Lecture 9 Derivation of functions determined by an equation

§ 1 Derivation of functions determined by $F(x,y,z)=0$

Theorem 1.1 Suppose $F(x,y,z)=0$ and $F_z \neq 0$. Then

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}.$$

Proof Differentiating the equation $F(x,y,z)=0$ with respect to x and y, respectively, we get

$$F_x + F_z \cdot \frac{\partial z}{\partial x} = 0 \quad \text{and} \quad F_y + F_z \cdot \frac{\partial z}{\partial y} = 0.$$

Hence

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Example 1.1 Suppose $z = f(x, y)$ is determined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution By differentiating the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

with respect to x,

we have that

$$\frac{2}{a^2}x+\frac{2}{c^2}zz_x=0,$$

whence

$$z_x=-\frac{c^2x}{a^2z}.$$

In a similar way,

$$z_y=-\frac{c^2y}{b^2z}.$$

Example 1.2 Suppose $F(x-y, y-z, z-x)=0$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution By differentiating the equation $F(x-y, y-z, z-x)=0$ with respect to x,

we have that

$$F_1'+F_2'(-z_x)+F_3'(z_x-1)=0,$$

whence

$$z_x=\frac{F_3'-F_1'}{F_3'-F_2'}.$$

In a similar way,

$$z_y = \frac{F_1' - F_2'}{F_3' - F_2'}.$$

§2 Derivation of functions determined by $F(x,y,z)=0$ and $G(x,y,z)=0$

Theorem 2.1 Suppose $z=z(x)$ and $y=y(x)$ are determined by

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}.$$

Then

$$y'(x) = -\frac{\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} \quad \text{and} \quad z'(x) = -\frac{\begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}}.$$

Proof By differentiating the equation system $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$

with respect to x, we have that

$$\begin{cases} F_x + F_y y' + F_z z' = 0 \\ G_x + G_y y' + G_z z' = 0 \end{cases},$$

whence

$$y'(x) = -\frac{\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}} \quad \text{and} \quad z'(x) = -\frac{\begin{vmatrix} F_y & F_x \\ G_y & G_x \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}}.$$

By using $\dfrac{D(F,G)}{D(y,z)} = \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}$, we have

$$y'(x) = -\frac{\dfrac{D(F,G)}{D(x,z)}}{\dfrac{D(F,G)}{D(y,z)}} \quad \text{and} \quad z'(x) = -\frac{\dfrac{D(F,G)}{D(y,x)}}{\dfrac{D(F,G)}{D(y,z)}}.$$

A similar argument as in the proof of Theorem 2.1 shows that

Theorem 2.2 Suppose the m functions $u_i(x_1, x_2, \cdots, x_n)$ $(i = 1, 2, \cdots, n)$ are determined by m equations:

$$F_i(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_m) = 0 \quad (i = 1, 2, \cdots, m).$$

Then

$$\frac{\partial u_i}{\partial x_j} = -\frac{\dfrac{D(F_1, \cdots, F_{j-1}, F_j, F_{j+1}, \cdots, F_m)}{D(u_1, \cdots, u_{j-1}, x_j, u_{j+1}, \cdots, u_m)}}{\dfrac{D(F_1, \cdots, F_{j-1}, F_j, F_{j+1}, \cdots, F_m)}{D(u_1, \cdots, u_{j-1}, u_j, u_{j+1}, \cdots, u_m)}}.$$

Example 2.1 Suppose $x=r\cos\theta, y=r\sin\theta$. Find

$$\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial y}.$$

Solution By differentiating the equation system

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

with respect to x we have that

$$\begin{cases} \frac{\partial r}{\partial x}\cos\theta - \frac{\partial \theta}{\partial x} r\sin\theta = 1 \\ \frac{\partial r}{\partial x}\sin\theta + \frac{\partial \theta}{\partial x} r\cos\theta = 0 \end{cases}.$$

Hence

$$\begin{cases} \dfrac{\partial r}{\partial x} = \cos\theta \\ \dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r} \end{cases}.$$

In a similar way, we can get

$$\begin{cases} \dfrac{\partial r}{\partial x} = \sin\theta \\ \dfrac{\partial \theta}{\partial x} = -\dfrac{\cos\theta}{r} \end{cases}.$$

Example 2.2 Suppose $\begin{cases} x+y+z+u+v=1 \\ x^2+y^2+z^2+u^2+v^2=2 \end{cases}$.

Find u_x and u_{x^2}.

Solution By differentiating the equation system

$$\begin{cases} x+y+z+u+v=1 \\ x^2+y^2+z^2+u^2+v^2=2 \end{cases}$$

with respect to x we have that

$$\begin{cases} u_x + v_x = -1 \\ 2uu_x + 2vv_x = -2x \end{cases}.$$

Hence

$$\begin{cases} u_x = \dfrac{x}{u-v} \\ v_x = -\dfrac{x}{u-v} \end{cases}.$$

By differentiating the equation $u_x = \dfrac{x}{u-v}$ with respect to x

we know that

$$u_{x^2} = \frac{(u-v)-x(u_x - v_x)}{(u-v)^2} = \frac{(u-v)^2 - 2x^2}{(u-v)^3}.$$

Example 2.3 Use the transformation $u=x+y, v=x-y$ and $w=xy-z$ to change the following equation:

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=0.$$

Solution It easily follows that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}=1 \text{ and } \frac{\partial v}{\partial y}=-1.$$

Hence all the second order partial derivatives of u and v are 0.

Since $z = xy - w$, we have the following equalities:

$$\frac{\partial z}{\partial x} = y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}, \quad \frac{\partial z}{\partial y} = x - \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v};$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\partial^2 w}{\partial u^2} - 2\frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial v^2}, \quad \frac{\partial^2 z}{\partial x \partial y} = 1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2};$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial v^2}.$$

Hence

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 2 - 4\frac{\partial^2 w}{\partial u^2}.$$

This yields

$$\frac{\partial^2 w}{\partial u^2} = \frac{1}{2}.$$

Example 2.4 Change the equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ by using the following transformation $\xi = x - at, \quad \eta = x + at$.

Solution Since

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = -a\frac{\partial u}{\partial \xi} + a\frac{\partial u}{\partial \eta},$$

we have that

$$\frac{\partial^2 u}{\partial t^2} = -a(-a\frac{\partial^2 u}{\partial \xi^2} + a\frac{\partial^2 u}{\partial \xi \partial \eta}) + a(-a\frac{\partial^2 u}{\partial \xi \partial \eta} + a\frac{\partial^2 u}{\partial \eta^2})$$

$$= a^2(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}).$$

Similarly, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}.$$

Hence the equation is changed into the form:

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

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Page 182: 7 (5); 10; 11; 13; 18(3).