Lecture 31 The computation of double integrals (V)

§ 1.5 Added examples (II)

Example 1.5.10 Change $\iint\limits_D f(uv)\,du\,dv$,

where $\begin{cases} 2 \le uv \le 3 \\ u \le v \le tu \end{cases}$, into an iterated integral.

Solution Let $\begin{cases} x = uv \\ y = \dfrac{u}{v} \end{cases}$. Then $J = -\dfrac{2u}{v}$

and

$$\iint_D f(uv)\,dudv = \int_2^3 dx\int_1^t f(x)\frac{1}{2y}\,dy = \frac{1}{2}\int_2^3 f(x)\,dx\int_1^t \frac{1}{y}\,dy.$$

Example 1.5.11 Find $I = \iint_{x^2+y^2\le x+y} (x+y)\,dxdy$.

Solution

$$I = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta\int_0^{\sin\theta+\cos\theta} (r\sin\theta + r\cos\theta)\,rdr$$

$$= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin\theta+\cos\theta)\,d\theta\int_0^{\sin\theta+\cos\theta} r^2 dr$$

$$=\frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}(\sin\theta+\cos\theta)^4\,d\theta$$

$$=\frac{1}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(1+\frac{1+4\cos 4\theta}{2}2\sin 2\theta\right)d\theta=\frac{\pi}{2}.$$

Example 1.5.12 Change $\iint\limits_D f(xy)\,dxdy$,

where $D: 1\le xy\le 2,\ x\le y\le 2x$, into a definite integral.

Solution Let $\begin{cases} u=xy \\ v=\dfrac{y}{x} \end{cases}$.

Then $J = \dfrac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = \dfrac{x}{2y} = \dfrac{1}{2v}$ and

$$\iint_D f(x,y)\,dxdy = \frac{1}{2}\ln 2 \int_1^2 f(u)\,du .$$

Example 1.5.13 Find $I = \iint_D e^{\frac{x-y}{x+y}} dxdy$, where D is bounded by $x=0$, $y=0$ and $x+y=1$.

Solution Let $\begin{cases} u = x-y \\ v = x+y \end{cases}$. Then $|J| = \dfrac{1}{2}$

and

$$I=\iint\limits_{D} e^{\frac{x-y}{x+y}}dxdy=\iint\limits_{D'} e^{\frac{u}{v}}dudv=\frac{1}{4}\left(e-\frac{1}{e}\right),$$

where D' is the domain bounded by the curves $u=-v$, $u=v$ and $v=1$.

Example 1.5.14 Find $\lim\limits_{n\to\infty}\sum\limits_{j=1}^{2n}\sum\limits_{i=1}^{n}\frac{2}{n^2}\left[\frac{2i+j}{n}\right]$, where $[x]$ means the maximal integer part of x which is not greater than x.

Solution Since $\left[\frac{2i+j}{n}\right]=\left[2\frac{i}{n}+2\frac{j}{2n}\right]$, we see that

$$I=\lim_{n\to\infty}\sum_{j=1}^{2n}\sum_{i=1}^{n}\frac{2}{n^2}\left[\frac{2i+j}{n}\right]=\lim_{n\to\infty}\sum_{j=1}^{2n}\sum_{i=1}^{n}\frac{1}{2n}\frac{1}{n}\left[2\frac{i}{n}+2\frac{j}{2n}\right]$$

$$=4\iint_D[2x+2y]\,dxdy,$$

where $D=[0,1;0,1]$. Hence

$$I=\iint_{D_2}dxdy+2\iint_{D_3}dxdy+3\iint_{D_4}dxdy=\frac{3}{2}.$$

Example 1.5.15 Suppose $f \in C[0,1]$ is positive and decreasing. Show that

$$\frac{\int_0^1 xf^2(x)dx}{\int_0^1 xf(x)dx} \le \frac{\int_0^1 f^2(x)dx}{\int_0^1 f(x)dx}.$$

Proof It follows from

$$\int_0^1 xf^2(x)dx\int_0^1 f(y)dy \le \int_0^1 xf(x)dy\int_0^1 f^2(y)dy$$

that

$$\iint_D xf(x)f(y)\left[f(y)-f(x)\right]dxdy \ge 0$$

and

$$\iint_D yf(y)f(x)\left[f(x)-f(y)\right]dxdy \geq 0.$$

Hence

$$\frac{1}{2}\iint_D f(x)f(y)[f(x)-f(y)](y-x)dxdy \geq 0.$$

Example 1.5.16 Suppose the sequence $\{f_n(x,y)\}$ uniformly converges to a function $f(x,y)$ on a bounded and closed domain D.

Show that

$$\iint_D f(x, y)dxdy = \lim_{n\to\infty} \iint_D f_n(x, y)dxdy .$$

Proof We divide the proof into two steps.

Step I We first prove the continuity of $f(x, y)$ on D.

It follows from the continuity of $f_n(x, y)$ and the uniform convergence of $\{f_n(x, y)\}$ to $f(x, y)$ that for any $\varepsilon > 0$, there are some $N > 0$ and $\delta > 0$ such that for all $n > N$ and all $(x, y) \in O((x_0, y_0), \delta)$,

$$\left|f_n(x,y)-f_n(x_0,y_0)\right|<\frac{\varepsilon}{3},$$

and all $(x,y)\in D$,

$$\left|f_n(x,y)-f(x,y)\right|<\frac{\varepsilon}{3}.$$

This shows that for any $(x,y)\in O((x_0,y_0),\delta)$,

$$\begin{aligned}\left|f(x,y)-f(x_0,y_0)\right| &\le \left|f(x,y)-f_{N+1}(x,y)\right| \\ &+\left|f_{N+1}(x,y)-f_{N+1}(x_0,y_0)\right|+\left|f_{N+1}(x_0,y_0)-f(x_0,y_0)\right| \\ &<\varepsilon\end{aligned}$$

Step I shows that $\iint\limits_D f(x,y)dxdy$ exists.

Step II $\iint_D f(x,y)dxdy = \lim_{n\to\infty}\iint_D f_n(x,y)dxdy$.

It follows from the uniform convergence of $\{f_n(x,y)\}$ to $f(x,y)$ that for any $\varepsilon>0$, there is some $N>0$ such that for all $n>N$ and all $(x,y)\in D$,

$$|f_n(x,y)-f(x,y)|<\frac{\varepsilon}{S(D)},$$

where $S(D)$ denotes the area of D.

Hence

$$\left|\iint_D f_n(x,y)dxdy-\iint_D f(x,y)dxdy\right|\le\iint_D |f_n(x,y)-f(x,y)|dxdy<\varepsilon.$$

Example 1.5.17 Use the double integral to prove the following inequality:

$$\left(\int_a^b f(x)\cos kx\,dx\right)^2 + \left(\int_a^b f(x)\sin kx\,dx\right)^2 \le \left(\int_a^b |f(x)|\,dx\right)^2.$$

Proof Since

$$\left(\int_a^b f(x)\cos kx\,dx\right)^2 = \iint_D f(x)f(y)\cos kx\cos ky\,dx\,dy$$

and

$$\left(\int_a^b f(x)\sin kx\,dx\right)^2 = \iint_D f(x)f(y)\sin kx\sin ky\,dx\,dy,$$

we have that

$$\left(\int_a^b f(x)\cos kx dx\right)^2+\left(\int_a^b f(x)\sin kx dx\right)^2$$

$$=\iint_D f(x)f(y)(\cos kx\cos ky+\sin kx\sin ky)dxdy$$

$$=\iint_D f(x)f(y)\cos k(x-y)dxdy$$

$$\le\iint_D |f(x)|\cdot|f(y)|dxdy$$

$$\le\left(\int_a^b |f(x)|dx\right)^2,$$

where $D=\{(x,y): a\le x\le b,\ a\le y\le b\}$.

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