Lecture 7 Partial derivatives and differentials (II)

§ 3 Higher order partial derivatives and differentials

3.1 Definition of higher order partial differentials

$$f_{x^2}(x, y) = \lim_{\Delta x \to 0} \frac{f_x(x+\Delta x, y) - f_x(x, y)}{x};$$

$$f_{xy}(x, y) = \lim_{\Delta y \to 0} \frac{f_x(x, y+\Delta y) - f_x(x, y)}{y};$$

$$f_{yx}(x, y) = \lim_{\Delta x \to 0} \frac{f_y(x + \Delta x, y) - f_y(x, y)}{x};$$

$$f_{y^2}(x, y) = \lim_{\Delta y \to 0} \frac{f_y(x, y\Delta y) - f_y(x, y)}{y}.$$

In the same way, we can define the following:

$$f_{x^3},\ f_{x^2y},\ f_{xy^2},\ f_{y^3}\ \text{etc.}$$

Example 3.1.1 Suppose $u = xye^x \cos y$. Find all its second order partial derivatives.

Solution Obviously,

$$u_x = ye^x \cos y + xye^x \cos y, \quad u_y = xe^x \cos y - xye^x \sin y.$$

Then

$$u_{x^2} = (x+2)ye^x \cos y;$$

$$u_{xy} = e^x \cos y - ye^x \sin y + xe^x \cos y - xye^x \sin y$$

$$= (x+1)(\cos y - y \sin y)e^x$$

(The second partial derivative with respect to variables x and y)

and

$$u_{y^2} = -(2\sin y + y\cos y)xe^x.$$

Example 3.1.2 Suppose

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}.$$

Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

Solution It follows from

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0$$

and

$$f_y(0,0)=\lim_{x\to 0}\frac{f(0,y)-f(0,0)}{y}=0$$

that $f_x(x,y)=\begin{cases}\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}y, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

and

$$f_y(x, y) = \begin{cases} \dfrac{x^4 - 4x^2y^2 - y^4}{(x^2+y^2)^2}x, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}.$$

Consequently,

$$f_{xy}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = -1$$

and

$$f_{yx}(0, 0) = \lim_{x \to 0} \frac{f_y(x, 0) - f_y(0, 0)}{x} = 1.$$

Remark 3.1.1 This implies that, in general,

$f_{xy}(x, y) \neq f_{yx}(x, y)$.

Theorem 3.1.2 If both f_{xy} and f_{yx} are continuous, then

$f_{xy} = f_{yx}$.

Proof Let $F = \varphi(x, y+\Delta y) - \varphi(x, y)$ and

$\varphi(x, y) = f(x+\Delta x, y) - f(x, y)$.

Then

$$F = \varphi_y(x, y+\theta_1 \Delta y)\Delta y$$

$$= [f_y(x+\Delta x, y+\theta_1 \Delta y) - f_y(x, y+\theta_1 \Delta y)]\Delta y$$

$= f_{yx}(x+\theta_2\Delta x, y+\theta_1\Delta y)\Delta x\Delta y$,

where $0<\theta_1, \theta_2<1$. Similarly

$$F = f_{xy}(x+\theta_3\Delta x, y+\theta_4\Delta y)\Delta x\Delta y,$$

where $0<\theta_3, \theta_4<1$. Now, the conclusion $f_{xy} = f_{yx}$ follows from the assumption that both f_{xy} and f_{yx} are continuous.

Corollary 3.1.3 Suppose $u = f(x, y)$ has all partial derivatives up to order n including that they are continuous. Then

$$f_{x^\lambda y^{k-\lambda}} = \frac{\partial^k f}{\partial x^\lambda \partial y^{k-\lambda}} \quad (0<k<n, \quad 0\le\lambda\le k).$$

3.2 A formula for higher order differentials

Theorem 3.2.1 Suppose $u = f(x, y)$. Then

$$d^n u = \sum_{n=0}^{n} C_n^k \frac{\partial^n f}{\partial x^{n-k} \partial y^k} dx^{n-k} dy^k .$$

Solution Obviously, $du = f_x(x, y)dx + f_y(x, y)dy$. This shows that the result holds when $n = 1$.

We assume that when $n = r$, the result holds. That means,

$$d^r u = \sum_{k=0}^{r} C_r^k \frac{\partial^r f}{\partial x^{r-k} \partial y^k} dx^{r-k} dy^k .$$

When $n=r+1$, we know that

$$d^{r+1}u = d(d^r u) = d\left(\sum_{k=0}^{r} C_r^k \frac{\partial^r f}{\partial x^{r-k}\partial y^k} dx^{r-k} dy^k\right)$$

$$= \sum_{k=0}^{r} C_r^k d\left(\frac{\partial^r f}{\partial x^{r-k}\partial y^k}\right) dx^{r-k} dy^k$$

$$= \sum_{k=0}^{r} C_r^k \left(\frac{\partial^{r+1} f}{\partial x^{r-k+1}\partial y^k} dx + \frac{\partial^{r+1} f}{\partial x^{r-k}\partial y^{k+1}} dy\right) dx^{r-k} dy^k$$

$$= \sum_{k=0}^{r+1} C_{r+1}^k \frac{\partial^{r+1} f}{\partial x^{r-k+1}\partial y^k} dx^{r+1-k} dy^k .$$

Example 3.2.1 Suppose $u = \frac{1}{2}\log(x^2 + y^2 + 1)$. Find d^2u.

Solution It follows from

$$u_x = \frac{x}{x^2 + y^2 + 1} \text{ and } u_y = \frac{y}{x^2 + y^2 + 1}$$

that

$$u_{xx} = \frac{-x^2 + y^2 + 1}{(x^2 + y^2 + 1)^2}, \quad u_{xy} = -\frac{2xy}{(x^2 + y^2 + 1)^2}$$

and

$$u_{yy} = \frac{x^2 - y^2 + 1}{(x^2 + y^2 + 1)^2}.$$

Hence

$$d^2u = \frac{-x^2+y^2+1}{(x^2+y^2+1)^2}dx^2 - \frac{4xy}{(x^2+y^2+1)^2}dxdy + \frac{x^2-y^2+1}{(x^2+y^2+1)^2}d^2y\,.$$

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