Lecture 30 The computation of double integrals (IV)

§ 1.5 Added examples (I)

Example 1.5.1 Show

$$\iint_S (ax+by+c)\,dxdy = 2\int_{-1}^{1}\sqrt{1-u^2}\,f\left(u\sqrt{a^2+b^2}+c\right)du,$$

where $S: x^2+y^2\le 1,\ a^2+b^2\neq 0$.

Solution Let $\begin{cases} u=\dfrac{1}{\sqrt{a^2+b^2}}(ax+by) \\ v=\dfrac{1}{\sqrt{a^2+b^2}}(bx-ay) \end{cases}$.

Then $|J|=1$ and

$$\iint_S (ax+by+c)\,dxdy = 2\int_{-1}^{1}\sqrt{1-u^2}\,f\left(u\sqrt{a^2+b^2}+c\right)du.$$

Example 1.5.2 Find $I=\iint_D e^{\frac{x-y}{x+y}}dxdy$, where D is bounded by $x=0$, $y=0$ and $x+y=1$.

Solution Let $\begin{cases} u=x-y \\ v=x+y \end{cases}$. Then $|J|=\frac{1}{2}$

and

$$I=\iint_{D} e^{\frac{x-y}{x+y}}dxdy=\iint_{D'} e^{\frac{u}{v}}dudv$$

$$=\frac{1}{4}\left(e-\frac{1}{e}\right),$$

where D' is the domain bounded by the curves $u=-v$, $u=v$ and $v=1$.

Example 1.5.3 Change $\iint_{D} f(xy)dxdy$, where D: $1\le xy\le 2,\ x\le y\le 2x$, into a definite integral.

Solution Let $\begin{cases} u = xy \\ v = \dfrac{y}{x} \end{cases}$. Then $J = \dfrac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \dfrac{x}{2y} = \dfrac{1}{2v}$

and

$$\iint_D f(x,y)\,dxdy = \frac{1}{2}\ln 2\int_1^2 f(u)\,du .$$

Example 1.5.4 Change $\iint_D f(uv)dudv$, where $\begin{cases} 2 \le uv \le 3 \\ u \le v \le tu \end{cases}$, into an iterated integral.

Solution Let $\begin{cases} x = uv \\ y = \dfrac{u}{v} \end{cases}$. Then $J = -\dfrac{2u}{v}$

and

$$\iint_D f(uv)\,du\,dv = \int_2^3 dx \int_1^t f(x)\frac{1}{2y}\,dy = \frac{1}{2}\int_2^3 f(x)\,dx \int_1^t \frac{1}{y}\,dy .$$

Example 1.5.5 Suppose $f \in C[a,b]$. Then

$$\int_a^b dx \int_a^x f(y)\,dy = \int_a^b (b-x) f(x)\,dx .$$

Proof $$\int_a^b dx \int_a^x f(y)\,dy = \iint_D f(y)\,dx\,dy = \int_a^b dy \int_a^y f(y)\,dx$$

$$= \int_a^b (y-a) f(y)\,dy = \int_a^b (x-a) f(x)\,dx .$$

Example 1.5.6 Find $\int_0^1 dy \int_1^y \left(e^{-x^2} + e^x \sin x\right) dx$.

Solution

$$\int_0^1 dy \int_1^y \left(e^{-x^2} + e^x \sin x\right) dx = -\iint_D \left(e^{-x^2} + e^x \sin x\right) dx dy$$

$$= \frac{e^{-1} - e \sin 1}{2}.$$

Example 1.5.7 Change the order of the following iterated integrals.

(1) $\int_0^1 dy \int_0^y f(x,y)\,dx$;

(2) $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$;

(3) $\int_1^e dx \int_0^{\log x} f(x,y)\,dy$;

(4) $\int_0^1 dy \int_0^{2y} f(x,y)\,dx + \int_1^3 dy \int_0^{3-y} f(x,y)\,dx$.

Solution (1) $\int_0^1 dy \int_0^y f(x,y)\,dx = \int_0^1 dx \int_x^1 f(x,y)\,dx$;

(2) $\int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx = \int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} f(x,y)\,dy$;

(3) $\int_1^e dx \int_0^{\log x} f(x,y)dy = \int_0^1 dy \int_{e^y}^e f(x,y)dx$;

(4) $\int_0^1 dy \int_0^{2y} f(x,y)dx + \int_1^3 dy \int_0^{3-y} f(x,y)dx = \int_0^2 dx \int_{\frac{x}{2}}^{3-x} f(x,y)dy$.

Example 1.5.8 Suppose

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} f(r\cos\theta, r\sin\theta)\,dr.$$

(1) Change the order of the iterated integrals;

(2) Find the two iterated integrals with respect to x and y.

Solution (1)

$$I=\int_0^{\sqrt{2}a}dr\int_{-\frac{\pi}{4}}^{\arccos\frac{r}{2a}}rf(r\cos\theta,r\sin\theta)d\theta$$
$$+\int_{\sqrt{2}a}^{2a}dr\int_{-\arccos\frac{r}{2a}}^{\arccos\frac{r}{2a}}rf(r\cos\theta,r\sin\theta)d\theta.$$

(2) $I=\int_0^a dy\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}}f(x,y)dx+\int_a^0 dy\int_{-y}^{a+\sqrt{a^2-y^2}}f(x,y)dx$

and

$$I=\int_0^a dx\int_{-x}^{\sqrt{2ax-x^2}}f(x,y)dy+\int_a^{2a}dx\int_{-\sqrt{2ax-x^2}}^{\sqrt{2ax-x^2}}f(x,y)dy.$$

Example 1.5.9 Suppose $f(x,y)=\frac{\partial^2 F(x,y)}{\partial x \partial y}$. Find

$$\int_0^A dx \int_0^B f(x,y)dy.$$

Solution $\int_0^A dx \int_0^B f(x,y)dy = \int_0^A dx \int_0^B \frac{\partial^2 F(x,y)}{\partial x \partial y} dy$

$$= \int_0^A \frac{\partial F(x,y)}{\partial x} \Big|_0^B dx$$

$$= \int_0^A \left[\frac{\partial F(x,B)}{\partial x} - \frac{\partial F(x,0)}{\partial x} \right] dx$$

$$= F(B,B) + F(0,0) - F(0,B) - F(A,0).$$

Example 1.5.10 Suppose $I=\iint_D [\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial x}\right)^2]dxdy$.

Let $\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$ which is regular (that's, all partial derivatives of $x(u,v)$ and $x(u,v)$ with respect to u and v are continuous) and maps the domain D onto Ω. If

$$\begin{cases} \frac{\partial x}{\partial u}=\frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial v}=-\frac{\partial y}{\partial u} \end{cases},$$

then

$$I=\iint\limits_{\Omega}\left[\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2\right]dudv\,.$$

Proof By substitution, we see that

$$I=\iint\limits_{\Omega}\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]\cdot\frac{\partial(x,y)}{\partial(u,v)}dudv\,.$$

It follows from

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial x}{\partial v}\right)^2,$$

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

that

$$\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2=\left[\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right]\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right|,$$

which implies

$$I=\iint_{\Omega}[\left(\frac{\partial f}{\partial u}\right)^2+\left(\frac{\partial f}{\partial v}\right)^2]dudv.$$

Homework Page 298: 10; 11