Lecture 33 The computation of triple integrals (II)

§ 1 Triple integrals (II)

1.4.2 Let $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$. Then $|J| = r^2\sin\varphi$.

Example 1.4.2 Find the volume of the solid bounded by $z = \sqrt{x^2+y^2}\cot\beta$ and $x^2+y^2+(z-a)^2 = a^2$, where $\beta \in \left(0, \frac{\pi}{2}\right)$ and $a > 0$.

Solution Let $\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\varphi \end{cases}$.

Then

$$I = \int_0^{2\pi} d\theta \int_0^{\beta} d\varphi \int_0^{2a\cos\varphi} r^2 \sin\varphi dr = \frac{4}{3}\pi a^3 \left(1 - \cos^4 \beta\right).$$

Example 1.4.3 Find $I = \iiint\limits_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dxdydz$, where

V is the solid bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution Let $\begin{cases} x = ar\cos\theta\sin\varphi \\ y = br\sin\theta\cos\varphi \\ z = cr\cos\varphi \end{cases}$.

Then

$$I = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{r} abcr^2 \sin\varphi r^2 dr = \frac{4}{5}abc.$$

Example 1.4.4 Find the volumes of the solids bounded by the following surfaces, respectively.

(1) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2 = \frac{x^2}{a^2}+\frac{y^2}{b^2}$;

(2) $\left(\frac{x}{a}+\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1\ \ (x>0,\ y>0,\ z>0,\ a,\ b,\ c>0)$;

(3) $z=x^2+y^2,\ \ z=2\left(x^2+y^2\right),\ \ xy=a^2,\ xy=2a^2,\ \ x=2y$ and $2x=y$, where $x>0,\ \ y>0$.

Solution (1) Let $\begin{cases} x=ar\cos\theta\sin\varphi \\ y=br\sin\theta\sin\varphi \\ z=cr\cos\varphi \end{cases}$.

Then $|J|=abcr^2\sin\varphi$ and

$$V=8abc\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}d\varphi\int_0^{\sin\varphi}r^2\sin\varphi dr=\frac{\pi^2}{4}abc.$$

(2) Let $\begin{cases} x = ar\cos^2\theta\cos\varphi \\ y = br\sin^2\theta\cos\varphi \\ z = cr\sin\varphi \end{cases}$.

Then $|J| = 2abcr^2\sin\varphi\cos\varphi\cos\theta$ and

$$V = 2abc\int_0^{\frac{\pi}{2}}\cos\theta d\theta\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi d\varphi\int_0^1 r^2 dr = \frac{1}{3}abc.$$

(3) Let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$.

Then $|J| = r$

and

$$V=\int_{\arctan\frac{1}{2}}^{\arctan 2} d\theta \int_{a\sqrt{\csc\theta\sec\theta}}^{\sqrt{2}a\sqrt{\csc\theta\sec\theta}} r dr \int_{r^2}^{2r^2} dz=\frac{9}{4}a^4 .$$

Example 1.4.5 Find $I=\iiint\limits_{\Omega} \frac{z\log\left(x^2+y^2+z^2+1\right)}{x^2+y^2+z^2+1} dV$,

where Ω is the domain bounded by $x^2+y^2+z^2=1$.

Solution Let $\begin{cases} x=r\cos\theta\sin\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\varphi \end{cases}$.

Then $|J| = r^2 \sin\varphi$ and

$$I = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \cos\varphi d\varphi \int_0^1 \frac{r \ln(r^2+1)}{r^2+1} dr = 0.$$

Example 1.4.7 Suppose $f(x,y,z)$ is continuous.

Change

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{x+y} f(x,y,z) dz$$

into the iterated integral in the form

$$\int_a^b dz \int_{c(z)}^{d(z)} dx \int_{e(z,x)}^{g(z,x)} f(x,y,z) dy.$$

Solution Since

$$\int_0^{1-x} dy \int_0^{x+y} f(x,y,z)dz = \int_0^x dz \int_0^{1-x} f dy + \int_x^1 dz \int_{z-x}^{1-x} f dy,$$

we see that

$$I = \int_0^1 dx \int_0^x dz \int_0^{1-x} f dy + \int_0^1 dx \int_x^1 dz \int_{z-x}^{1-x} f dy$$

$$= \int_0^1 dz \int_z^1 dx \int_0^{1-x} dy + \int_0^1 dz \int_0^z dx \int_{z-x}^{1-x} dy.$$

Example 1.4.10 Find the volume of the solid bounded

by $\left(\frac{x}{a}\right)^{\frac{2}{5}} + \left(\frac{y}{b}\right)^{\frac{2}{5}} + \left(\frac{z}{c}\right)^{\frac{2}{5}} = 1.$

Solution Let $u=\left(\frac{x}{a}\right)^{\frac{1}{5}}, v=\left(\frac{y}{b}\right)^{\frac{1}{5}}, w=\left(\frac{z}{c}\right)^{\frac{1}{5}}$. Then

$$\frac{D(x,y,z)}{D(u,v,w)}=125abc(uvw)^4.$$

And let $\begin{cases} x=r\sin\varphi\sin\theta \\ y=r\sin\varphi\cos\theta \\ z=r\cos\varphi \end{cases}$. Then

$$V=\frac{5}{504}abc\pi.$$

Example 1.4.11 Find $I=\iiint\limits_V xyz\,dxdydz$, where

$$V=\left\{(x,y,z):1\le\frac{yz}{x}\le 2, y\le zx\le 2y, z\le xy\le 2z\right\}.$$

Solution Let $u=\dfrac{yz}{x}, v=\dfrac{zx}{y}, w=\dfrac{xy}{z}$.

Then

$$\frac{D(u,v,w)}{D(x,y,z)}=4 \quad\text{and}\quad xyz=uvw.$$

These show that

$$I=\iiint\limits_V xyz\,dxdydz=\frac{1}{4}\iiint\limits_{V^*} uvw\,dudvdw=\frac{1}{4}\left(\int_1^2 udu\right)^3=\frac{27}{32},$$

where $V^* = \{(u, v, w): 1 \le u \le 2, 1 \le v \le 2, 1 \le w \le 2\}$.

Example 1.4.12 Find

$$\iiint_{x^4+y^4+z^4\le 1} |xyz|(x^4+y^4+z^4+1)dxdydz.$$

Solution Obviously,

$$\iiint_{x^4+y^4+z^4\le 1} |xyz|(x^4+y^4+z^4+1)dxdydz$$

$$= 4\iiint_{\substack{x^4+y^4+z^4\le 1\\ x\ge 0,\ y\ge 0,\ z\ge 0}} |xyz|(x^4+y^4+z^4+1)dxdydz \cdot$$

Let $\begin{cases} x = \sqrt{u} \\ y = \sqrt{v} \\ z = \sqrt{w} \end{cases}$.

Then $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \dfrac{1}{8\sqrt{uvw}}$ and

$$\iiint\limits_{x^4+y^4+z^4\le 1} |xyz|(x^4 + y^4 + z^4 + 1)dxdydz$$

$$= \iiint\limits_{\substack{u^2+v^2+w^2\le 1 \\ u\ge 0,\, v\ge 0,\, w\ge 0}} \frac{\sqrt{uvw}}{8\sqrt{uvw}}(u^2 + v^2 + w^2 + 1)dudvdw$$

$$= \iiint\limits_{\substack{u^2+v^2+w^2\le 1\\ u\ge 0,\, v\ge 0,\, w\ge 0}} (u^2+v^2+w^2+1)\,du\,dv\,dw\,.$$

Let $\begin{cases} u = r\cos\theta\sin\varphi \\ v = r\sin\theta\sin\varphi \\ w = r\cos\varphi \end{cases}$.

Then

$$\iint\limits_{x^4+y^4+z^4\le 1} |xyz|(x^4+y^4+z^4+1)dxdy$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi \int_0^1 r^2(r^2+1)dr$$

$$= \frac{4}{15}\pi\,.$$

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