Chapter 17

Lecture 21 Integrals with parameter(s) (I)

§ 1 The concept of integrals with parameter(s)

Defintion 1.1 Suppose $f \in C[a,b;c,d]$.

Then

$$I(y)=\int_a^b f(x,y)dx$$

is called the integral with the parameter y.

§ 2 Properties

Proposition 2.1 Suppose $f \in C[a,b;c,d]$. Then $I(y) \in C[c,d]$.

Proof It's Obvious that

$$I(y+\Delta y) - I(y) = \int_a^b f(x, y+\Delta y)dx - \int_a^b f(x,y)dx$$

$$= \int_a^b [f(x, y+\Delta y) - f(x,y)]dx.$$

Since $f \in C[a,b;\ c,d]$, it is uniformly continuous. Hence for any $\varepsilon > 0$, there must exist some δ such that for any pairs (x_1, y_1) and (x_2, y_2), if

$$|x_1 - x_2| < \delta \text{ and } |y_1 - y_2| < \delta,$$

then $$\left|f(x_1,y_1)-f(x_2,y_2)\right|<\varepsilon.$$

Hence

$$\left|I(y+\Delta y)-I(y)\right|\le\int_a^b\left|f(x,y+\Delta y)-f(x,y)\right|dx<(b-a)\varepsilon.$$

This shows that

$$\lim_{y\to y_0}\int_a^b f(x,y)dx=\int_a^b\lim_{y\to y_0}f(x,y)dx.$$

Proposition 2.2 Suppose both f and f_y are continuous on $[a,b;\ c,d]$. Then

$$\frac{d}{dy}\int_a^b f(x,y)dx=\int_a^b f_y(x,y)dx=\int_a^b\frac{\partial f(x,y)}{\partial y}dx.$$

Proof The following is obvious.

$$\frac{I(y+\Delta y)-I(y)}{\Delta y}=\frac{\int_a^b f(x,y+\Delta y)-f(x,y)dx}{\Delta y}$$

$$=\int_a^b \frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}dx$$

$$=\int_a^b f_y(x,y+\Delta y)dx.$$

where $0<\theta<1$.

Proposition 2.1 shows that

$$I'(y)=\int_a^b f_y(x,y)dx.$$

Proposition 2.3 Suppose that $f\in C[a,b;c,d]$, and that both $a(y)$ and $b(y)\in C[c,d]$, $a\le a(y)\le b$, $a\le b(y)\le b$.

Then

$$F(y)=\int_{a(y)}^{a(y)} f(x,y)\,dx$$

is continuous on $[c,d]$.

Proof Since

$$F(y+\Delta y)-F(y)=\int_{a(y+\Delta y)}^{b(y+\Delta y)} f(x,y+\Delta y)\,dx-\int_{a(y)}^{b(y)} f(x,y)\,dx$$

$$=-\int_{a(y)}^{a(y+\Delta y)} f(x,y+\Delta y)\,dx+\int_{b(y)}^{b(y+\Delta y)} f(x,y+\Delta y)\,dx$$

$$+\int_{a(y)}^{b(y)}\left[f(x,y+\Delta y)-f(x,y)\right]dx,$$

we see that

$$\lim_{\Delta y \to 0}\left[F(y+\Delta y)-F(y)\right]=0.$$

Proposition 2.4 Suppose $f,\ f_y \in C[a,b;\ c,d]$, $a'(y)$, $b'(y) \in C[c,d]$, $a \le a(y)$ and $b(y) \le b$.

Then

$$F(y)=\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)dx+\int_{a(y)}^{b(y)} f_y dx+f[b(y),y]b'(y)-f[a(y),y]a'(y).$$

Proof For any fixed y_0, let

$$F(y)=\int_{a(y_0)}^{b(y_0)} f(x,y)dx+\int_{b(y_0)}^{b(y)} f(x,y)dx-\int_{a(y_0)}^{a(y)} f(x,y)dx$$

$$=F_1(y)+F_2(y)-F_3(y).$$

Then

$$F_1'(y_0) = \int_{a(y_0)}^{b(y_0)} f_y dx \;;$$

$$F_2'(y_0) = \lim_{y \to y_0} \frac{F_2(y) - F_2(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{\int_{b(y_0)}^{b(y)} f(x, y) dx}{y - y_0}$$

$$= \lim_{y \to y_0} \left[\frac{b(y) - b(y_0)}{y - y_0} f(\bar{x}, y) \right]$$

$$= f(b(y_0), y_0) b'(y_0)$$

and

$$F_3'(y_0) = f(a(y_0), y_0) a'(y_0).$$

Hence

$$F'(y) = \int_{a(y)}^{b(y)} f_y(x, y) dx + f(b(y), y) b'(y) - f(a(y), y) a'(y).$$

Example 2.1 Suppose $F(y)=\int_y^{y^2}\frac{\sin yx}{x}dx$. Find $F'(y)$.

Solution $$F'(y)=\int_y^{y^2}\cos yx dx+\frac{2\sin y^3}{y}-\frac{\sin y^2}{y}$$

$$=\frac{3\sin y^3-2\sin y^2}{y}.$$

Example 2.2 Show that $\lim_{y\to 0^+}F(y)=\int_y^{y^2}\frac{\frac{\sin yx}{x}}{y^2}dx=-1$.

Proof $$\lim_{y\to 0^+}F(y)=\lim_{y\to 0^+}\frac{\int_y^{y^2}\frac{\sin yx}{x}dx}{y^2}=\lim_{y\to 0^+}\frac{\frac{3\sin y^3-2\sin y^2}{y}}{2y}=-1.$$

Example 2.3 Find $I(\theta)=\int_0^{\pi}\ln(1+\theta\cos x)dx \quad (\theta\in(-1,1))$.

Solution Let

$$f(\theta,x)=1+\theta\cos x,\quad f_{\theta}(\theta,x)=\frac{\cos x}{1+\theta\cos x}.$$

Then

$$f,f_{\theta}\in C[0,\pi;\ a,b],$$

where $0<b<1$.

Proposition 2.2 shows that

$$I'(\theta)=\int_0^{\pi}\frac{\cos x}{1+\theta\cos x}dx=\frac{\pi}{\theta}-\frac{1}{\theta}\int_0^{\pi}\frac{1}{1+\theta\cos x}dx.$$

Since

$$\int \frac{dx}{1+\theta\cos x} = \frac{2}{\theta\sqrt{1-\theta^2}}\arctan\left(\sqrt{\frac{1-\theta}{1+\theta}}\tan x\right)+C \quad \left(t=\tan\frac{x}{2}\right),$$

we see that

$$I'(\theta)=\frac{\pi}{\theta}-\frac{2}{\theta\sqrt{1-\theta^2}}\frac{\pi}{2}=\pi\left(\frac{1}{\theta}-\frac{1}{\theta\sqrt{1-\theta^2}}\right).$$

It follows that

$$I(\theta)=\pi\left(\log\theta+\log\frac{1+\sqrt{1-\theta^2}}{\theta}\right)+C=\pi\log\left(1+\sqrt{1-\theta^2}\right)+C.$$

The hypothesis $I(0)=0$ implies $C=-\pi\log 2$. Hence

$$I(\theta)=\pi\log\left(1+\sqrt{1-\theta^2}\right)-\pi\log 2=\pi\frac{\log\left(1+\sqrt{1-\theta^2}\right)}{2}.$$

Homework Page 249: 1; 2. Page 250: 5 (1)