Lecture 13 Added examples

§ 1 Added examples (II)

Example 7 Suppose

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}.$$

Find the directional derivative at $(0,0)$ along the direction $\vec{l}=\{\cos\alpha,\sin\alpha\}$.

Solution By definition,

$$\left.\frac{\partial f}{\partial \vec{l}}\right|_{(0,0)}=\lim_{t\to 0}\frac{f(t\cos\alpha,t\sin\alpha)-f(0,0)}{t}=\lim_{x\to 0}\frac{t^2}{t^2}\cos\alpha\sin\alpha=\frac{\sin 2\alpha}{2}.$$

Remark Obviously, $f_x(0,0)=0$ and $f_y(0,0)=0$.

Then

$$\left.\frac{\partial f}{\partial \vec{l}}\right|_{(0,0)}\neq f_x(0,0)\cos\alpha+f_y(0,0)\sin\alpha$$

except $\sin 2\alpha=0$.

The reason for this is that $f(x,y)$ is not differentiable at $(0,0)$.

Example 8 Suppose

$$f(x,y)=\begin{cases}\dfrac{\sin^2 x+\sin^2 y+\sin^2(x+y)}{x\sin x+y\sin y+(x+y)\sin(x+y)}, & x^2+y^2\neq 0\\ 1, & x^2+y^2=0\end{cases}.$$

Find $f_x(0,0)$ and $f_{x^2}(0,0)$.

Solution $f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)}{x}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{x}-1}{x}=0$.

Obviously,

$$f_x(x,y)=\begin{cases}\dfrac{g(x,y)-h(x,y)}{\left[x\sin x+y\sin y+(x+y)\sin(x+y)\right]^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases},$$

where

$$g(x, y) = [\sin 2x + \sin 2(x+y)]\left[x\sin x + y\sin y + (x+y)\sin(x+y)\right]$$

and

$$h(x, y) = \left[\sin^2 x + \sin^2 y + \sin^2(x+y)\right] \cdot \left[\sin x + x\cos x + \sin(x+y) + (x+y)\cos(x+y)\right].$$

Then

$$f_{x^2}(0, 0) = \lim_{x\to 0} \frac{g(x, y) - h(x, y)}{\left[x\sin x + y\sin y + (x+y)\sin(x+y)\right]^2} = -\frac{5}{6}.$$

Example 9 Suppose $f(x, y)$ is continuous at $M_0(x_0, y_0)$ and $g(x, y)$ is differentiable at M_0 with $g(M_0)=0$. Show that $f(x, y)g(x, y)$ is differentiable at M_0.

Proof Since $g(x, y)$ is differentiable at M_0 with $g(M_0)=0$, we see that

$$g(x, y)=g_x(M_0)(x-x_0)+g_y(M_0)(y-y_0)+o(\sqrt{(x-x_0)^2+(y-y_0)^2})$$

and

$$f(x, y)g(x, y)=f(M_0)g_x(M_0)(x-x_0)$$
$$+f(M_0)g_y(M_0)(y-y_0)+h(x, y),$$

where

$$h(x, y) = \big(f(x, y) - f(M_0)\big)\big(g_x(M_0)(x - x_0) + g_y(M_0)(y - y_0)\big)$$

$$+ o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right).$$

It follows from

$$\left|\big(f(x, y) - f(M_0)\big)\big(g_x(M_0)(x - x_0) + g_y(M_0)(y - y_0)\big)\right|$$

$$\le \left|f(x, y) - f(M_0)\right| \cdot \sqrt{g_x(M_0)^2 + g_y(M_0)^2} \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

that

$$h(x, y) = o\left(\sqrt{(x - x_0)^2 + (y - y_0)^2}\right).$$

Hence $f(x, y)g(x, y)$ is differentiable at M_0.

Example 10 Suppose $f(x, y)$ is differentiable, $f(1,1)=1$, $f_x(1,1)=a$ and $f_y(1,1)=b$.

Let $\varphi(x)=f\{x, f[x, f(x,x)]\}$. Find $\varphi(1)$ and $\varphi'(1)$.

Solution Since

$$f[1, f(1,1)]=f(1,1)=1$$

and

$$f\{1, f[1, f(1,1)]\}=f(1,1),$$

we get

$$\varphi(1) = 1 .$$

Since

$$\varphi'(x) = f_x\left\{x, f\left[x, f(x,x)\right]\right\} + f_y\left\{x, f\left[x, f(x,x)\right]\right\}\left\{f\left[x, f(x,x)\right]\right\}'$$

$$= f_x\left\{x, f\left[x, f(x,x)\right]\right\} + f_y\left\{x, f\left[x, f(x,x)\right]\right\} \cdot \left\{f_x\left[x, f(x,x)\right] + f_y\left[x, f(x,x)\right]\left[f(x,x)\right]'\right\}$$

$$= f_x\{x, f[x, f(x,x)]\} + f_y\{x, f[x, f(x,x)]\} \cdot$$

$$\{f_x[x, f(x,x)] + f_y[x, f(x,x)][f_x(x,x) + f_y(x,x)]\},$$

we have that

$$\varphi'(1) = f_x(1,1) + f_y(1,1) \cdot \{f_x(1,1) + f_y(1,1)[f_x(1,1) + f_y(1,1)]\}$$

$$= a + ab + ab^2 + b^3.$$

Example 11 Suppose $f_x(x_0, y_0)$ exists and $f_y(x, y)$ is continuous at (x_0, y_0). Show that $f(x, y)$ is differentiable at (x_0, y_0).

Proof Let Δx and Δy be the increments of x and y, respectively.

Then the corresponding increment Δz of z is

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

$$= f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) + f(x_0+\Delta x, y_0) - f(x_0, y_0).$$

Since

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) = f_y(x_0+\Delta x, y_0+\theta\,\Delta y)\Delta y$$

$$= f_y(x_0, y_0)\Delta y + \varepsilon\Delta y,$$

where $\varepsilon = f_y(x_0 + \Delta x, y_0 + \theta\,\Delta y) - f_y(x_0, y_0)$ and $\theta \in (0,1)$

and

$$f(x_0 + \Delta x, y_0) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + o(\Delta x).$$

Hence

$$\Delta z = f_y(x_0, y_0)\Delta y + f_x(x_0, y_0)\Delta x + \varepsilon\Delta y + o(\Delta x).$$

The continuity of $f_y(x, y)$ at (x_0, y_0) implies that

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \varepsilon = 0.$$

These yield

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{o(\Delta x) + \varepsilon \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = 0,$$

which implies that $f(x, y)$ is differentiable at (x_0, y_0).