Lecture 25 Special class for exercises in Chapters 17 and 18

Example 1 (Page 250: 5(2)) Find $\int_0^{\pi} \log\left(1-2a\cos x+a^2\right)dx$.

Solution Let

$$I(a)=\int_0^{\pi} \log\left(1-2a\cos x+a^2\right)dx$$

and

$$f(a,x)=\log\left(1-2a\cos x+a^2\right).$$

If $|a|<1$, then

$$I'(a)=\int_0^{\pi}\frac{2a-2\cos x}{1-2a\cos x+a^2}dx$$

$$=\frac{1}{a}\int_0^{\pi}\left(1+\frac{a^2-1}{1-2a\cos x+a^2}\right)dx$$

$$=0.$$

It follows from $I(0)=0$ that $I(a)=0$ when $|a|<1$.

If $|a|>1$, then

$$I(a)=I(\frac{1}{b})$$

$$=\int_0^{\pi}\log\left(1-\frac{2}{b}\cos x+\frac{1}{b^2}\right)dx$$

$$=\int_0^\pi \log\left(b^2-2b\cos x+1\right)dx-\int_0^\pi \log b^2 dx$$

$$=2\pi\log|a|.$$

If $a=1$, then

$$I(1)=\int_0^\pi \log(2-2\cos x)\,dx=0.$$

If $a=-1$, then

$$I(-1)=\int_0^\pi \log(2+2\cos x)\,dx=0.$$

Hence

$$I(a)=\begin{cases}2\pi\log|a|, & |a|>1\\ 0, & |a|\le 1\end{cases}.$$

Example 2 (Page 264: 4(4)) Discuss the uniform convergence of $\int_0^1 x^{p-1}\log^2 x dx$ under the following conditions, respectively

(1) $p \geq p_0 > 0$;

(2) $p > 0$.

Solution (1) For $P \geq P_0 > 0$ and $0 < x \leq 1$,

$$0 < x^{p-1}\log^2 x \leq x^{p_0-1}\log^2 x .$$

Then

$$\int_0^1 x^{p_0-1}\log^2 x dx = \frac{1}{p_0} x^{p_0} \log x \Big|_0^1 - \int_0^1 \frac{2}{p_0} x^{p_0-1} \log x dx$$

$$= \frac{2}{p_0^3} x^{p_0} \Big|_0^1 = \frac{2}{p_0^3}.$$

Hence $\int_0^1 x^{p-1}\log^2 x dx$ is uniformly convergent for $p \geq p_0 > 0$.

(2) Obviously, $x^{p-1}\log^2 x$ is decreasing for $0 < p < 1$ in $(0,1]$. Hence for any η: $0 < \eta < \frac{1}{2}$,

$$\int_{\frac{1}{2}\eta}^{\eta} x^{p-1}\log^2 x dx \geq \eta^{p-1}\log^2\eta(\eta-\frac{1}{2}\eta)=\frac{1}{2}\eta^{p}\log^2\eta .$$

The fact $\lim_{p\to+0}\eta^{p}=1$ implies that there exists $0<p_0<1$ such that $\eta^{p_0}>\frac{1}{2}$.

$$\int_{\frac{1}{2}\eta}^{\eta} x^{p_0-1}\log^2 x dx > \frac{1}{2}\log^2\frac{1}{4}.$$

Hence $\int_0^1 x^{p-1}\log^2 x dx$ is not uniformly convergent for $p>0$.