Lecture 12 Added examples

§ 1 Added examples (I)

Example 1 Suppose $f(x, y) = y\sqrt{|x|}$.

(1) Find $f_x(0,0)$ and $f_y(0,0)$;

(2) Show that $f(x,y)$ is not differentiable at $(0,0)$.

Solution (1) Obviously, $f_x(0,0)=0$ and $f_y(0,0)=0$.

(2) Since

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y\sqrt{|x|}}{\sqrt{x^2+y^2}}$$

does not exist, we see that $f(x, y)$ is not differentiable at $(0, 0)$.

Example 2 Suppose

$$f(x,y)=\begin{cases} \dfrac{x^2 y}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}.$$

Show that

(1) $f(x, y)$ is continuous at $(0, 0)$;

(2) both $f_x(0, 0)$ and $f_y(0, 0)$ exist;

(3) both $f_x(x, y)$ and $f_y(x, y)$ are not continuous at $(0, 0)$;

(4) $f(x, y)$ is not differentiable at $(0, 0)$.

Solution (1) It follows from $\left|\frac{x^2 y}{x^2 + y^2}\right| \le |y|$ that

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^2 + y^2} = 0,$$

which shows the continuity of $f(x, y)$ at $(0, 0)$.

(2) Obviously, $f_x(0, 0) = 0$ and $f_y(0,0) = 0$.

(3) Since

$$f_x(x, y) = \begin{cases} \dfrac{2xy^3}{\left(x^2+y^2\right)^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

and

$$f_y(x, y) = \begin{cases} \dfrac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} x^2, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases},$$

we know that the two limits $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{2xy^3}{\left(x^2+y^2\right)^2}$ and

$\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} x^2$ do not exist.

This shows the discontinuity of $f_x(x, y)$ and $f_y(x, y)$ at $(0, 0)$.

(4) Since the limit

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x, y)-f(0, 0)-f_x(0, 0)x-f_y(0, 0)y}{\sqrt{x^2+y^2}} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

does not exist, we see that $f(x, y)$ is not differentiable at $(0, 0)$.

Example 3 Apply the transformation

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

to change the following equation:

$$x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial x} = 0.$$

Solution Differentiating the equation system $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

with respect to x we get

$$\begin{cases} \dfrac{\partial r}{\partial x}\cos\theta - \dfrac{\partial \theta}{\partial x} r\sin\theta = 1 \\ \dfrac{\partial r}{\partial x}\sin\theta + \dfrac{\partial \theta}{\partial x} r\cos\theta = 0 \end{cases},$$

which implies that

$$\begin{cases} \dfrac{\partial r}{\partial x} = \cos\theta \\ \dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r} \end{cases}.$$

In a similar way, we have that

$$\begin{cases} \dfrac{\partial r}{\partial y} = \sin\theta \\ \dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r} \end{cases}.$$

It follows from

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial x} \\ \dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial y} \end{cases}$$

that

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r}\right) r\cos\theta - \left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\cdot\frac{\sin\theta}{r}\right) r\sin\theta = \frac{\partial u}{\partial \theta}.$$

Hence the equation $x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial x}=0$ is changed into the form:

$$\frac{\partial u}{\partial \theta}=0.$$

Example 4 Suppose $x^2=uw$, $y^2=vw$, $z^2=uv$ and $f(x,y,z)=F(u,v,w)$.

Show that

$$xf_x+yf_y+zf_z=uF_u+vF_v+wF_w.$$

Solution Differentiating the equations $x^2=uw$, $y^2=vw$ and $z^2=uv$ with respect to x

we get

$$\begin{cases} \dfrac{\partial u}{\partial x} w + \dfrac{\partial w}{\partial x} u = 2x \\ \dfrac{\partial v}{\partial x} w + \dfrac{\partial w}{\partial x} v = 0 \\ \dfrac{\partial u}{\partial x} v + \dfrac{\partial v}{\partial x} u = 0 \end{cases},$$

which yields

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{x}{w} \\ \dfrac{\partial v}{\partial x} = -\dfrac{v}{uw} x \\ \dfrac{\partial w}{\partial x} = \dfrac{x}{u} \end{cases}.$$

Similar computations show that

$$\begin{cases} \dfrac{\partial u}{\partial y} = -\dfrac{u}{vw} y \\ \dfrac{\partial v}{\partial y} = \dfrac{y}{w} \\ \dfrac{\partial w}{\partial y} = \dfrac{y}{v} \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial u}{\partial z} = \dfrac{z}{v} \\ \dfrac{\partial v}{\partial z} = \dfrac{z}{u} \\ \dfrac{\partial w}{\partial z} = -\dfrac{w}{uv} z \end{cases}.$$

It follows from

$$\begin{cases} f_x = F_u \dfrac{\partial u}{\partial x} + F_v \dfrac{\partial v}{\partial x} + F_w \dfrac{\partial w}{\partial x} \\ f_y = F_u \dfrac{\partial u}{\partial y} + F_v \dfrac{\partial v}{\partial y} + F_w \dfrac{\partial w}{\partial y} \\ f_z = F_u \dfrac{\partial u}{\partial z} + F_v \dfrac{\partial v}{\partial z} + F_w \dfrac{\partial w}{\partial z} \end{cases}$$

that

$$\begin{cases} xf_x = uF_u - vF_v + wF_w \\ yf_y = -uF_u + vF_v + wF_w \\ zf_z = uF_u + vF_v - wF_w \end{cases}.$$

Hence

$$xf_x + yf_y + zf_z = uF_u + vF_v + wF_w.$$

Example 5 Suppose $F(x, y, x-z, y^2-w)=0$ and all its second order partial derivatives are continuous. Suppose $F_4 \neq 0$. Find $\frac{\partial w}{\partial y}$ and $\frac{\partial^2 w}{\partial y^2}$.

Solution Differentiating the equation

$F(x, y, x-z, y^2-w)=0$ with respect to y we get

$$F_2+F_4(2y-\frac{\partial w}{\partial y})=0,$$

which yields

$$\frac{\partial w}{\partial y}=2y+\frac{F_2}{F_4}.$$

Differentiating the equation $\frac{\partial w}{\partial y}=2y+\frac{F_2}{F_4}$ with respect to

y we get

$$\frac{\partial^2 w}{\partial y^2} = 2 + \frac{F_4 \frac{\partial F_2}{\partial y} - F_2 \frac{\partial F_4}{\partial y}}{(F_4)^2}.$$

Since

$$\frac{\partial F_2}{\partial y} = F_{22} + F_{24}\left(2y - \frac{\partial w}{\partial y}\right) = \frac{F_4 F_{22} - F_2 F_{24}}{F_4}$$

and

$$\frac{\partial F_4}{\partial y} = F_{24} + F_{44}\left(2y - \frac{\partial w}{\partial y}\right) = \frac{F_4 F_{24} - F_2 F_{44}}{F_4},$$

we have that

$$\frac{\partial^2 w}{\partial y^2} = 2 + \frac{F_4^2 F_{22} - 2F_2 F_4 F_{24} + F_2^2 F_{44}}{(F_4)^3}.$$

Example 6 Suppose $f(x, y)$ satisfies $f(y^2, y) = 1$ and $f_x(x, y) = x^2 + 2y$. Find $f(x, y)$.

Solution It follows from

$$f_x(x, y) = x^2 + 2y$$

that $f(x, y) = \frac{1}{3}x^3 + 2xy + g(y)$, where $g(y)$ denotes a function depending only on y.

$f(y^2, y) = 1$ implies that

$$g(y) = 1 - 2y^3 - \frac{1}{3}y^6.$$

Hence

$$f(x, y) = \frac{1}{3}x^3 + 2xy + 1 - 2y^3 - \frac{1}{3}y^6.$$