Lecture 28 Center of a mass system , Average value, and Special class for exercises in Chapter 8

§ 1 The center of a mass system

Theorem 1.1 Suppose $l: y = f(x)(a \le x \le b)$ is smooth.

Then its coordinates of the center of a mass system are

$$\bar{x} = \frac{\int_a^b x\sqrt{1+y'^2}\,dx}{\int_a^b \sqrt{1+y'^2}\,dx}, \quad \bar{y} = \frac{\int_a^b y\sqrt{1+y'^2}\,dx}{\int_a^b \sqrt{1+y'^2}\,dx}.$$

Example 1.1 Find the coordinates of the center of a mass system of the curve $l:\ x^2 + y^2 = r^2 \quad (y \ge 0)$.

Solution By the symetriness of l, we see that $\bar{x}=0$, and

$$\bar{y}=\frac{\int_a^b y\sqrt{1+y'^2}\,dx}{\int_a^b \sqrt{1+y'^2}\,dx}=\frac{2r}{\pi}.$$

§ 2 Average value

Theorem 2.1 Suppose f is continuous on $[a,b]$. Then its average value is

$$\bar{y}=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example 2.1 Find the average value of $y = xe^x$ on $[0,1]$.

Solution Theorem 2.1 tells us that $\bar{y} = \int_0^1 xe^x dx = 1$.

§ 2 Exercises in Chapter 8

Example 3.1 (Page 335: 3) Let Ω be the solid obtained by rotating along the $y-$axis, the domain bounded by the curves: $x = a$, $x = b$, $x-$axis and

$$y = f(x) \geq 0$$

which is continuous. Show that the volume of Ω is

$$V = 2\pi \int_a^b xy(x)dx.$$

Proof Obviously, for any $x \in (a, b)$, the volume element dV of the part of the solid bounded between x and

$x + \Delta x$ is as follows:

$$dV = \pi\left[(x+\Delta x)^2 - x^2\right]y(x) \approx 2\pi xy(x)\Delta x.$$

Hence $V = 2\pi\int_a^b xy(x)dx$.

Example 3.2 Let Ω be the solid obtained by rotating along the $y-$axis, the domain bounded by the curves:

$x=0$, $x=\pi$, $x-$axis and $y=\sin x$. Find the volume of Ω.

Solution By **Example** 3.1, we know that

$$V = 2\pi\int_0^{\pi} x\sin x\,dx = 2\pi^2 .$$

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