b g(x)dx = 0$, then $\int_a^b f(x)g(x)dx = 0$. This implies that we can take any point in [a,b] as ξ;

(2) If $\int_a^b g(x)dx > 0$, then $m \le \dfrac{\int_a^b g(x)f(x)dx}{\int_a^b g(x)dx} \le M$. Hence there is some $\xi \in [a,b]$ such that

$$f(\xi) = \frac{\int_a^b g(x)f(x)dx}{\int_a^b g(x)dx}.$$

Our result follows.

Remark 1.10 If $g=1$, then $f(\xi)=\frac{1}{b-a}\int_a^b f(x)dx$ -- The average value of $f(x)$ on $[a,b]$.

Theorem 1.10 If $f\in R[a,b]$, let $F(x)=\int_a^x f(x)\,dx$. Then $F\in C[a,b]$.

Proof It follows from $f\in R[a,b]$ that there is some $M>0$ such that for all $x\in[a,b]$,

$$|f(x)\le M|.$$

We deduce that

$$|F(x+\Delta x)-F(x)| = \left|\int_x^{x+\Delta x} f(x)dx\right| = |f(\xi)\Delta x| \le M|\Delta x|.$$

The proof is finished.

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