Lecture 32 The computation of triple integrals (I)

§ 1 Triple integrals (I)

1.1 Definition

Suppose that the function $f(x,y,z)$ which is defined on V is bounded in $\bar{R}^3$, and I is a constant.

If for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any partition $P = \{\Delta_i\}_{i=0}^{n}$ and for any $M_i(\xi_i, \eta_i, \zeta_i) \in \Delta_i$, whenever $d < \delta$, we have

$$\left|\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta V_i - I\right| < \varepsilon,$$

then $f(x,y,z)$ is called integrable on V, denoted by $f \in R(V)$, and the limit is called the triple integral of $f(x,y,z)$ on V, which is denoted by

$$I = \iiint_V f(x,y,z)\,dxdydz .$$

Proposition 1.1 If $f \in C(V)$, where V is bounded, then $f \in R(V)$.

Proposition 1.2 If the set of discontinuity of $f(x,y,z)$ is contained in a set whose measure is 0, then $f \in R(V)$

1.2 The computations of triple integrals

Theorem 1.2.1 If $f \in R[a,b;c,d;e,f]$ and for all $x \in [a,b]$,

$$I(x) = \iint_D f(x,y,z)\,dydz$$

exist, where $D = [c,d;e,f]$, then $\int_a^b dx \iint_D f(x,y,z)\,dydz$

exists and

$$\iiint_V f dxdydz = \int_a^b dx \iint_D f(x,y,z)\,dydz.$$

Proof Let P be a partition of V by using the planes which are parallel to the coordinate planes. Then $V = \bigcup V_{ijk}$, where $V_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i]$.

Let

$$M_{ijk} = \sup_{M \in V_{ijk}} \{f(M)\} \text{ and } m_{ijk} = \inf_{M \in V_{ijk}} \{f(M)\}.$$

Then for any $\xi_i \in [x_{i-1}, x_i]$,

$$m_{ijk}\Delta y_j \Delta z_k \le \iint_{[y_{j-1},y_j]\times[z_{k-1},z_k]} f(\xi_i, y, z)\,dydz \le M_{ijk}\Delta y_j \Delta z_k.$$

This gives

$$\sum m_{ijk}\Delta y_j \Delta z_k \le \iint_D f(\xi_i, y, z)\,dydz \le \sum M_{ijk}\Delta y_j \Delta z_k$$

and

$$\sum_{i,j,k} m_{ijk}\Delta x_i \Delta y_j \Delta z_k \le \sum_i I(\xi_i)\Delta x_i \le \sum_{i,j,k} M_{ijk}\Delta y_j \Delta z_k.$$

When $d \to 0$, we see that $\lim_{d\to 0}\sum_i I(\xi_i)\Delta x_i$ exists and this limit is independent of the choice of ξ_i and the partition P. This shows that

$$I(x) = \iint_D f(x,y,z)\,dydz$$

is integrable. Hence

$$\int_a^b I(x)\,dx = \iiint_V f(x,y,z)\,dxdydz\,.$$

The proof is complete.

As a consequence of Theorem 1.2.1, the following is obvious.

Corollary 1.2.2

$$\iiint\limits_V f(x,y,z)\,dxdydz=\int_a^b dx\iint\limits_D f(x,y,z)\,dydz$$
$$=\int_a^b dx\int_c^d dy\int_e^h f(x,y,z)\,dz$$

Theorem 1.2.3 In Theorem 1.2.1, if the domain is replaced by

$a\le x\le b$, $f_1(x)\le y\le f_2(x)$ and $z_1(x,y)\le z\le z_2(x,y)$,

then

$$\iiint\limits_V f(x,y,z)\,dxdydz=\int_a^b dx\int_{f_1(x)}^{f_2(x)} dy\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz.$$

Example 1.2.1 Find $I=\iiint_V \frac{dxdydz}{x^2+y^2}$, where V is bounded by $x=1,\ x=2,\ z=0,\ y=x$ and $z=y$.

Solution

$$I=\int_1^2 dx\int_0^x dy\int_0^y \frac{dz}{x^2+y^2}=\frac{1}{2}\int_1^2 dx\int_0^x \frac{2y}{x^2+y^2}dy$$

$$=\frac{1}{2}\int_1^2 \log\left(x^2+y^2\right)\Big|_0^x dx$$

$$=\frac{1}{2}\int_1^2 \log\left(2x^2\right)dx+\int_1^2 \log x^2 dx=\frac{1}{2}\log 2+\int_1^2 \log x dx$$

$$=\frac{1}{2}\log 2.$$

1.3 The change of variables in the triple integrals

Theorem 1.3.1 Let

$$\begin{cases} u = u(x,y,z) \\ v = v(x,y,z) \\ w = w(x,y,z) \end{cases}$$

be one-to-one from $V \to V^*$.

Suppose that $f(x,y,z)$ and all partial derivatives of u, v, w are continuous on V, and that $\frac{1}{J} = \frac{D(u,v,w)}{D(x,y,z)} \neq 0$ and is continuous.

Then

$$\iiint_V f(x,y,z)\,dxdydz$$
$$=\iiint_{V^*} f\big(x(u,v,w),y(u,v,w),z(u,v,w)\big)|J|\,dudvdw\,.$$

1.4 Some special changes

1.4.1 Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$. Then $|J| = r$.

Example 1.4.1 Find $I=\iiint_V (x^2+y^2)\,dxdydz$, where V

is bounded by

$$z=2\left(x^2+y^2\right) \text{ and } z=4.$$

Solution Let

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}.$$

Then $\dfrac{D(x,y,z)}{D(u,v,w)}=r$. It follows that

$$I=\int_0^{\pi} d\theta \int_0^{\sqrt{2}} dr \int_{r^2}^{4} r^3 dz=\frac{8}{3}\pi.$$

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