Lecture 20 Computation of definite integrals

§1 Computation

1.1 Newton-Leibniz's Formula

Theorem 1.1.1 If $f \in C[a,b]$ and $F(x) = \int_a^x f(x)dx$ is derivable, then

$$F'(x) = f(x).$$

Proof For any $x \in [a,b]$, without loss of generality, we may assume that $x \in (a,b)$. Then

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{\int_x^{x+\Delta x} f(x)dx}{\Delta x}.$$

Then there is some ξ between x and $x+\Delta x$ such that

$$\frac{F(x+\Delta x)-F(x)}{\Delta x}=f(\xi).$$

Since $\lim_{\Delta x\to 0}\xi=x$, we see that

$$\lim_{\Delta x\to 0}\frac{F(x+\Delta x)-F(x)}{\Delta x}=f(x).$$

The proof is completed.

As the first application of Theorem 1.1.1 we have that

Corollary 1.1.2 If $f\in C[a,b]$ and $u(x)$ is derivable on $[\alpha,\beta]$ with $u(\alpha)=a$ and $u(\beta)=b$, then

$$F(x)=\int_a^{u(x)}f(t)dt$$

is derivable and

$$F'(x) = f(u(x))u'(x).$$

Proof Let $y = u(x)$ and $G(y) = \int_a^y f(t)dt$. Then

$$G'(y) = f(y).$$

Since $F(x) = G(y)$, we see that

$$F'(x) = G'(y)y' = f'(u(x))u'(x).$$

Corollary 1.1.3 Suppose that $f(x)$ is continuous, $u(x)$ and $v(x)$ are derivable.

Then

$$F(x)=\int_{v(x)}^{u(x)} f(t)dt$$

is derivable and

$$F'(x)=f(u(x))u'(x)-f(v(x))v'(x).$$

The proof easily follows from

$$F(x)=\int_{a}^{u(x)} f(t)dt-\int_{a}^{v(x)} f(t)dt.$$

As the second application of Theorem 1.1.1, we get

Theorem 1.1.4 (Newton-Leibniz's Formula) If

$f\in C[a,b]$ and $F(x)$ is one of its anti-derivative, then

$$\int_{a}^{b} f(x)\,dx=F(x)|_{a}^{b}.$$

Proof Let $G(x)=\int_a^x f(x)dx$. Then

$$G(x)=F(x)+c \quad \text{and} \quad c=-F(a).$$

It follows that

$$\int_a^b f(x)dx = F(b)-F(a)$$
$$= F(x)\Big|_a^b .$$

Example 1.1.1 Find $\int_0^{1/2} e^{2x}dx$.

Solution $\int_0^{1/2} e^{2x}dx=\frac{1}{2}e^{2x}\Big|_0^{\frac{1}{2}}=\frac{1}{2}(e-1)$.

Example 1.1.2 Find $\int_0^2 \frac{x}{\sqrt{1+x^2}}dx$.

Solution $\int_0^2 \frac{x}{\sqrt{1+x^2}}dx = \frac{1}{2}\int_0^2 \frac{d(1+x^2)}{\sqrt{1+x^2}} = \sqrt{1+x^2}\Big|_0^2 = \sqrt{5}-1$.

Example 1.1.3 Suppose $F(x)=\int_x^{x^2} \sin x dx$. Find $F'(x)$.

Solution $F'(x) = 2x\sin x^2 - \sin x$.

§ 2 Integration by substitution

Theorem 2.1 Suppose $f \in C[a,b]$. Let $x=\varphi(t)$, where $\varphi(t)$ satisfies that

(1) $\varphi'(t)$ is continuous on $[a,b]$;

(2) $\varphi(\alpha)=a, \varphi(\beta)=b$ and $a \le \varphi(t) \le b$.

Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Proof Let $G(x)$ be one of the anti-derivative of $f(x)$. For example, let

$$G(x)=\int_a^x f(x)dx.$$

Then $G'(x) = f(x)$ and obviously, $G(\varphi(t))$ is an anti-derivative of $f(\varphi(t))\varphi'(t)$.

It follows from

$$\int_a^b f(x)dx = G(b) - G(a)$$

that

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = G(\varphi(t))\Big|_\alpha^\beta = G(b) - G(a).$$

This yields that

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

The proof is finished.

Example 2.1 Find $\int_0^a \sqrt{a^2 - x^2}\,dx$.

Solution Let $x = a\sin t$. Then

$$\int_0^a \sqrt{a^2 - x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t}\,a\cos t\,dt$$

$$= \frac{\pi}{4}a^2.$$

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