Lecture 24 Special class for exercises in Chapter 7

Example 1.1 (Page 298: 3) Discuss the relations of integrability of the functions on $[a, b]$: $f(x)$, $|f(x)|$ and $f^2(x)$.

Solution Since

$$\big||f(x)|-|f(y)|\big| \le |f(x)-f(y)|,$$

we see if $f \in R[a, b]$ then $|f| \in R[a, b]$. Also $f^2 \in R[a, b]$ since $f^2 = |f| \cdot |f|$.

The example shows that the converse is not true.

Let

$$f(x)=\begin{cases}-1, & x \text{ is rational}\\ 1, & x \text{ is irrational}\end{cases}.$$

Obviously, $|f|\in R[a,b]$ and $f^2\in R[a,b]$, but $f\notin R[a,b]$.

The following holds: $|f|\in R[a,b]$ if and only if $f^2\in R[a,b]$

The proof easily follows from the following result.

Lemma 1.1 Suppose $\varphi\in C[A,B]$ and $f\in R[a,b]$ with $A\le f(x)\le B$. Then

$$\varphi\circ f\in R[a,b].$$

Proof For any $\varepsilon > 0$, it follows from the continuity of $\varphi(x)$ on $[A, B]$ that there is some $\eta > 0$ such that for any closed subinterval of $[A, B]$ with its length less than η, the oscillation of $\varphi(x)$ on this subinterval is not greater than $\frac{\varepsilon}{2(b-a)}$. We use Ω to denote the oscillation of $\varphi(x)$ on $[A, B]$.

It follows from $f \in R[a, b]$ that there is some $\delta > 0$ such that for any partition $P = \{x_i\}_{i=0}^{n}$ of $[a, b]$ with $\lambda(P) < \delta$,

$$\sum_{i=1}^{n} \omega_i(f)\Delta x_i < \frac{\eta\varepsilon}{2\Omega}.$$

Let $I' = \{i' : \omega_{i'}(f) < \eta\}$ and $I'' = \{i'' : \omega_{i''}(f) \geq \eta\}$. Then

$$\sum_{i=1}^{n} \omega_i(\varphi \circ f)\Delta x_i = \sum_{i'} \omega_{i'}(\varphi \circ f)\Delta x_{i'} + \sum_{i''} \omega_{i''}(\varphi \circ f)\Delta x_{i''}$$

$$\leq \frac{\varepsilon}{2(b-a)} \sum_{i'} \Delta x_{i'} + \Omega \sum_{i''} \Delta x_{i''}$$

$$\leq \frac{\varepsilon}{2} + \Omega \sum_{i''} \Delta x_{i''}.$$

Since

$$\frac{\eta\varepsilon}{2\Omega} > \sum_{i=1}^{n} \omega_i(f)\Delta x_i = \sum_{i'} \omega_{i'}(f)\Delta x_{i'} + \sum_{i''} \omega_{i''}(f)\Delta x_{i''}$$

$$\geq \sum_{i''} \omega_{i''}(f)\Delta x_{i''} \geq \eta \sum_{i''} \Delta x_{i''} ,$$

we see that

$$\sum_{i''} \Delta x_{i''} < \frac{\varepsilon}{2\Omega} ,$$

whence

$$\sum_{i=1}^{n} \omega_i(\varphi \circ f)\Delta x_i < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

The proof is complete.

Remark The following example shows if $\varphi \in R[A, B]$ and $f \in R[a,b]$ with $A \le f(x) \le B$.

Then the composite function $\varphi \circ f$ may not be integrable.

Let $[a,b]=[0,1]$,

$$\varphi(x)=\begin{cases}0, & x=0\\ 1, & x\neq 0\end{cases}$$

and

$$f(x)=\begin{cases}\dfrac{1}{p}, & x=\dfrac{q}{p}\\ 0, & x \text{ is irrational}\end{cases}.$$

Then

$$\varphi(f(x)) = \begin{cases} 1, & x = \dfrac{q}{p} \\ 0, & x \text{ is irrational} \end{cases}.$$

Obviously, $\varphi \circ f$ is not integrable on $[0,1]$ since the oscillation of $\varphi \circ f$ on any closed subinterval is 1.

Example 1.2 (Page 298: 4) Suppose $f \in R[a,b]$. Show that there is a sequence $\{\varphi_n(x)\}$ of piecewise linear functions such that

$$\int_a^b f(x)dx = \lim_{n \to \infty} \int_a^b \varphi_n(x)dx.$$

Proof Let $P=\{x_i\}_{i=0}^{n}$, where $x_i=a+\frac{i}{n}(b-a)$. For each n and for any $x\in(x_i,x_{i+1})$ $(i=0,1,\cdots,n-1)$,

let

$$\varphi_i(x)=k_i(x-x_i)+f(x_i),$$

where $k_i=\dfrac{f(x_i)-f(x_{i-1})}{\Delta x_i}$ and $\Delta x_i=\dfrac{b-a}{n}$.

Obviously, each $\varphi_i(x)$ is piecewise linear and

$$\int_a^b \varphi_n(x)dx=\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\varphi_i(x)dx.$$

$$=\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left[k_i(x-x_i)+f(x_i)\right]dx$$

$$=\frac{1}{2}\sum_{i=1}^{n}\left[f(x_{i-1})+f(x_i)\right]\frac{b-a}{n}.$$

The result easily follows.

Example 1.3 (Page 299: 5) Suppose $f\in R[A,B]$. Show

$$\lim_{h\to 0}\int_a^b\left|f(x+h)-f(x)\right|dx=0,$$

where $A<a<b<B$.

Proof Let $M = \omega_{[A, B]}(f)$ and let $P = \{x_i\}_{i=0}^{n}$, where $x_i = a + \frac{i}{n}(b-a)$ and let ω_i be the oscillation of $f(x)$ on $[x_i, x_{i+1}]$.

Then for any $\varepsilon > 0$ there is some $N > 0$ such that for all $n > N$,

$$\frac{b-a}{n}\sum_{i=0}^{n-1}\omega_i < \frac{\varepsilon}{4}, \quad \frac{b-a}{n}M < \frac{\varepsilon}{2}$$

$$a - \frac{b-a}{n} > A \text{ and } b + \frac{b-a}{n} < B.$$

Claim I $\lim_{h\to 0+}\int_a^b |f(x+h)-f(x)|\,dx = 0$.

Let $0 < h < \dfrac{b-a}{n}$. Then for each $i \in \{0, 1, \cdots, n-2\}$ and any $x \in [x_i, x_{i+1}]$,

$$|f(x+h)-f(x)| \le \omega_i + \omega_{i+1},$$

for any $x \in [x_{n-1}, x_n]$,

$$|f(x+h)-f(x)| \le \omega_{n-1} + M.$$

It follows that

$$\int_a^b |f(x+h)-f(x)|\,dx = \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} |f(x+h)-f(x)|\,dx$$

$$\le \sum_{i=0}^{n-2} \int_{x_i}^{x_{i+1}} (\omega_i + \omega_{i+1}) dx + \int_{b-\frac{b-a}{n}}^{b} |f(x+h) - f(x)|$$

$$\le 2\sum_{i=0}^{n-1} \omega_i \frac{b-a}{n} + \frac{b-a}{n} M$$

$$< \varepsilon .$$

Claim I is proved.

In a similar way, we can prove

Claim II $\lim_{h \to 0-} \int_a^b |f(x+h) - f(x)| dx = 0$.

It follows from Claims I and II that

$$\lim_{h \to 0} \int_a^b |f(x+h) - f(x)| dx = 0 .$$