Lecture 23 Computation of definite integrals (IV)

§ 5 Added examples (II)

Example 5.16 Suppose $\int_0^{\pi} \frac{\cos x}{(x+2)^2} dx = A$. Find $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{x+1} dx$.

Solution $\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{x+1} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{x+1} dx = \int_0^{\pi} \frac{\sin x}{x+2} dx$

$$= -\frac{\cos x}{x+2}\Big|_0^{\pi} - \int_0^{\pi} \frac{\cos x}{(x+2)^2} dx = \frac{1}{\pi+2} + \frac{1}{2} - A.$$

Example 5.17 Suppose $f \in C[0,1]$ and $f(x) > 0$. Show that

$$\log\left(\int_0^1 f(x)dx\right) \geq \int_0^1 \log f(x)dx.$$

Proof Since $(\log x)'' < 0$ on $(0,1)$, we see that

$$\log\left(\frac{1}{n}\sum_{i=1}^n f\left(\frac{i}{n}\right)\right) \geq \frac{1}{n}\sum_{i=1}^n \log f\left(\frac{i}{n}\right).$$

It follows from $f \in C[0,1]$ that

$$\log\left(\int_0^1 f(x)dx\right) \geq \int_0^1 \log f(x)dx.$$

Example 5.18 Suppose $f(x) > 0$. Show that for all $x > 0$,

$$\varphi(x)=\frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}$$

is increasing.

Proof Since

$$\varphi'(x)=\frac{\int_0^x (x-t)f(x)f(t)dt}{(\int_0^x f(t)dt)^2},$$

we know that

$$\varphi'(x)\geq 0,$$

whence $\varphi(x)=\dfrac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}$ is increasing.

Example 5.19 Suppose $f\in C^2[a,b]$ and $f(a)=f(b)=0$. Show that

$$|\int_a^b f(x)dx|\le\frac{(b-a)^3}{12}\max_{x\in[a,b]}|f''(x)|.$$

Proof By parts, we have that

$$\int_a^b f(x)dx=(x-a)f(x)\Big|_a^b-\int_a^b(x-a)f'(x)dx=-\int_a^b(x-a)f'(x)dx.$$

Further,

$$\int_a^b f'(x)(x-a)dx=\int_a^b f(x)dx-\int_a^b f''(x)(x-a)(x-b)dx.$$

Hence

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b f''(x)(x-a)(x-b)dx.$$

It follows that

$$\left|\int_a^b f(x)dx\right| \le \frac{(b-a)^3}{12}\max\{|f''(x)|\}.$$

Example 5.20 Suppose $f,\ g \in R[a,\ b]$. Show that

$$\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx.$$

Proof For any partition $P = \{x_i\}_{i=0}^n$ of $[a, b]$,

$$\left(\int_a^b f(x)g(x)\right)^2 = \left(\lim_{\lambda\to 0}\sum_{i=1}^n f(\xi_i)g(\xi_i)\Delta x_i\right)^2$$

$$=[\lim_{\lambda\to 0}\sum_{i=1}^{n}(f(\xi_i)\sqrt{\Delta x_i})(g(\xi_i)\sqrt{\Delta x_i})]^2$$

$$\le \lim_{\lambda\to 0}\sum_{i=1}^{n}(f(\xi_i)\sqrt{\Delta x_i})^2\cdot\lim_{\lambda\to 0}\sum_{i=1}^{n}(g(\xi_i)\sqrt{\Delta x_i})^2$$

$$=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)^2\Delta x_i\cdot\lim_{\lambda\to 0}\sum_{i=1}^{n}g(\xi_i)^2\Delta x_i\,.$$

This implies that

$$(\int_a^b f(x)g(x)dx)^2\le\int_a^b f^2(x)dx\int_a^b g^2(x)dx\,.$$

Example 5.21 (1) Suppose $f\in R[a,\ b]$. Then

$$\left[\int_a^b f(x)dx\right]^2\le(b-a)\int_a^b f^2(x)dx\,.$$

(2) Suppose $f \in R[a, b]$ and $f(x) \geq m > 0$. Then

$$\int_a^b f(x)dx \int_a^b \frac{1}{f(x)}dx \geq (b-a)^2.$$

(3) Suppose $f, g \in R[a, b]$. Then

$$[\int_a^b (f(x)+g(x))^2 dx]^{\frac{1}{2}} \leq [\int_a^b f^2(x)dx]^{\frac{1}{2}} + [\int_a^b g^2(x)dx]^{\frac{1}{2}}.$$

The proofs of (1) and (2) easily follow from Example 4.15.

The proof of (3)

$$\int_a^b (f(x)+g(x))^2 dx = \int_a^b f^2(x)dx + 2\int_a^b f(x)g(x)dx + \int_a^b g^2(x)dx$$

$$\le \int_a^b f^2(x)dx + 2[\int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx]^{\frac{1}{2}} + \int_a^b g^2(x)dx,$$

$$= [(\int_a^b f^2(x)dx)^{\frac{1}{2}} + (\int_a^b g^2(x)dx)^{\frac{1}{2}}]^2.$$

It follows that

$$[\int_a^b (f(x)+g(x))^2 dx]^{\frac{1}{2}} \le [\int_a^b f^2(x)dx]^{\frac{1}{2}} + [\int_a^b g^2(x)dx]^{\frac{1}{2}}.$$

Example 5.22 Suppose $f \in C[0, A]$. Then

$$\lim_{h\to 0}\frac{1}{h}\int_a^x [f(t+h)-f(t)]dt = f(x)-f(a) \quad (a < x < A).$$

Proof

$$\lim_{h\to 0}\frac{\int_a^x [f(t+h)-f(t)]dt}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt - \int_a^{a+h} f(t)dt}{h} = f(x)-f(a).$$

Example 5.23 Suppose $f(x)$ is continuous and $f(x+y)=f(x)+f(y)$. Find $f(x)$.

Solution Obviously,

$$\int_0^x f(x+y)dx=\int_0^x f(x)dx+\int_0^x f(y)dx=\int_0^x f(x)dx+xf(y).$$

Since

$$\int_0^x f(x+y)dx=\int_y^{x+y} f(t)dt,$$

we see that

$$xf(y)=\int_0^{x+y} f(t)dt-\int_0^y f(t)dt-\int_0^x f(t)dt.$$

This shows that for all x and y, $xf(y) = yf(x)$ which implies that for all $x \neq 0,\ y \neq 0$

$$\frac{f(y)}{y} = \frac{f(x)}{x}.$$

This implies that for $x \neq 0$, $f(x) = kx$.

It is obvious that $f(0) = 0$. Hence

$$f(x) = kx.$$

Example 5.24 Suppose

$$x_n = \sqrt[n]{[1+(\frac{1}{n})^2][1+(\frac{2}{n})^2]\ldots[1+(\frac{n}{n})^2]},\quad y_n = \sum_{k=1}^{n} \frac{1}{k} \sin \frac{k\pi}{n+1}$$

and

$$z_n = \sum_{k=1}^{n} \tan^2 \frac{1}{\sqrt{n+k}} .$$

Find $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} z_n$.

Solution (1) Since $\log x_n = \frac{1}{n}\sum_{i=1}^{n}\left[\log 1+\left(\frac{i}{n}\right)^2\right]$ and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log\left[1+\left(\frac{i}{n}\right)^2\right]=\int_0^1 \log(1+x)^2 dx = \log 2 - 2 + \frac{\pi}{2}.$$

Hence

$$\lim_{n\to\infty} x_n = 2e^{\frac{\pi}{2}-2} .$$

(2) Since $y_n = \frac{1}{n+1}\sum_{k=1}^{n}\frac{n+1}{k}\sin\frac{k\pi}{n+1}$, we have that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=1}^{n}\frac{n+1}{k}\sin\frac{k\pi}{n+1} = \int_0^1 \frac{\sin \pi x}{x}dx.$$

(3) It follows from

$$\tan^2 x - x^2 = \frac{2}{3}x^4 + o(x^4)$$

that

$$0 \le \tan^2\frac{1}{\sqrt{n+k}} - \left(\frac{1}{\sqrt{n+k}}\right)^2 \le \frac{4}{3}\left(\frac{1}{\sqrt{n+k}}\right)^4,$$

whence

$$0 < z_n = \sum \tan^2\frac{1}{\sqrt{n+k}} = \sum_{k=1}^{n}\left[\tan^2\frac{1}{\sqrt{n+k}} - \left(\frac{1}{\sqrt{n+k}}\right)^2\right] + \sum\frac{1}{n+k}.$$

Since $0 \le \sum_{k=1}^{n}[\tan^2 \frac{1}{\sqrt{n+k}} - \left(\frac{1}{\sqrt{n+k}}\right)^2] \le \frac{4}{3}\sum \frac{1}{n^2}$, we have that

$$\lim_{n\to\infty} z_n = \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \frac{1}{n} = \int_0^1 \frac{dx}{1+x} = \log 2.$$

The other two limits can be found in a similar way.

Example 5.25 Suppose that $f(x)$ is continuous and decreasing, and that $f(x) > 0$.

Let

$$a_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)dx.$$

Show that $\{a_n\}$ is convergent.

Proof $$a_n = f(1) + \sum_{k=2}^{n}\int_{k-1}^{k} f(k)dx - \sum_{k=2}^{n}\int_{k-1}^{k} f(x)dx$$

$$= f(1) + \sum_{k=2}^{n}\int_{k-1}^{k} [f(k) - f(x)]dx.$$

It follows that

$$a_{n+1} - a_n = \int_{n}^{n+1} [f(n+1) - f(x)]dx \le 0,$$

whence $\{a_n\}$ is decreasing.

Since

$$a_n = \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1}\int_{k}^{k+1} f(x)dx \ge \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n-1} f(k) = f(n) > 0,$$

we see that $\{a_n\}$ is convergent.

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