Chapter 20

Lecture 27 The computation of double integrals (I)

§ 1 The computation of double integrals

1.1 Reduction of a double integral to an iterated integral

Theorem 1.1 If $f \in R[a,b;c,d]$ and for all $x \in [a,b]$, the integral $F(x) = \int_c^d f(x,y)dy$ $\left(resp.\ G(y) = \int_a^b f(x,y)dx\right)$ exists, then

$$\iint\limits_{[a,b;c,d]} f(x,y)dxdy = \int_a^b dx \int_c^d f(x,y)dy$$

$$\left[resp. \quad \iint\limits_{[a,b;c,d]} f(x,y)dxdy = \int_c^d dy \int_a^b f(x,y)dx \right].$$

Proof We divide our discussions into two cases.

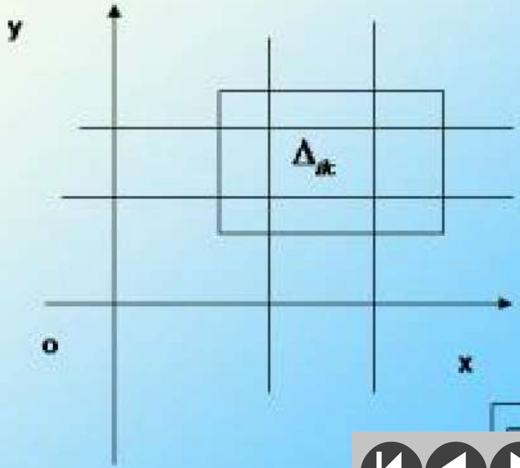
Case 1.1.1 D is a rectangle.

(1) Partition Take the partition $\{\Delta_{ik}\}$ D as the figure shown.

(2) Summation Let $M_{ik} = \sup\limits_{\Delta_{ik}}\{f\}$, $m_{ik} = \inf\limits_{\Delta_{ik}}\{f\}$ and

$\xi_i \in [x_{i-1}, x_i]$. Then

$$m_{ik}\Delta y_k \le \int_{y_{k-1}}^{y_k} f(\xi_i, y)dy \le M_{ik}\Delta y_k .$$



It easily follows that

That is,

$$\sum_k m_{ik}\Delta y_k \le F(\xi_i) \le \sum_k M_{ik}\Delta y_k .$$

Hence

$$\sum_{i,k} m_{ik}\Delta x_i\Delta y_k \le \sum F(\xi_i)\Delta x_i \le \sum_{i,k} M_{ik}\Delta x_i\Delta y_k .$$

which implies that $F(x)$ is integrable on $[a,b]$.

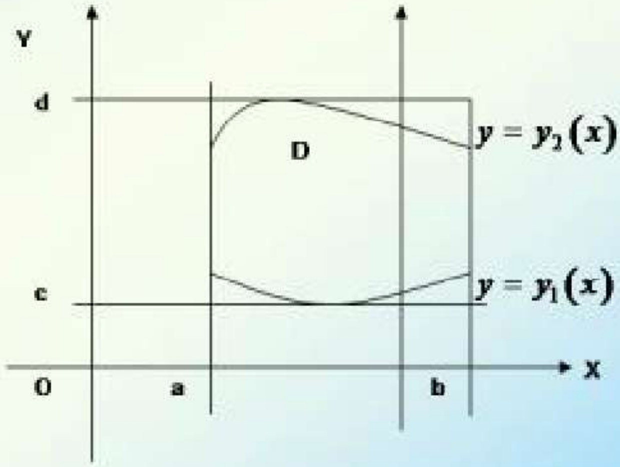
(3) Limit Let $d = \max\{diam\Delta x_{ik}\}$. Then by taking the limit when $d \to 0$, we see that

$$\int_a^b dx\int_c^d f(x,y)dy = \iint_{[a,b;c,d]} f(x,y)dxdy .$$

Note $f \in R(D)$ if and only if $\sum_{i,k} \omega_{i,j}\Delta x_i\Delta_k < \varepsilon$.

where $\omega_{ij} = \max_{M, M' \in \Delta_{ik}} \{|f(M) - f(M')|\}$.

Case 1.1.2 For general case



Define $F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$.

Then $F \in R[a,b;c,d]$ and

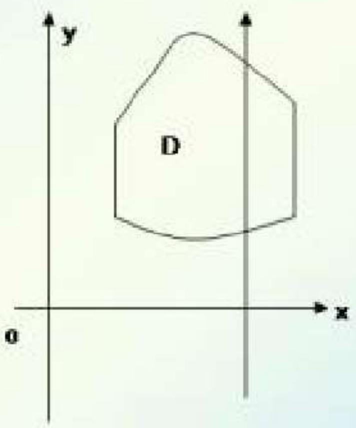
$$\iint_{[a,b;c,d]} F(x,y)\,dxdy = \iint_D f(x,y)\,dxdy.$$

1.2 $X-$domains and $Y-$domains

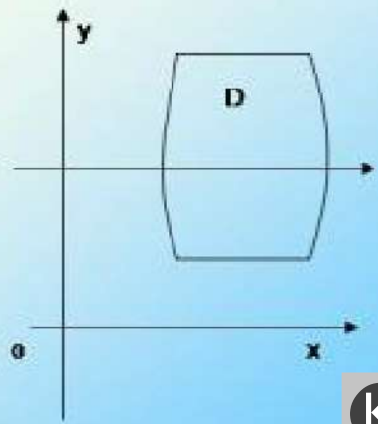
If $D=\{(x,y):\ a\le x\le b,\ f_1(x)\le y\le f_2(x)\}$, then D is an $X-$domain;

If $D=\{(x,y):\ c\le y\le d,\ g_1(y)\le x\le g_2(y)\}$ then D is a $Y-$domain.

X-Type domain



Y-Type domain



Theorem 1.2.1 (1) If D is an $X-$domain, i.e.,

$$D=\{(x,y): a\le x\le b, f_1(x)\le y\le f_2(x)\},$$

then

$$\iint_D f(x,y)dxdy=\int_a^b dx\int_{f_1(x)}^{f_2(x)} f(x,y)dy.$$

(2) If D is a $Y-$domain, i.e.,

$$D=\{(x,y):\ c\le y\le d,\ g_1(y)\le x\le g_2(y)\},$$

then

$$\iint_D f(x,y)dxdy=\int_c^d dy\int_{g_1(y)}^{g_2(y)} f(x,y)dx.$$

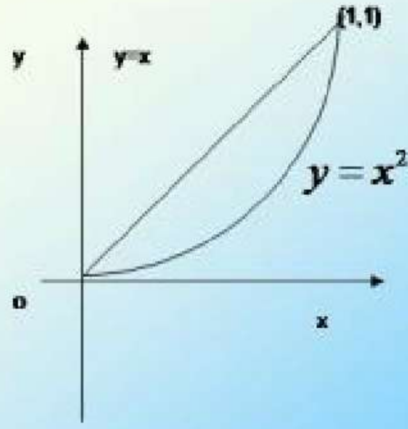
Example 1.1 Let $f(x,y)=\frac{1}{2}(2-x-y)$ and D be a domain as shown in the figure. Find $\iint_D f(x,y)dxdy$.

Solution (I) If we regard D as an $X-$domain, then

$$\iint_D f(x,y)dxdy=\int_0^1 dx\int_{x^2}^{x}\frac{1}{2}(2-x-y)dy$$

$$=\frac{1}{4}\int_0^1\left(4x-7x^2+2x^3+x^4\right)dx$$

$$=\frac{11}{120}.$$



(II) If we regard D as a $Y-$domain, then

$$\iint_D f(x,y)dxdy = \int_0^1 dy\int_y^{\sqrt{y}} \frac{1}{2}(2-x-y)dx$$

$$= \frac{1}{4}\int_0^1\left(4y^{\frac{1}{2}} - 5y - 2y^{\frac{3}{2}} + 3y^2\right)dy$$

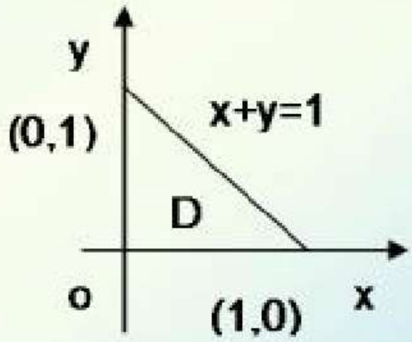
$$= \frac{11}{120}.$$

Example 1.2 Find the volume of the solid bounded by

$z = x + y$, $z = xy$, $x + y = 1$, $x = 0$ and $y = 0$.

Solution Obviously,

$$V_1 = \iint_D xy\,dxdy$$



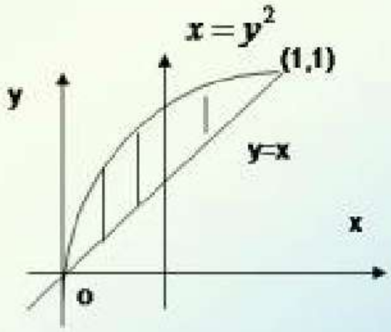
and

$$V_2 = \iint_D (x+y)\,dxdy.$$

Then

$$V = V_2 - V_1 = \frac{7}{24}.$$

Example 1.3 Find $I=\iint_D \frac{\sin y}{y}dxdy$, where D is show as in the figure



Solution $I=\int_0^1 dy\int_y^{y^2}\frac{\sin y}{y}dx=1-\sin 1$.

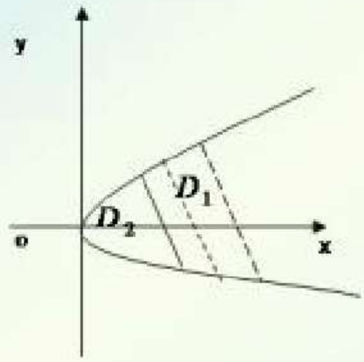
Example 1.4 Find $I=\iint_D (x+y)dxdy$, where D is bounded by $y^2=2x$, $x+y=4$, $x+y=12$.

Solution Obviously,

$$I=\iint_{D_1}(x+y)dxdy-\iint_{D_2}(x+y)dxdy.$$

Since

$$\iint_{D_1}(x+y)dxdy=\int_{-6}^{4}dy\int_{\frac{y^2}{2}}^{12-y}(x+y)dx$$



and

$$\iint_{D_2}(x+y)dxdy=\int_{-4}^{2}dy\int_{\frac{y^2}{2}}^{4-y}(x+y)dx\,,$$

it follows that

$$I=543\frac{11}{15}.$$

Homework Page 296: 1 (2) 2; 3 (1, 3)