Lecture 35 The applications and Improper multiple integrals

§ 1 Applications of triple integrals

1.1 The center of a solid in space

Suppose that Ω denotes a solid in space whose density is $\rho(M)$ for any $M \in \Omega$.

Suppose $\rho(M)$ is continuous in Ω. Then the coordinates of the center of Ω are as follows.

$$\begin{cases} x = \dfrac{\int_\Omega x\rho(M)d\Omega}{\int_\Omega \rho(M)d\Omega} = \dfrac{\int_\Omega x\rho dm}{\int_\Omega \rho dm} \\ y = \dfrac{\int_\Omega y\rho(M)d\Omega}{\int_\Omega \rho(M)d\Omega} = \dfrac{\int_\Omega y\rho dm}{\int_\Omega \rho dm} \\ z = \dfrac{\int_\Omega z\rho(M)d\Omega}{\int_\Omega \rho(M)d\Omega} = \dfrac{\int_\Omega z\rho dm}{\int_\Omega \rho dm} \end{cases}.$$

If $\Omega \subset R^3$, then

$$\begin{cases} x = \dfrac{\iiint\limits_{\Omega} x\rho(x, y, z)dxdydz}{\iiint\limits_{\Omega} \rho(x, y, z)dxdydz} \\ y = \dfrac{\iiint\limits_{\Omega} y\rho(x, y, z)dxdydz}{\iiint\limits_{\Omega} \rho(x, y, z)dxdydz} \\ z = \dfrac{\iiint\limits_{\Omega} z\rho(x, y, z)dxdydz}{\iiint\limits_{\Omega} \rho(x, y, z)dxdydz} \end{cases}.$$

Example 1.1 Suppose

$$\Omega = \left\{(x, y, z) : x^2 + y^2 + z^2 \le 1, z \ge 0\right\}$$

which is well-distributed. Find the coordinates of the center of Ω.

Solution By the symmetry of Ω, we see that $\bar{x}=0, \bar{y}=0$ and

$$\bar{z}=\frac{\iiint\limits_{\Omega} z\rho(x,y,z)dxdydz}{\iiint\limits_{\Omega} \rho(x,y,z)dxdydz}=\frac{3}{16}.$$

Hence the coordinates of the center of Ω is $(0,0,\frac{3}{16})$.

1.2 Rotational inertia

Suppose that Ω denotes a solid in R^3, whose density is $\rho(M)$ for any $M \in \Omega$. Suppose $\rho(M)$ is continuous in Ω. Then the following triple integrals are called the inertia of Ω with respect to yz-plane, zx-plane and xy-plane, respectively:

$$\iiint_{\Omega} x^2 \rho(x, y, z)dxdydz, \quad \iiint_{\Omega} y^2 \rho(x, y, z)dxdydz$$

and $\iiint_{\Omega} z^2 \rho(x, y, z)dxdydz$.

And the following triple integrals are called the inertia of Ω with respect to $z-$axis, $x-$axis and $y-$axis, respectively:

$$I_{oz} = \iiint_{\Omega} (x^2 + y^2) \rho(x, y, z)dxdydz,$$

$$I_{ox} = \iiint_{\Omega} (y^2 + z^2) \rho(x, y, z)dxdydz$$

and

$$I_{oy} = \iiint_{\Omega} \left(z^2 + x^2\right) \rho(x, y, z) dxdydz .$$

1.3 Gravitation

Suppose that Ω denotes a solid in R^3, whose density is $\rho(M)$ for any $M \in \Omega$., and that $M(x_0, y_0, z_0)$ is a point outside Ω. If $\rho(M)$ is continuous in Ω,

then the components of the gravitation of Ω to

$M(x_0, y_0, z_0)$ in $x-$axis, $y-$axis and $z-$axis are as

follows:

$$F_x = k\iiint_{\Omega} \frac{\rho(M)(x - x_0)}{r^3} dxdydz ,$$

$$F_y = k\iiint_{\Omega} \frac{\rho(M)(y - y_0)}{r^3} dxdydz$$

and

$$F_z = k\iiint_{\Omega} \frac{\rho(M)(z - z_0)}{r^3} dxdydz ,$$

where k denotes the gravitation constant and

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} .$$

§ 2 Improper multiple integrals

2.1 Double integrals with unbounded domains

2.1.1 Definition

Suppose $D \subset R^2$ is an unbounded domain and a function $f(M)$ is well-defined in D. Let σ denote

the subdomain of D which is bounded by a smooth

closed curve. If the double integral

$$\iint_{\sigma} f(M)dxdy$$

exists and the limit

$$\lim_{\sigma \to D} \iint_{\sigma} f(M)dxdy$$

exists and have the same value I for any smooth closed curve σ in D, then I is called the improper double integral of $f(M)$ on an unbounded domain, which is denoted by

$$I = \iint_{D} f(M)dxdy .$$

Also we call that $\iint_{D} f(M)dxdy$ converges.

2.1.2 The relations between integrability and absolute integrability

Theorem 2.1 $\iint\limits_D f(M)dxdy$ converges if $\iint\limits_D |f(M)|dxdy$ converges.

2.1.3 Cauchy's test

Theorem 2.2 Suppose $D \subset R^2$ is an unbounded domain and

$$\iint\limits_{\sigma} f(M)dxdy$$

exists for any bounded subdomain $\sigma \subset D$. Let r denotes the distance from $M \in D$ to the origin o.

If for any sufficiently large r,

$$|f(M)| \le \frac{c}{r^p},$$

where then c is a constant and $p > 2$, then

$\iint\limits_D f(M)dxdy$ converges.

Proof Let

$$F(x, y) = \begin{cases} f(x, y), & (x, y) \in D \\ 0, & (x, y) \notin D \end{cases}$$

and $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$.

Then

$$\iint_D |f(x,y)|\,dxdy = \iint_{R^2} |F(x,y)|\,dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^{+\infty} r\,|F(r\cos\theta, r\sin\theta)|\,dr\,.$$

Since for sufficiently large r (for instance $r \geq r_0$),

$$r \cdot |F(r\cos\theta, r\sin\theta)| \leq \frac{c}{r^{p-1}},$$

it follows that

$$\iint_D |f(x,y)|\,dxdy \le \int_0^{2\pi} d\theta \int_0^{r_0} r\,|F(r\cos\theta, r\sin\theta)|\,dr$$

$$+\int_0^{2\pi} d\theta \int_0^{+\infty} \frac{c}{r^{p-1}}\,dr$$

Hence $\iint_D |f(x,y)|\,dxdy$ converges. Theorem 2.1 shows that $\iint_D f(x,y)\,dxdy$ converges.

2.2 Double integrals with unbounded integrants

2.1.1 Definition

Suppose $D \subset R^2$ is a domain and a function $f(M)$ has some irregular points or irregular curves in D. Let γ be a smooth curve in D and $f(M)$ is well-defined in the domain Σ bounded by γ and the boundary ∂D of D.

If the double integral

$$\iint_{\Sigma} f(M)dxdy$$

always exists and has the same value I for any smooth curve γ in D, then I is called the improper double integral of $f(M)$ with unbounded integrant, which is denoted by

$$I = \iint_D f(M)dxdy.$$

Also we call that $\iint_D f(M)dxdy$ converges.

2.1.2 Cauchy's test

Theorem 2.2 Suppose $D \subset R^2$ is a domain and $f(M)$ has an irregular point B in $D \subset R^2$. If for any point M in D which sufficiently close to B, the following is satisfied:

$$|f(M)| \leq \frac{c}{r^p},$$

where r denotes the distance from M to B, c is a constant and $p<2$, then $\iint_D f(M)dxdy$ converges.

§ 3 Examples

Example 3.1 Discuss the integrability of the following improper integral:

$$\iint\limits_{[0,1;\ 0,1]} \frac{ydxdy}{\sqrt{x}}.$$

Solution Since

$$\iint\limits_{[\varepsilon,1;\ 0,1]} \frac{ydxdy}{\sqrt{x}} = \int_0^1 ydy \int_\varepsilon^1 \frac{dx}{\sqrt{x}}$$

$$= \sqrt{x}\Big|_\varepsilon^1 \to 1,$$

as $\varepsilon \to 0+$. This shows that $\iint\limits_{[0,1;\,0,1]} \frac{ydxdy}{\sqrt{x}}$ is convergent.

Example 3.2 Find

$$I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy$$

and show

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Solution By Cauchy's test, this improper integral converges.

Let $D_R = \{(x, y): x^2 + y^2 \le R^2\}$. Then

$$\iint_{D_R} e^{-(x^2+y^2)} dxdy = \int_0^{2\pi} d\theta \int_0^R r e^{-r^2} dr = \pi(1 - e^{-R}).$$

Hence

$$\lim_{R\to\infty} \iint_{D_R} e^{-(x^2+y^2)} dxdy = \pi,$$

which shows

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy = \pi.$$

By the relation

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dxdy = \lim_{t\to+\infty}\left(\int_{-t}^{t} e^{-y^2}dy\int_{-t}^{t} e^{-x^2}dx\right) = \lim_{t\to+\infty}\left(\int_{-t}^{t} e^{-x^2}dx\right)^2,$$

we know that

$$\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

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