Chapter 20

Lecture 28 The computation of double integrals (II)

1.3 Computation by polar coordinates

Let $D: r_1(\theta) \le r \le r_2(\theta),\ \theta \in [\alpha, \beta]$, and $r_1(\theta),\ r_2(\theta) \in C[\alpha, \beta]$.

Then $\Delta\sigma = \frac{1}{2}\left[(r+\Delta r)^2 - r^2\right]\Delta\theta = \left(r + \frac{1}{2}\Delta r\right)\Delta r \Delta\theta \approx r\Delta r\Delta\theta$

and

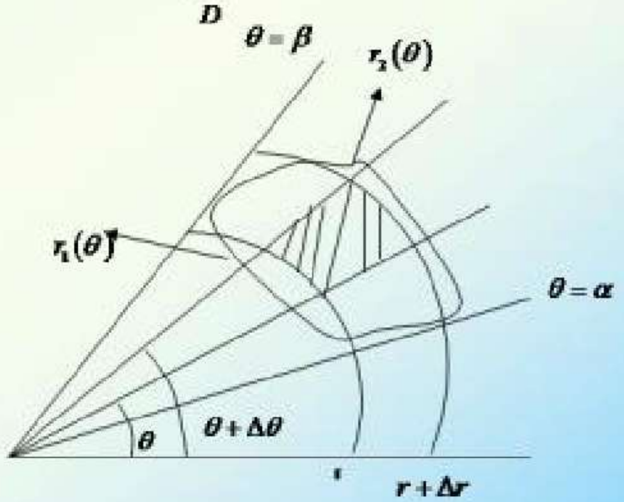
$$\sum_{i=1}^{n} f(M_i)\Delta\sigma_i \approx \sum_{i=1}^{n} f(r_i\cos\theta_i, r_i\sin\theta_i) r_i \Delta\theta_i \Delta\sigma_i .$$

As both Δr and $\Delta\theta$ are sufficient small,

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta_i \sigma_i \approx \sum_{i=1}^{n} f(r_i\cos\theta_i, r_i\sin\theta_i) r_i \Delta\theta_i \Delta r_i .$$

Hence

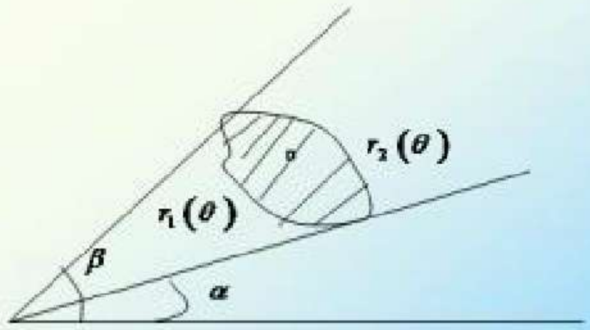
$$\iint_D f(x,y)\,dxdy = \iint_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta .$$



In the application of the above formula, the key is to determine the boundary curves $\theta=\alpha$, $\theta=\beta$, $\Gamma_1(\theta)$ and $\Gamma_2(\theta)$.

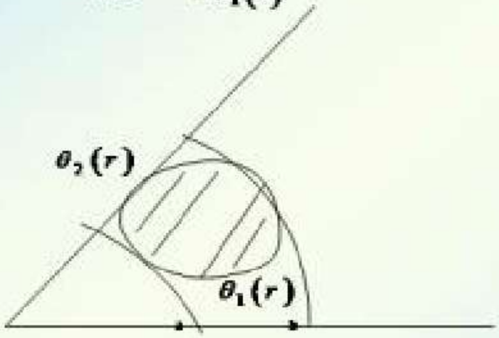
(1) If the boundary curves are $\alpha\le\theta\le\beta$, $r(\theta_1)\le r\le r(\theta_2)$, then

$$\iint_D f(x,y)\,dxdy=\int_\alpha^\beta d\theta\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta,r\sin\theta)\,rdr.$$



(2) If the boundary curves are $a \le r \le b,\ \theta_1(r) \le \theta \le \theta_2(r)$, then

$$\iint_D f(x,y)\,dxdy = \int_a^b dr \int_{\theta_1(r)}^{\theta_2(r)} f(r\cos\theta, r\sin\theta)\,r\,d\theta .$$



(3) If the origin lies in D, then

$$\iint_D f(x,y)\,dxdy = \int_0^{2\pi} d\theta \int_0^{r(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr .$$

Example 1.3.1 Find $I = \iint\limits_{x^2+y^2\le 1} e^{-x^2-y^2}dxdy$.

Solution Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $dxdy = rdrd\theta$ and

$$I = \int_0^{2\pi} d\theta \int_0^1 re^{-r^2}dr = \pi(1-e^{-1}).$$

Example 1.3.2 Suppose $f(x,y)$ is continuous. Find

$$\lim_{p\to 0}\frac{1}{\pi p^2}\iint\limits_{x^2+y^2\le p^2} f(x,y)dxdy.$$

Solution Let $x = r\cos\theta$ and $y = r\sin\theta$.

Then $dxdy = rdrd\theta$ and

$$\lim_{p\to 0}\frac{1}{\pi p^2}\iint_{x^2+y^2\le p^2} f(x,y)dxdy = \lim_{p\to 0}\frac{1}{\pi p^2}\int_0^p dr\int_0^{2\pi} rf(r\cos\theta, r\sin\theta)d\theta$$

$$= \lim_{p\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(p\cos\theta, p\sin\theta)d\theta$$

$$= f(0,0).$$

Example 1.3.3 (Weak form) Suppose $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = a$. Find

$$\lim_{p\to 0}\frac{1}{\pi p^2}\iint_{x^2+y^2\le p^2} f(x,y)dxdy.$$

Solution It follows from

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = a$$

that for any $\varepsilon > 0$, there is some $\delta > 0$ such that for all $(x, y) \in O(\hat{o}, \delta)$,

$$|f(x, y) - a| < \varepsilon .$$

This shows that

$$\left| \frac{1}{\pi p^2} \iint_{x^2+y^2 \le p^2} f(x,y)dxdy - a \right| \le \frac{1}{\pi p^2} \iint_{x^2+y^2 \le p^2} |f(x,y) - a|\, dxdy$$

$$< \varepsilon .$$

Hence

$$\lim_{p \to 0} \frac{1}{\pi p^2} \iint_{x^2+y^2 \le p^2} f(x,y)dxdy = a .$$

Example 1.3.4 Find $F'(t)$, where $F(t)=\iint\limits_{x^2+y^2\le t^2} f(x^2+y^2)d\sigma$,

$t>0$ and $f(u)$ is derivable.

Solution Let $x=r\cos\theta$ and $y=r\sin\theta$. Then

$dxdy=rdrd\theta$ and

$$F(t)=\int_0^{2\pi}d\theta\int_0^t rf(r^2)dr=2\pi\int_0^t rf(r^2)dr.$$

Then

$$F'(t)=2\pi tf(t^2).$$

Example 1.3.5 Find $\iint_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,d\sigma$,

where $D=\{(x,y):0\le r\le 1, 0\le\theta\le\frac{\pi}{2}\}$.

Solution Let $x=r\cos\theta$ and $y=r\sin\theta$. Then $dxdy=rdrd\theta$

and

$$\iint_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,d\sigma=\int_0^{2\pi}d\theta\int_0^1 r\sqrt{\frac{1-r^2}{1+r^2}}\,dr=\frac{\pi}{2}(\pi-1).$$

Example 1.3.6 Find $\int_0^a dx\int_0^x \sqrt{x^2+y^2}\,dy$.

Solution Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $dxdy = rdrd\theta$

and

$$\int_0^a dx\int_0^x \sqrt{x^2+y^2}\,dy = \int_0^{\frac{\pi}{4}} d\theta\int_0^{\frac{a}{\cos\theta}} r^2 dr = \frac{\sqrt{2}+\log(\sqrt{2}+1)}{6}.$$

Example 1.3.7 Find the area of the D bounded by

$$xy = a^2 \text{ and } x+y=\frac{5}{2}a \quad (a>0).$$

Solution We have known that $S=\iint\limits_D dxdy$. Hence

$$S=\iint_D dxdy=\int_{\frac{a}{2}}^{2a}\left(\frac{5}{2}a-x-\frac{a^2}{x}\right)dx$$

$$=\left(\frac{15}{8}-\ln 4\right)a^2.$$

Example 1.3.8 Find $I=\iint_D (x+y)\operatorname{sgn}(x-y)dxdy$, where

$D=\{(x,y):0\le x\le 1, 0\le y\le 1\}$.

Solution $I=\int_0^1 dx\int_0^x (x+y)dy-\int_0^1 dx\int_x^1 (x+y)dy$

$$=\frac{4}{3}\pi+8\ln\left(1+\sqrt{3}\right)-4\log 2.$$

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