Optimum Structural Design

结构优化设计

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简要回顾：上一节课所讲方法思路
Review: basic idea of methods in the previous Chapters

- Chapter 2
  - Single va

What are the common characteristics of the transformed methods in Chapter 3 ?

- Chapter 3
  - Multi variables
  - Usually, **transform** to single variable
  - Common characteristics: Optimization along a line

Characteristics of the problem in this Chapter

- Chapter 4 Linear Programming 第四章 线性规划
- Constrained optimization
- A special type
  - both the objective and the constraint functions are linear functions of optimization variables
- Classified as linear programming
- Denoted as LP for short

Chapter 4 Linear Programming
第四章 线性规划

- 4.1 简介
Introduction
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- **4.5定义及原理**
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- 4.6 一般方程组的转轴运算

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- 4.7 单纯形法思想的形成

Motivation to simplex Method

- 4.8 单纯形法

simplex Algorithm

4.1 Introduction 简介

- Originating of LP
- Some ea ly example applications of linear programmi
  - (i) In tl
    diesel fuel and gasoline)
  - (ii) in a manufacturing firm (take into account the various cost and loss factors and arrive at the most profitable production plan)
  - (iii) in a communication network and the routing of aircraft and ships

Who originated LP ?

- The applications of linear programming in civil engineering :
- (I) A linear programming can be applied to allocate the amount of procurement on bricks from different manufacturers at different prices and with different strengths, minimizing total cost.
- (ii) A linear programming can also be used to determine the amount of buildings to be tendered for construction with different profit level and different resource consumption so as to maximize total construction profit.
- Though numerous of applications can be found in civil engineering, this book will only concentrate on basic theory of linear programming and present several typical example in structural design.

Linear Programming

- Example of modeling

	Building type 1	Building type 2	available
Labor	1500	2100	23200
Machine	3200	1600	31500
Finance	5500	8600	92500
Profit	26000	31400	

- **Maximize the profit** under given available resource

4.2
线形规划问题的一般形式

GENERAL FORM OF A LINEAR PROGRAMMING PROBLEM

- A general form of linear programming problem can be stated as
- (i) *In scalar form.*

min or max

$$f(x_1, x_2, \cdots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad (4.2.1)$$

s.t.

$$\left.\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ &\vdots \\ &a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \le b_k \end{aligned}\right\} \qquad (4.2.2a)$$

$$\left.\begin{aligned} &d_{11}x_1 + d_{12}x_2 + \cdots + d_{1n}x_n \ge e_1 \\ &d_{21}x_1 + d_{22}x_2 + \cdots + d_{2n}x_n \ge e_2 \\ &\vdots \\ &d_{l1}x_1 + d_{l2}x_2 + \cdots + d_{ln}x_n \ge e_l \end{aligned}\right\} \qquad (4.2.2b)$$

$$\left.\begin{aligned} &p_{11}x_1 + p_{12}x_2 + \cdots + p_{1n}x_n = q_1 \\ &p_{21}x_1 + p_{22}x_2 + \cdots + p_{2n}x_n = q_2 \\ &\vdots \\ &p_{s1}x_1 + p_{s2}x_2 + \cdots + p_{sn}x_n = q_s \end{aligned}\right\} \qquad (4.2.2b)$$

GENERAL FORM OF A LINEAR PROGRAMMING PROBLEM

- c_j, b_j, e_j, q_j, a_{ij}, d_{ij} and p_{ij} (i = 1, 2, …, n) and known constants
- x_j are the
- general
  - objective func… …mizing or maximizing
  - constraint fun… …ns: equalities or inequalities
- characteristics of the general form of LP
  - both the objective and the constraint functions are linear

What are the common characteristics of the transformed methods in Chapter 3 ?

(ii) *In summation form.* *The linear programming problem in scalar form may also be stated in a compact form by using the summation sign as:*

min or max $$f(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} c_j x_j \qquad (4.2.3)$$

s.t. $$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, 2, \cdots, k \qquad (4.2.4a)$$

$$\sum_{j=1}^{n} d_{tj} x_j \ge e_t \quad t = 1, 2, \cdots, l \qquad (4.2.4b)$$

$$\sum_{j=1}^{n} p_{rj} x_j = q_r \quad r = 1, 2, \cdots, s \qquad (4.2.4c)$$

- (iii) *In matrix form*

min or max $$C^T X \qquad (4.2.5)$$

s.t.

$$AX \le B \qquad (4.2.6a)$$

$$DX \ge E \qquad (4.2.6b)$$

$$PX = Q \qquad (4.2.6c)$$

Where

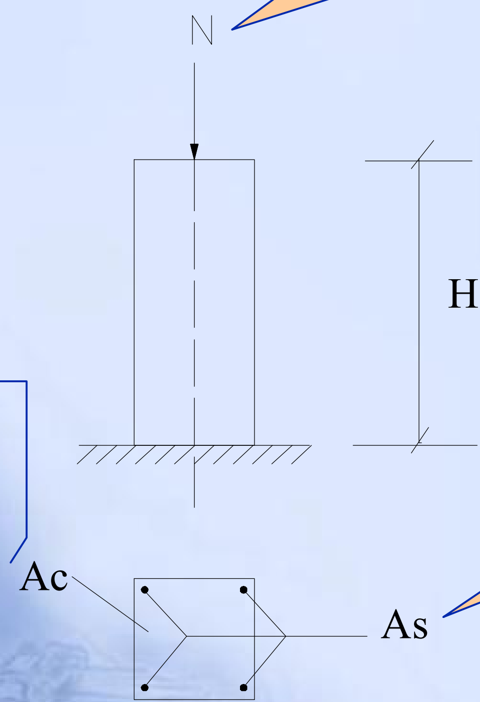
$$X = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \quad a = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \quad B = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{Bmatrix} \quad C = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix}$$

$$D = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & & & \\ d_{l1} & d_{l2} & \cdots & d_{ln} \end{bmatrix} \quad E = \left\{ \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_l \end{matrix} \right\} \quad P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & & & \\ p_{s1} & p_{s2} & \cdots & p_{sn} \end{bmatrix} \quad Q = \left\{ \begin{matrix} q_1 \\ q_2 \\ \vdots \\ q_s \end{matrix} \right\}$$

and the superscript *T* is used to indicate the transpose.

- ***An example of model of linear programming***.

Given an RC short column carrying an axial loading *N*, as shown in Fig 4.2.1, derive a mathematic minimizes the cost of the column.



Total cost of the main materials in the RC column can be expressed as

$$c_t = c_c A_c H + c_s A_s H G_s \quad (4.2.7)$$

The compression strength requirement of the RC column should be satisfied

$$N \le f_c A_c + f_y A_s \quad (4.2.8)$$

and

$$A_s \ge \rho_{\min} A_c \quad (4.2.9)$$

$$A_s \le \rho_{\max} A_c \quad (4.2.10)$$

Where $\rho_{\min}$ is the minimum reinforcement ratio.

$\rho_{\max}$ is the maximum reinforcement ratio.

- Minimize this transformed cost, the mathematical model of the RC column can be derived

min $$z = c_c A_c + c_s A_s G_s \qquad (4.2.11)$$

s.t
$$\left.\begin{aligned} N &\le f_c A_c + f_y A_s \\ A_s &\ge \rho_{\min} A_c \\ A_s &\le \rho_{\max} A_c \end{aligned}\right\} \qquad (4.2.12)$$

Given $N = 1640 kN$, C25 $f_c = 11.9 N/mm^2$,HRB335 $f_y = 300 N/mm^2$

$\rho_{\min} = 0.4\%$ $\rho_{\max} = 5\%$ $C_c = 300 \times 10^{-9}\ yuan / mm^3$

$C_s = 3.1\ yuan / kg$ $G_s = 7.8 \times 10^{-6}\ kg/mm^3$

- Substituting all these parameters into Eqs. (4.2.11)~(4.2.12), yields

Find , $A_s, \ A_c$ (4.2.13)

min $z = 3.0\times10^{-7} A_c + 2.418\times10^{-5} A_s$ (4.2.14)

s.t
$$\left.\begin{aligned} & A_s + 3.967\times10^{-2} A_c \geq 5.467\times10^{3} \\ & A_s \geq 0.004 A_c \\ & A_s \leq 0.05 A_c \end{aligned}\right\} \quad (4.2.15)$$

- Eqs. (4.2.13)~(4.2.14) are typical expressions of general form of linear programming problem.

4.3线形规划问题的标准形式

Standard Form of a Linear Programming Problem

- The standard form of linear programming problem can be stated in scalar, summation or matrix form. To save space, only two forms will be listed as follows:

(i) *In scalar form.*

$$\min \quad f(x_1, x_2, \cdots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \quad (4.3.1)$$

$$\text{s.t} \quad \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \quad (4.3.2a)$$

$$\left.\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \\ \mathrm{M} \\ x_n \geq 0 \end{array}\right\} \qquad (4.3.2b)$$

(ii) *In matrix form*.

$$\min \quad C^T X \qquad (4.3.3)$$

$$\text{s.t} \quad AX = B \qquad (4.3.4a)$$

$$X \geq 0 \qquad (4.3.4b)$$

- The characteristics of the linear programming problem stated in the standard form are:
- 1. The objective function is of the minimization type.
- 2. All the constraints are of the equality type.
- 3. All the decision variables are nonnegative.

- Any linear programming problem can be put in the standard form by the use of the transformations.

1. The maximization of a function is equivalent to the minimization of the negative of the same function .For example, the objective function.

minimize $$f = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

is equivalent to

maximize $$f' = -f = -c_1 x - c_2 x_2 - \cdots - c_n x_n$$

Consequently, the objective function can be stated in the minimization form in any linear programming problem.

2. In most of the engineering optimization problems, the decision variables represent some physical dimensions and hence the variables *xj* have to be nonnegative. Thus , if *xj* is unrestricted in sign, it can be written as

$$x_j = x_j^{'} - x_j^{''}$$

where

$x_j^{'} \geq 0$ and $x_j^{''} \geq 0$

It can be seen that *xj* will be negative, zero or positive depending on whether is greater than, equal to or less than .

3. If a constraint appears in the form of a 'less than' type of inequality as

$$a_{k_1} x_1 + a_{k_2} x_2 + \text{☹} + a_{k_n} x_n \leq b_k$$

it can be converted into the equality form by adding a nonnegative slack variable as follows:

$$a_{k_1} x_1 + a_{k_2} x_2 + \text{☹} + a_{k_n} x_n + x_{n+1} = b_k$$

where x_{n+1} is a nonnegative variable known as the surplus variable.

It can be seen that there are *m* equations in *n* decision variables in a linear programming problem.

Example 4.3.1

Consider the problem of maximizing

$$F = 2x_1 - x_2 + 5x_3$$

the constraints：

$$x_1 - 2x_2 + x_3 \le 8$$

$$3x_1 - 2x_2 \ge -18$$

$$2x_1 + x_2 - 2x_3 \le 4$$

$$x_3 \ge 0$$

State this linear programming problem in standard form.

- *Solution*

Since x1 and x2 are not restricted to be nonnegative we write then as

$$x_1 = x_1^+ - x_1^-$$

$$x_2 = x_2^+ - x_2^-$$

where $x_1^+ \geq 0, x_1^- \geq 0, x_2^+ \geq 0,$ and $x_2^- \geq 0$

Thus the problem becomes:

$$F = 2x_1^+ - 2x_1^- - x_2^+ + x_2^- + 5x_3$$

- subject to：

$$x_1^+ - x_1^- - 2x_2^+ + 2x_2^- + x_3 \le 8$$
$$3x_1^+ - 3x_1^- - 2x_2^+ + 2x_2^- \ge -18$$
$$2x_1^+ - 2x_1^- + x_2^+ - x_2^- - 2x_3 \le 4$$
$$x_1^+ \ge 0, x_1^- \ge 0, x_2^+ \ge 0, x_2^- \ge 0, x_3 \ge 0$$

- This can be stated as a minimization problem by taking the new objective as $-F$ and the constraints can be stated as equalities by introducing a slack or surplus variable． Thus the problem can be stated in standard form as:

- Minimize $f = -F = -(2x_1^+ - 2x_1^- - x_2^+ + x_2^- + 5x_3)$
  subject to

$$x_1^+ - x_1^- - 2x_2^+ + 2x_2^- + x_3 - 8 + y_1 = 0$$

$$3x_1^+ - 3x_1^- - 2x_2^+ + 2x_2^- + 18 - y_2 = 0$$

$$2x_1^+ - 2x_1^- + x_2^+ - x_2^- - 2x_3 - 4 + y_3 = 0$$

$$x_1^+ \geq 0, x_1^- \geq 0, x_2^+ \geq 0, x_2^- \geq 0, x_3 \geq 0,$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0,$$

- where y_1 and y_3 are the slack variables and y_2 is the surplus variable.

- **Example 4.3.2**

  State the linear programming problem of Eqs. (4.2.13)~(4.2.15) in standard form.

- *Solution*

  Since the design variables and are nonnegative, and the objective function are in the minimization form, one need only transform the constraints into equalities by introducing slack or surplus variables. This transformation will directly result in the standard form

- Find , A_s, A_c

  min $z = 3.0\times10^{-7} A_c + 2.418\times10^{-5} A_s$

  s.t $A_s + 3.967\times10^{-2} A_c - x_1 = 5.467\times10^{3}$

  $A_s - 0.004 A_c - x_2 = 0$

  $A_s - 0.05 A_c + x_3 = 0$

where x_1 and x_2 are the surplus variables and x_3 is the slack variable.

4.4线性几何规划问题的图解法

Geometry of Linear Programming problem

A linear programming problem with only two variables presents a simple case for which the solution can be obtained by using a rather elementary graphical method. Apart from the solution, the graphical method gives a physical picture of certain geometrical characteristics of linear programming problem. The following example is considered to illustrate the graphical method of solution.

- **Example 4.4.1**
- A construction team tenders for two types of plastering work
  - Sprayed granite-like coating
  - Terrazzo
- Consumes some material, and requires grinding machines and high pressure pumping machines
- different machining required for each plastering
- profit on each plastering type given in table

Type of machine	Machining time required for the machine part granite coating I	terrazzo II	Maximum time available per week (minutes)
Material consumption (kg)	10	5	2500
Grinding machines (hour)	4	10	2000
Pressure pump (hour)	1	1.5	450
Profit per unit	50 Yuan	100 Yuan	

- Determine the number of plastering I and II manufactured per week be denoted by *x* and *y* respectively. The constraints due to the maximum time limitations on the various machines are given by

$$10x + 5y \leq 2500 \qquad (E_1)$$

$$4x + 10y \leq 2000 \qquad (E_2)$$

$$x + 1.5y \leq 450 \qquad (E_3)$$

- Since the variables *x* and *y* cannot take negative valued, we have

$$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \end{array}\right\} \qquad (E_4)$$

- The total profit is given by

$$f(x, y) = 50x + 100y \qquad (E_5)$$

- Thus the problem is to determine the nonnegative values of *x* and *y* which satisfy the constraints stated in Eqs.(E1) to (E3) and maximize the objective function given by Eq.(E5).
- Solution

The set of inequalities given by Eqs.(E1) to (E3) can easily be represented on a graph. By taking the coordinate axes as *x* and *y*, the equality

$$10x + 5y = 2500 \qquad (E_6)$$

is shown by the line *AB* and the region corresponding to the inequality

$$10x + 5y \leq 2500 \qquad (E_1)$$

is shown by the shaded area in Fig. 4.3.1.

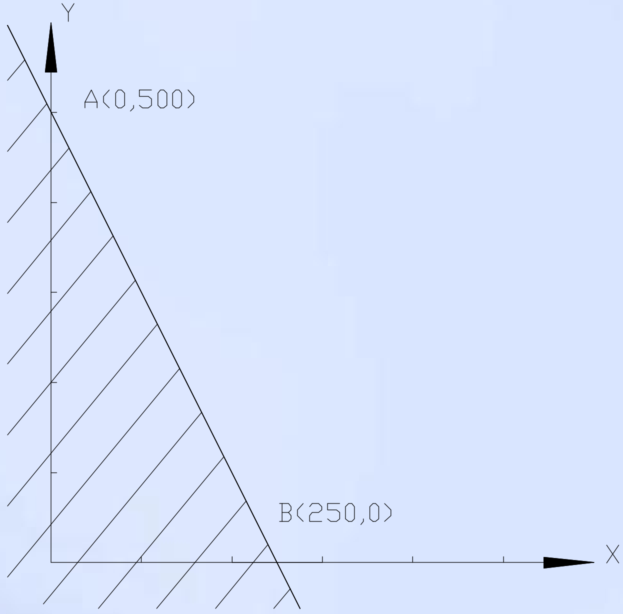


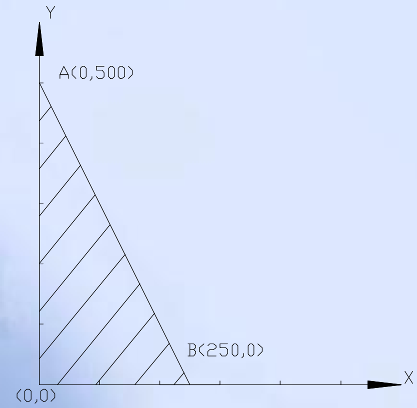
Figure 4.3.1 Feasible region given by Eq.(E1)

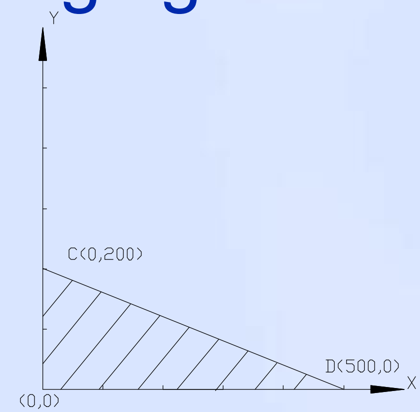
- To obtain the *x* and *y* intercepts for the inequality (E1), we proceed as follows:

Put *x*=0 in Eq. (E6) to get *y*=500 (point *A*).

Put *y*=0 in Eq. (E6) to obtain *x*=250 (point *B*).

- Similarly ,we can get the following figs:





- Figure 4.3.2 Feasible region given by Eqs.(E1) and (E4)

Figure 4.3 Feasible region given by Eqs.(E2) and (E4)

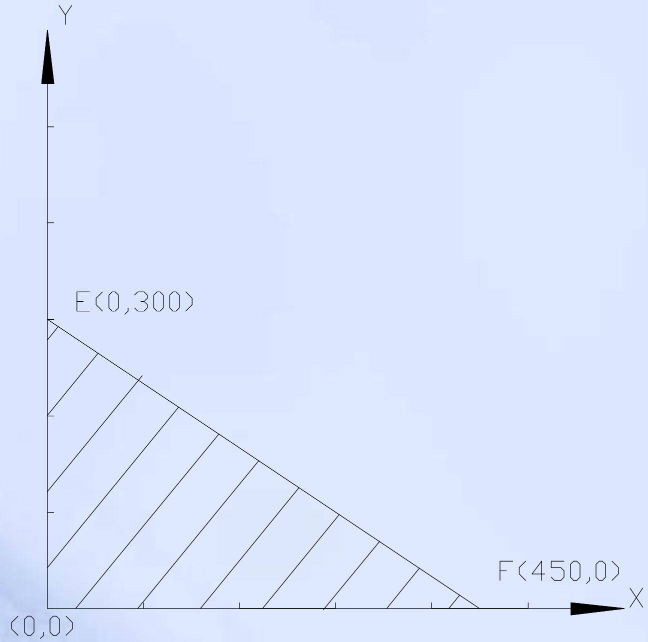
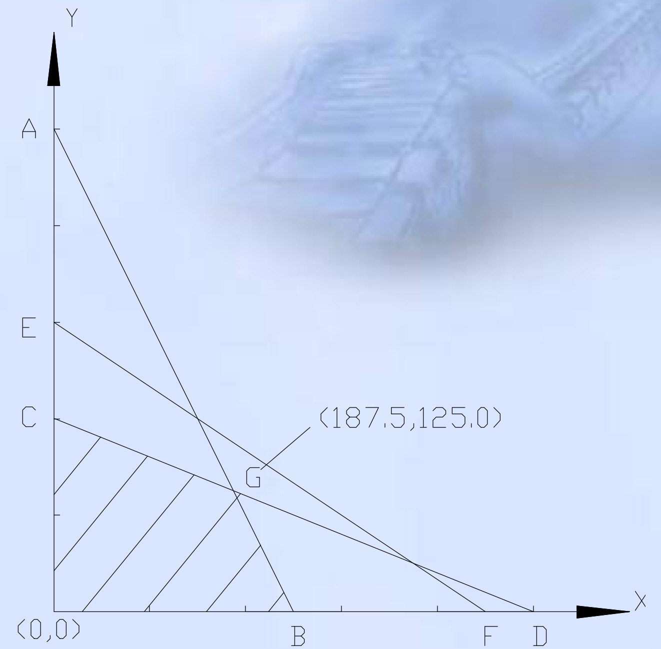
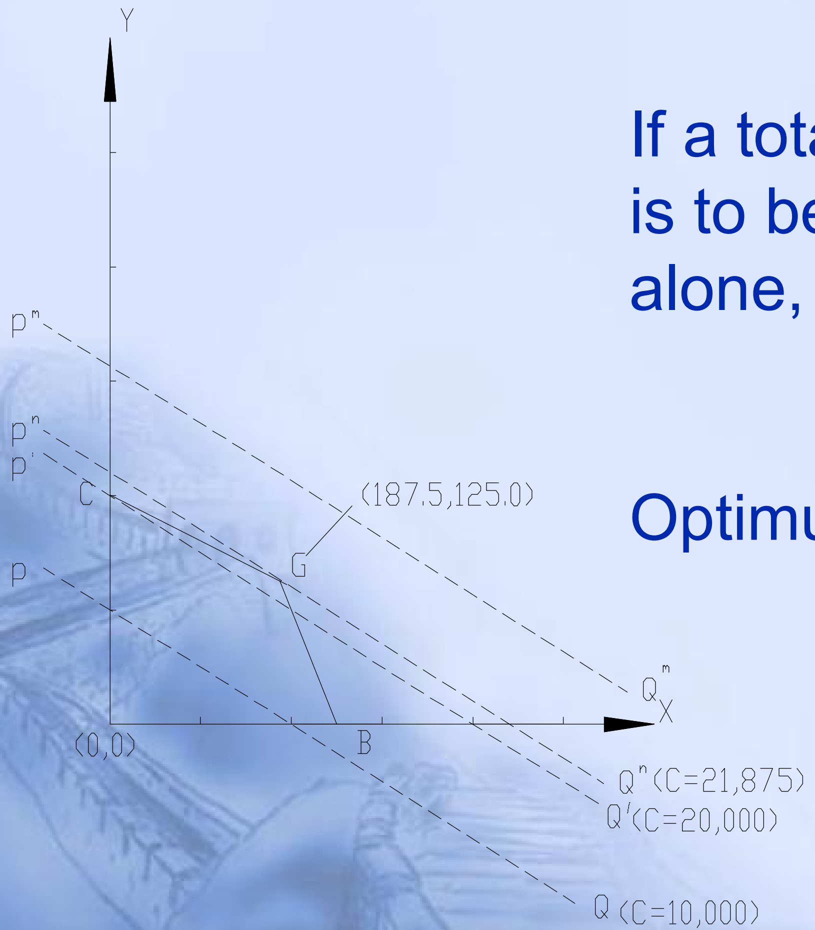


Figure 4.4 Feasible region given by Eqs.(E3) and (E4)



Feasible region given by Eqs.(E1) and (E4)

we obtain Fig. 4.5, the shaded area of which represents the region of all possible solutions for the problem. Our objective is to find at least one point out of the infinite points shown in the shaded region of fig. 4.5 which maximizes the profit function (E5).



If a total profit of, say, 10,000 *Yuan,* is to be obtained by producing type I alone, the value of *x* will be 200.

Optimum point $x^*=187.5$, $y^*=125.0$

- A linear programming problem may have

  (i) a unique and finite optimum solution

  (ii) an infinite number of optimal solutions

  (iii) an unbounded solution

  (iv) no solution

  (v) a unique feasible point.

- Assuming that the linear programming problem is properly formulated, the following general geometrical characteristics can be noted from the graphical solution:

  (i). The feasible region is a convex polygon.

  (ii). The optimum value occurs at an extreme point or vertex of the feasible region

4.5 定义及定理
Definitions and Theorems

- The geometrical characteristics of linear programming problems stated in the previous section can be proved mathematically. Some of the more powerful methods for solving linear programming problems take advantage of these characteristics. The terminology used in linear programming and some of the important theorems are considered in this section.

- **Point in n-dimensional space:** A point X in an n-dimensional space is characterized by an ordered set of n values or coordinates $(x_1, x_2, \cdots, x_n)$.
- **Line segment in n-dimensions (L):** If the coordinates of two points A and B are given by $x_j^{(1)}$ and $x_j^{(2)} (j = 1,2,\cdots,n)$,the line segment (L) joining these points is the collection of points $X(\lambda)$ whose coordinates are given by $x_j = \lambda x_j^{(1)} + (1-\lambda) x_j^{(2)}$, where $0 \leqslant \lambda \leqslant 1$.
- **Hyperplane:** In n-dimensional space, the set of points whose coordinates satisfy a linear equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = a^T X = b$$

is called a hyperplane.

- **Convex set :** It is a collection of points such shat if X(1) and X(2) are any two points in the collection, the line segment joining them is also in the collection. If S denotes the convex set, it can be defined mathematically as follows :

  If $X^{(1)}, X^{(2)} \in S$ then $X \in S$ 4

  where

  $$X = \alpha X^{(2)} + (1-\alpha)X^{(1)}, 0 \leq \alpha \leq 1$$

- **Convex polyhedron :** It is a set of points common to one or more halfspaces.

- Vertex (extreme point): This is a point in the convex set which does not lie on a line segment joining two other points of the set.

- feasible solution: In a linear programming problem, any solution which satisfies th. Constraints $aX = b$ and $X \geq 0$ is called a feasible solution.
- Basic solution: This a solution in which variables are set equal to zero.
- Basis: The collection of variables not set equal to zero to obtain the basic solution is the basis.

- **Basic feasible solution:** This is a basic solution which satisfied the nonnegativity conditions.

- **Nondegenerate basic feasible solution:** This is a basic feasible solution which has got exactly m positive xi.

- **Optimal solution:** Feasible solution which optimizes the objective function is called an optimal solution.

- **optimal basic solution:** This is a basic feasible solution for which the objective function is optimal.

- The basic theorems in linear programming
- THEOREM 4.1: The intersection of any number of convex sets is also convex.
- proof
- THEOREM 4.2: The feasible region of a linear programming problem is convex.
- proof
- THEOREM 4.3: Any local minimum solution is global for a linear programming problem.
- proof
- THEOREM 4.4 :Every basic feasible solution is an extreme point of the convex set of feasible solutions.
- proof

- THEOREM 4.5 :Let S be a closed, bounded convex polyhedron with X_i^e i =1 to p as the set of its extreme points. Then any vector $X \in S$ can be written as

$$X = \sum_{i=1}^{p} \lambda_i X_i^e \quad \lambda_i \geq 0 \qquad \sum_{i=1}^{p} \lambda_i = 1$$

- proof
- THROREM 4.6 ： Let S be a closed convex polyhedron. Then the minimum of a linear function over S is attained at an extreme point of S.
- proof

4.6 PIVOTAL REDUCTION OF LINEAR EQUATION SYSTEM

- 4.6.1 Solution of a system of Linear Simultaneous Equations 线性联立方程组的解
- Review: some of the elementary concepts of linear equations

- Consider the following system of n-equations in *n* unknowns :

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \qquad (E_1) \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad (E_2) \\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \qquad (E_3) \\
\vdots & \qquad\qquad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \qquad (E_n)
\end{aligned}
\qquad (4.5.1)
$$

Assuming that this set of equations possesses a unique solution, one way of solving the system consists of reducing the equations to a form known as canonical form.

- It is well known from elementary algebra that the solution of Eqs.(4.5.1) will not be altered under the following elementary operations:

(i) any equation E_r is replaced by the equation kE_r where k is a nonzero constant

(ii) any equation E_r is replaced by the equation E_r+kE_s where E_s is any other equation of the system.

By making use of these elementary operations, the system of Eqs.(4.5.1) can be reduced to a convenient equivalent form as follows.

- The resulting system of equations can be written as

$$
\begin{aligned}
&a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1,i-1}x_{i-1} + 0\cdot x_i + a'_{1,i+1}x_{i+1} + \cdots + a'_{1n}x_n = b'_1 \\
&a'_{21}x_1 + a'_{22}x_2 + \cdots + a'_{2,i-1}x_{i-1} + 0\cdot x_i + a'_{2,i+1}x_{i+1} + \cdots + a'_{2n}x_n = b'_2 \\
&\vdots \\
&a'_{j-1,1}x_1 + a'_{j-1,2}x_2 + \cdots + a'_{j-1,i-1}x_{i-1} + 0\cdot x_i + a'_{j-1,i+1}x_{i+1} + \cdots + a'_{j-1,n}x_n = b'_{j-1} \qquad (4.5.2) \\
&a'_{j,1}x_1 + a'_{j,2}x_2 + \cdots + a'_{j,i-1}x_{i-1} + 1\cdot x_i + a'_{j,i+1}x_{i+1} + \cdots + a'_{j,n}x_n = b'_j \\
&a'_{j+1,1}x_1 + a'_{j+1,2}x_2 + \cdots + a'_{j+1,i-1}x_{i-1} + 0\cdot x_i + a'_{j+1,i+1}x_{i+1} + \cdots + a'_{j+1,n}x_n = b'_j \\
&\vdots \\
&a'_{n1}x_1 + a'_{n2}x_2 + \cdots + a'_{n,i-1}x_{i-1} + 0\cdot x_i + a'_{n,i+1}x_{i+1} + \cdots + a'_{nn}x_n = b'_n
\end{aligned}
$$

- where the primes indicate that the a'_{ij} and b'_j are changed from the original system.

- A *pivot operation*: The procedure of eliminating a particular variable from all but one equations

- we take the system of Eqs.(4.5.2) and perform a new pivot operation by eliminating $x_s, s \neq i,$ in all the equations except in *i*th equation $i \neq j,$ he zeroes or the 1 in the *i*th column will not be disturbed . This pivotal operations can be repeated by using a different variable and equation each tine until the system of Eqs.(4.5.1) is reduced to the form

- 1

$$1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 + \cdots + 0\cdot x_n = b_1''$$
$$0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 + \cdots + 0\cdot x_n = b_2'' \qquad (4.5.3)$$
$$0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 + \cdots + 0\cdot x_n = b_3''$$

This system of Eqs.(4.5.3) is said to be in canonical form and has been obtained fter carrying out *n* pivot operations, From the canonical form, the solution vector can be directly obtained as

$$x_i = b_i'' \quad i = 1, 2, \cdots, n \qquad (4.5.4)$$

- Since the set of Eqs.(4.5.3) has been obtained from Eqs.(4.5.1) only through elementary operations, the system of Eqs.(4.5.3) is equivalent to the system of Eqs.(4.5.1). Thus the solution given in Eqs.(4.5.4) is the desired solution for Eqs.(4.5.1).

- **Example 4.6.1**
- Solve the following system of equations by using pivot operations.

$$(\mathrm{I}0) \quad 4x_1 + 3x_2 + x_3 = 13$$

$$(\mathrm{II}0) \quad 3x_1 + 0 \cdot x_2 + 7x_3 = 24$$

$$(\mathrm{III}0) \quad x_1 + 2x_2 + 3x_3 = 14$$

- Solution:
- Since $a_{11} \neq 0$ in Eq. (Ⅰ0), we can pivot on the element $a_{11} = 4$ to get

$$x_1 + \frac{3}{4}x_2 + \frac{1}{4}x_3 = \frac{13}{4} \qquad \mathrm{I}1 = \frac{1}{4}\mathrm{I}0$$

$$0 - \frac{9}{4}x_2 + \frac{25}{4}x_3 = \frac{57}{4} \qquad \mathrm{II}1 = \mathrm{II}0 - 3\,\mathrm{I}1$$

$$0 + \frac{5}{4}x_2 + \frac{11}{4}x_3 = \frac{43}{4} \qquad \mathrm{III}1 = \mathrm{III}0 - \mathrm{I}1$$

- Now pivoting on the element $a'_{22} = -9/4$ in Eq. (Ⅱ1), we obtain

$$x_1 + 0 + \frac{7}{3}x_3 = 8 \qquad \text{Ⅰ2= Ⅰ1} - \frac{3}{4}\text{Ⅱ2}$$

$$0 + x_2 - \frac{25}{9}x_3 = -\frac{57}{9} \qquad \text{Ⅱ2} = -\frac{4}{9}\text{Ⅱ1}$$

$$0 + 0 + \frac{56}{9}x_3 = \frac{56}{3} \qquad \text{Ⅲ2=Ⅲ1} - \frac{5}{4}\text{Ⅱ2}$$

- Finally by carrying out the pivotal operation on $a''_{33} = 56/9$

$$x_1 + 0 + 0 = 1 \qquad \text{Ⅰ3= Ⅰ2} - \frac{3}{7}\text{Ⅲ2}$$

$$0 + x_2 + 0 = 2 \qquad \text{Ⅱ3= Ⅱ2+} \frac{25}{9}$$

$$0 + 0 + x_3 = 3 \qquad \text{Ⅲ3=} \frac{9}{56}\text{Ⅲ2}$$

- The Eqs.(Ⅰ3), (Ⅱ3)and (Ⅲ3) are in canonical form from which the solution can be ready obtained as

$$x_1 = 1, x_2 = 2, x_3 = 3$$

4.6.2 Pivotal Reduction of a General System of Equations

- In stead of a square system, let us consider a system of *m* equations in *n* variables with . This system of equations is assumed to be consistent so that it will have at least one solution.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\qquad (4.6.5)
$$

- The solution vector(s) *X* which satisfy the Eqs. (4.6.5) are not evident from the Eqs. (4.6.5).

- If pivotal operations, with respect to any *m* variables, say, $x_1, x_2, \cdots, x_m$ are carried, the resulting set of equations can be written as follows:

Canonical system with pivotal variables $x_1, x_2, \cdots, x_m$		
$1 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_m + a''_{1,m+1} x_{m+1} + \cdots + a''_{1n} x_n = b'_1$ $0 \cdot x_1 + 1 \cdot x_2 + \cdots + 0 \cdot x_m + a''_{2,m+1} x_{m+1} + \cdots + a''_{2n} x_n = b'_2$ $\vdots$ $0 \cdot x_1 + 0 \cdot x_2 + \cdots + 1 \cdot x_m + a''_{m,m+1} x_{m+1} + \cdots + a''_{mn} x_n = b'_m$		(4.6.6)
Pivotal variables	Nonpivotal or independent variables	Constants

- One special solution which can always be deduced from the system of Eqs. (4.6.6) is

$$\left.\begin{aligned} x_i &= b_i'', \; i = 1,2,\cdots,m \\ x_i &= 0, \;\; i = m+1, m+2, \cdots, n \end{aligned}\right\} \qquad (4.6.7)$$

- This solution is called a basic solution since the solution vector contains no more than *m* nonzero terms.
- The pivotal variables $x_i, \; i = 1,2,\cdots,m$ are called basic variables, and other variables $x_i, \; i = m+1, m+2, \cdots, n$ are called nonbasic variables
- If all $b_i'' \; i = 1,2,\cdots,m$ in the solution given by Eqs.(4.6.7) are nonnegative, it satisfies Eqs. (4.1.3) in addition to Eqs.(4.1.2), and hence it can be called a basic feasible solution.

- It is possible to obtain the other basic solutions from the canonical system of Eqs. (4.6.6). We can perform an additional pivotal operation on the system after it is in canonical form, using a''_{pq} (which is nonzero) as the pivot term as the pivot term, $q > m$, and using any row *p* (among 1, 2, …*m*).
- The new system will x_q still be in canonical form, but with as the pivotal variable in place of x_p. The variable x_p, which was a basic variable in the original canonical form, will no longer be a basic variable in the new canonical form.

- This new canonical system yields a new basic solution (which may or may not be feasible) similar to that of Eqs.(4.6.7)
- It is to be noted that the values of all the basic variables change, in general, as we go from one basic solution to another, but only one zero variable (which is nonbasic in the original canonical form) becomes nonzero (which is basic in the new canonical system) and vice versa.

- **Example 4.6.2**
- Reduce the system of equations

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1 \qquad \text{(I0)}$$

$$x_1 + x_2 + x_3 + 3x_4 = 6 \qquad \text{(II0)}$$

$$x_1 - x_2 + x_3 + 5x_4 = 4 \qquad \text{(III0)}$$

into a canonical form with x_1, x_2 and x_3 as the basic variables.

- *Solution*:
- By pivoting on the element $a_{11} = 2$, we obtain

$$x_1 + \frac{3}{2}x_2 - x_3 - \frac{7}{2}x_4 = \frac{1}{2}$$

$$0 - \frac{1}{2}x_2 + 2x_3 + \frac{13}{2}x_4 = \frac{11}{2}$$

$$0 - \frac{5}{2}x_2 + 2x_3 + \frac{17}{2}x_4 = \frac{7}{2}$$

I1 = $\frac{1}{2}$ I0

II1 = II0-I1

III1 = III0-I1

and then if we pivot on $a'_{22} = -\frac{1}{2}$, we get

$$x_1 + 0 + 5x_3 + 16x_4 = 17$$

$$0 + x_2 - 4x_3 - 13x_4 = -11$$

$$0 + 0 - 3x_3 - 24x_4 = -24$$

I2 = I1- $\frac{3}{2}$ II2

II2 = -II1

III2 = III1+ $\frac{5}{2}$ II2

Finally we pivot on a'_{33} to obtain the required canonical form as

$$x_1 + x_4 = 2$$

$$x_2 - x_4 = 1$$

$$x_3 + 3x_4 = 3$$

I3 = I2-5III3

II3 = II2+4III3

III3 = $-\frac{1}{8}$ III2

$x_2 = 1 + x_4$

- From this canonical form, we can readily write the solution of x_1, x_2 and x_3 in terms of the other variable x_4 as

$$x_1 = 2 - x_4$$
$$x_2 = 1 + x_4$$
$$x_3 = 3 - 3x_4$$

- If Eqs. (I0), (II0) and (III0) are the constraints of a linear programming problem, the solution obtained by setting the independent variable equal to zero is called a basic solution. In the present case, the basic solution is given by

$$x_1 = 2 \quad x_2 = 1 \quad x_3 = 3 \quad \text{(basic variables)}$$

and $x_4 = 0$ (nonbasic or independent variable).

- Since this basic solution has all $x_j \geq 0$, $(j = 1,2,3,4)$ it is a basic feasible solution.

- If we want to move to a neighbouring basic solution, we can proceed from the canonical form given by Eqs. (I3), (II3) and (III3). Thus, if a canonical form in terms of the variables x_1 , x_2 and x_4 is required, we have to bring x_4 into the basis in place of the original basic variable x_3. Hence we pivot on a''_{34} in Eq. (III3). This gives the desired canonical form as

$$x_1 - \frac{1}{3}x_3 = 1$$

I4 = I3-III4

$$x_2 + \frac{1}{3}x_3 = 2$$

II4 = II3+II3

$$x_4 + \frac{1}{3}x_3 = 1$$

III4 = $\frac{1}{3}$ III3

- This canonical system fives the solution of x_1, x_2 and x_4 in terms of as x_3

$$x_1 = 1 + \frac{1}{3}x_3$$

$$x_2 = 2 - \frac{1}{3}x_3$$

$$x_4 = 1 - \frac{1}{3}x_3$$

and the corresponding basic solution is given by

$x_1 = 1 \; x_2 = 2 \quad x_4 = 1$ (basic variables)

$x_3 = 0$ (nonbasic variable)

•This basic solution can also be seen to be basic feasible solution.

- If we want to move to the next basic solution with x_1, x_3 and x_4 as basic variables, we have to bring into the current basis in place of . Thus we have to pivot of in Eq. (II4). This leads to the following canonical system:

$$x_1 + x_2 = 3 \qquad \text{I5=I4} + \frac{1}{3}\text{II5}$$

$$x_3 + 3x_2 = 6 \qquad \text{II5 = 3II4}$$

$$x_4 - x_2 = -1 \qquad \text{III5 = III4-} \frac{1}{3}\text{II5}$$

- The solution for x_1 , x_3 and

$$x_1 = 3 - x_2$$

$$x_2 = 6 - 3x_2$$

$$x_4 = -1 + x_2$$

$x_1 = 3$ $x_3 = 6$ $x_4 = -1$(basic variables)
$x_2 = 0$ (nonbasic variable)
. Since all the x_j are not nonnegative, this basic solution is not feasible.

- Finally, to obtain the canonical form in terms of the basic variables x_2 , x_3 and x_4 we pivot on a''_{12} in Eq.(I5), thereby bringing x_2 into the current basis in place of x_1 . This gives

$$x_2 + x_1 = 3$$
I6=I5

$$x_3 - 3x_1 = -3$$
II6 = II5-3I6

$$x_4 + x_1 = 2$$
III6 = III5 + I6

This canonical form gives the solution for x_2 , x_3 and x_4 in terms of x_1 as

$$x_2 = 3 - x_1$$

$$x_3 = -3 + 3x_1$$

$$x_4 = 2 - x_1$$

and the corresponding basic solution is

$x_2 = 3$ $x_3 = -3$ $x_4 = 2$ (basic variables)

$x_1 = 0$ (nonbasic variable)

4.7单纯形法思想的形成

MOTIVATION TO THE SIMPLEX METHOD

- Given a system in canonical form corresponding to basic solution, we have seen how to move to a neighbouring basic solution by a pivot operat

This can be done because the optimal solution, always occurs at an extreme point or vertex of the feasible domain.

- One way to find the optimal soluti of the given linear programming problem → generate all the basic solutions → pick the one which is feasible and corresponds to the optimal value of the objective function.

- If there are *m* equality constraints in *n* variables with , a basic solution can be obtained by setting any of the variables equal to zero. The number of basic solutions to be inspected is thus equal to the number of ways in which *m* variables can be selected from a group of *n* variables, i.e.,

$$\frac{n!}{(n-m)!m!} = \binom{n}{m}$$

> Usually, we do not have to inspect all these basic solutions since many of them will be infeasible.

- For example, if $n=10$ and $m=5$, we have [illegible] basic solutions and if $n=20$ and $m=10$, we have approximately 184700 basic solutions.

- However, for large *n* and *m*. this is still a very large number for inspecting one by one. Hence, what we really need is a computational scheme that examines a sequence of basic feasible solutions, each of which corresponds to a lower value of *f* until a minimum is reach

> The process is repeated until, in a finite number of steps, an optimum is found

- The simplex method of Dantzig is a po ul scheme for obtaining a basic feasible solution; if e solution is not optimal, the method provides for fin ng a neighbouring basic feasible solution which has a lower or equal value of *f*..

- In practice, the first step involved in the simplex method is to construct an auxiliary problem by introducing certain variables known as artificial variables into the standard form of the linear programming problem.

↓

The primary aim of adding the artificial variables is to bring the resulting auxiliary problem into a canonical form from which its basic feasible solution can be immediately obtained.

- Starting from this canonical form, the optimal solution of the original linear programming problem is sought in two phases:

(ⅰ) The first phase is intended to find a basic feasible solution to the original linear programming problem.

It consists of a sequence of pivot operations which produces a succession of different canonical forms from which the optimal solution of the auxiliary problem can be found

This also enables us to find a basic feasible solution, if one exists, of the original linear programming problem.

(ⅱ)The second phase is intended to find the optimal solution of the original linear programming problem.

It consists of a second sequence of pivot operations which enables us to move from one basic feasible solution to the next of the original linear programming problem. In this process, the optimal solution of the problem, if one exists, will be identified.

In this process, the optimal solution of the problem, if one exists, will be identified.

- The sequence of different canonical forms that is necessary in both the phases of the simplex method is generated according to the simplex algorithm described in the next section. That is, the simplex algorithm forms the main subroutine of the simplex method.

4.8单纯形法
SIMPLEX ALGORITHM

- The starting point of the simplex algorithm is always a set of equations, which includes the objective function along with the equality constraints of the problem in canonical form.

- the objective of the simplex algorithm is to find the vector X⩾0 which minimizes the functionand satisfies the equations:

a''_{ij} a''_j b''_i, and f''_0 are constants

$$
\begin{aligned}
&1\cdot x_1 + 0\cdot x_2 + \cdots + 0\cdot x_m + a''_{1,m+1}x_{m+1} + \cdots + a''_{1n}x_n = b''_1 \\
&0\cdot + 1\cdot + \cdots + 0\cdot + a''_{2,m+1} + \cdots + a''_{2n} = b''_2 \\
&\vdots \\
&0\cdot + 0\cdot + \cdots + 1\cdot + a''_{m,m+1} + \cdots + a''_{mn} = b''_m \\
&0\cdot + 0\cdot + \cdots + 0\cdot - f + c''_{m+1} + \cdots + c''_{mn} = -f''_0
\end{aligned}
\qquad (4.8.1)
$$

is treated as a basic variable in the canonical form

The basic solution which can be readily deduced from Eqs.(4.8.1) is

$$
\left.
\begin{aligned}
&x_i = b''_i, i = 1, 2, \ldots, m \\
&f = f''_0 \\
&x_i = 0, i = m+1, m+2, \ldots, n
\end{aligned}
\right\} \qquad (4.8.2)
$$

If this basic solution is also feasible, the values of x_i $i = 1, 2, \ldots, n$

are nonnegative and hence

$$b''_i \geq 0 \quad i = 1, 2, \ldots, m \qquad (4.8.3)$$

In phase Ⅰ of the simplex method, the basic solution corresponding to the canonical form obtained after the introduction of the artificial variables will be feasible for the auxiliary problem.

As has been stated earlier, the phase Ⅱ of the simplex method starts with a basic feasible solution of the original linear programming problem.

the initial canonical form at the start of the simplex algorithm will always be a basic feasible solution.

Let S be a closed convex polyhedron. Then the minimum of a linear function over S is attained at an extreme point of S.

- We know from Theorem 4.5.6 that the optimal solution of a linear programming problem lies at one of the basic feasible solutions. Since the simplex algorithm is intended to move form one basic feasible solution to the other through pivotal operations
- We have to make sure that the present basic feasible solution is not the optimal solution before moving to the next basic feasible solution. By merely glancing at the numbers c''_j , $j = 1, 2, \cdots, n$, we can tell whether the present basic feasible solution is optimal or not. The following theorem provides a means of identifying the optimal point.

- ***Identifying an Optimal Point***

- ***Theorem 4.8.1 A basic feasible solution is an optimal solution with a minimum objective function value of*** f''_0 ***if all the cost coefficients*** c''_j***,*** $j = m+1, m+2, \ldots, n$ ***in Eqs. (4.8.1) are nonnegative.***

- ***If, after testing for optimality, the current basic feasible solution is found to be nonoptimal, an improved basic solution is obtained from the present canonical form as aollow.***

- ***Improving a nonoptimal basic feasible solution.***

From the last row of Eqs.(4.8.1), we can write the objective function as

$$\left.\begin{aligned} f &= f_0'' + \sum_{i=1}^{m} c_i'' x_i + \sum c_j'' x_j \\ &= f_0'' \quad \text{for the solution given by Eqs.(4.8.2)} \end{aligned}\right\} \qquad (4.8.5)$$

If at least one c_j'' is negative, the value of f can be reduced by making the corresponding $x_j > 0$. In other words, the nonbasic x_j variable , for which the cost coefficient c_j'' is negative, is to be made a basic variable in order to reduce the value of the objective function.

- At the same time, due to the pivotal operation, one of the current basic variables will become nonbasic and hence the values of the new basic variables are to be adjusted in order to bring the value of f less than f_0'' .

  If there are more than one $c_j'' < 0$,the index *s* of the nonbasic variable x_s which is to be made basic is chosen such that (4.8.6).

$$c_s'' = \min c_j'' < 0$$

  If there is a tie in applying Eq.(4.8.6), i.e., if more than one c_j'' have the same minimum value, we select one of them as c_s'' arbitrarily.

- Having decided on the variable to become basic, we increase it from zero holding all other nonbasic variables zero, and observe the effect on the current basic variables

By Eqs. (4.8.1), these

$$x_1 = b_1'' -$$

$$x_2 = b_2'' - a_{2s}'' x_s , b_2'' \geq 0$$

$$\vdots$$

$$x_m = b_m'' - a_{ms}'' x_s , b$$

(4.8.7)

$$f = f_0'' + c_s'' x_s , c_s'' < 0$$

(4.8.8)

However, in the process of increasing the value of x_s, some of the variables $x_i (i = 1,2,\cdots,m)$ in Eqs.(4.8.7) may become negative.

Since $c_s'' < 0$, suggests that the value of should be made as large as possible in order to reduce the value of *f* as much as possible.

- It can be seen that if all the coefficients $a_{is}'' \leq 0$, $i = 1,2,\cdots,m$; then can be made infinitely large without making any $x_i < 0 (i = 1,2,\cdots,n)$
- **In such a case, the minimum value of *f* is minus infinity and the linear programming problems is said to have an unbounded solution.**

On the other hand, if at least one a''_{is} is positive, the maximum value that x_s can take without making x_i negative is (b''_i / a''_{is}) . If there are more than one $a''_{is} > 0$,the largest value x_s^* that can x_s take is given by the minimum of the ratios (b''_i / a''_{is}) for which $a''_{is} > 0$ .Thus

$$x_s^* = \frac{b''_r}{a''_{rs}} = \min_{a''_{is} > 0} (b''_i / a''_{is}) \qquad (4.8.9)$$

The choice of *r* in the case of a tie, assuming that all , is $b''_i > 0$ arbitrary.

If any for which $a''_{is} > 0$is zero in Eqs. (4.8.7), then x_s cannot be increased by any amount. Such a solution is called a degenerate solution.

- In the case of a nondegenerate basic feasible solution, a new basic feasible solution can be constructed with a lower value of the objective function as follows. By substituting the value of given by Eq.(4.8.9)into Eqs.(4.8.7) and (4.8.8), we obtain

$$\left.\begin{array}{l} x_s = x_x^* \\ x_i = b_i'' - a_{is}'' x_s^* \geq 0, i = 1,2,\text{☹}, m \text{ and } \\ \qquad i \neq r \quad x_r = 0 \\ x_j = 0, j = m+1, m+2, \text{☹}, n \text{ and } \\ \qquad j \neq s \end{array}\right\} \quad (4.8.10)$$

$$f = f_0'' + c_s'' x_s^* \leq f_0'' \quad (4.8.11)$$

- which can readily be seen to be a feasible solution different from the previous one . Since $a_{rs} > 0$ in Eq.(4.8.9), a single pivot operation on the element a''_{rs} in the system of Eqs.(4.8.10) can easily be deduced. Also, Eq.(4.8.11) shows that this basic feasible solution corresponds to a lower objective function value compared to that of Eqs.(4.8.2). This basic feasible solution can again be tested for optimality by seeing whether all $c''_i > 0$ in the new canonical form.

- If the solution is not optimal, the whole procedure of moving to another basic feasible solution from the present one has to be repeated.

- In the simplex algorithm, this procedure is repeated in an iterative manner until the algorithm finds either

  (i) a class of feasible solutions for which $f \to -\infty$ or

  (ii) an optimal basic feasible solution with all $c_i'' \geq 0, i = 1, 2, \ldots, n.$

- Since there are only a finite number of ways to choose a set of *m* basic variables out of *n* variables, the iterative process of the simplex algorithm will terminate in finite number of cycles. The iterative process of the simplex algorithm is shown as a flow chart in Fig.4.16

- **Example 4.8.1**
- Maximize $F = x_1 + 2x_2 + x_3$ subject to

$$2x_1 + x_2 - x_3 \le 2$$
$$-2x_1 + x_2 - 5x_3 \ge -6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_i \ge 0, i = 1,2,3$$

- *Solution* :
- The problem can be stated as:
- Minimize $f = -x_1 - 2x_2 - x_3$ subject to

$$2x_1 + x_2 - x_3 \le 2$$
$$2x_1 - x_2 + 5x_3 \le 6$$
$$4x_1 + x_2 + x_3 \le 6$$
$$x_i \ge 0 \quad i = 1,2,3$$

- By introducing the slack variables $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$, the system of equations can be stated in canonical form as

where x_4 , x_5 , x_6 and $-f$ can be taken as basic variables.

$$\left.\begin{aligned} 2x_1 + x_2 - x_3 + x_4 \\ 2x_1 - x_2 + 5x_3 + x_4 = 6 \\ 4x_1 + x_2 + x_3 + x_4 = 6 \\ -x_1 - 2x_2 - x_3 - f = 0 \end{aligned}\right\}$$

(E1)

- The basic solution c … by

Since the cost coefficients corresponding to nonbasic variables in Eqs.(E1) are negative $(c_1'' = -1, c_2'' = -2, c_3'' = -1)$ the present solution (E2) is not optimum.

$$x_4 = 2, x \ldots$$

$$x_1 = x_2 = x_3 = \ldots$$ (nonbasic variables) (E2)

and $f = 0$

which can also be seen to be feasible.

- To improve the present basic feasible solution, we first decide the variable (x_s) to be brought into the basic as

$$c_s'' = \min(c_j'' < 0) = c_2'' = -2$$

- Thus x_2 enters the next basic set. To obtain the new canonical form, we select the pivot element a_{rs}'' such that

$$\left(\frac{b_r''}{a_{rs}''}\right) = \min_{a_{rs}''>0}\left(\frac{b_i''}{a_{is}''}\right)$$

- In the present case, $s = 2$ and a_{12}'' and a_{32}'' are ≥ 0 . Since $(b_1''/a_{12}'')2/1$ and $(b_3''/a_{32}'') = 6/1$, $x_r = x_1$ .By pivoting on a_{12}'' the new system of equations can be obtained as

$$\left.\begin{aligned} 2x_1 + 1\cdot x_2 - x_3 + x_4 &= 2 \\ 4x_1 + 0\cdot x_2 + 4x_3 + x_4 + x_5 &= 8 \\ 2x_1 + 0\cdot x_2 + 2x_3 - x_4 + x_6 &= 4 \\ 3x_1 + 0\cdot x_2 - 3x_3 + 2x_4 - f &= 4 \end{aligned}\right\} \qquad \text{(E3)}$$

- The basic feasible solution corresponding to this canonical form is

$$x_2 = 2, x_5 =$$

$$x_1 = x_3 = x_4 = 0 \quad \text{(nonbasic variables)}$$

and $f = -4$

(E4)

Since $c''_3 = -3$, the present solution is not optimum.

- As $c''_s = \min(c''_i < 0) = c''_3$, $x_s = x_3$. Enters the next basis. To find the pivot element a''_{rs} we find the ratios (b''_i / a''_{is}) for $a''_{is} > 0$. In Eqs.(E3), only a''_{13} and a''_{33} are > 0, and hence

$$b''_2 / a''_{23} = 8/4 \text{ and } b''_3 / a''_{33} = 4/2$$

- Since both these ratios are same,we arbitrarily select a''_{23} as the pivot element.

- Pivoting on a''_{23} gives the following canonical system of equations:

$$\left.\begin{aligned}
&3x_1 + 1\cdot x_2 + 0\cdot x_3 + \frac{5}{4}x_4 + \frac{1}{4}x_5 = 4 \\
&1\cdot + 0\cdot + 1\cdot + \frac{1}{4}\cdot + \frac{1}{4}\cdot = 2 \\
&0\cdot + 0\cdot + 0\cdot - \frac{3}{2} - \frac{1}{2} + = 0 \\
&6 + 0\cdot + 0\cdot + \frac{11}{4} + \frac{3}{4} - f = 10
\end{aligned}\right\} \quad \text{(E5)}$$

$$\left.\begin{aligned}
&x_2 = 4, x_3 = 2, x_6 = 0 \quad \text{(basic variables)} \\
&x_1 = x_4 = x_{54} = 0 \quad \text{(nonbasic variables)} \\
&\text{and } f = -10
\end{aligned}\right\} \quad \text{(E6)}$$

- Since all $c''_i \geq 0$ in the present canonical form, the solution given in (E6) will be optimum.

- Usually, starting with Eqs.(E1),all the computations are done in a tableau form as shown below:

Basic Variables	Variables								
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	b_i''	(b_i'' / a_{is}'') for $a_{is}'' > 0$
x_4	2	① Pivot element	-1	1	0	0	0	2	2-smaller one (x_4 drops from nest basic)
x_5	2	-1	5	0	1	0	0	6	
x_6	4	1	1	0	0	1	0	6	6
$-f$	-1	-2	-1	0	0	0	1	0	

Most negative (enters nest basis)

(Most negative

- Result of pivoting

x_2	2	1	-1	1	0	0	0	2	
x_5	4	0	④ Pivot element	1	1	0	0	8	(select this arbitrarily . x_5 drops from nest basis)
x_6	2	0	2	-1	0	1	0	4	2
$-f$	3	0	-3	2	0	0	1	4	

Most negative c_i'' (x_3 enters nest basis)

Result of pivoting

x_2	3	1	0	$5/4$	$1/4$	0	0	4
x_3	1	0	1	$1/4$	$1/4$	0	0	2
x_6	0	0	0	$-3/2$	$-1/2$	1	0	0
$-f$	6	0	0	$11/4$	$3/4$	0	1	10

All c_i'' are > 0 and hence the present solution is optimum.

- **Example 4.8.2**
- Find the minimum solution of the axially compressed RC column expressed by Eqs. (4.2.13)-(4.2.15).
- *Solution*:

  The standard form was formulated in Example 4.3.2 given by

Find $A_s \quad A_c$

min $z = 3.0\times10^{-7} A_c + 2.4$

s.t. $A_s + 3.967\times10^{-2} A_c - x_1 \ldots 667\times10^{3}$

$A_s - 0.004 A_c - x_2 \ldots 0$

$A_s - 0.05 A_c + x_3 = 0$

where x_1 and x_2 are the surplus variables and x_3 is the slack variable.

- And then introduce artificial variables ξ_1, ξ_2, ξ_3 . Then constraints are given by:

$$\xi_1 + A_s + 0.03967 A_c - x_1 = 5466.7$$

$$\xi_2 + A_s - 0.004 A_c - x_2 = 0$$

$$\xi_3 + A_s - 0.05 A_c + x_3 = 0$$

Then objective function can be given by

$$50 \times 10^{-5} \xi_1 + 50 \times 10^{-5} \xi_2 + 50 \times 10^{-5} \xi_3 + 2.418 \times 10^{-5} A_s + 3 \times 10^{-7} A_c = z$$

- Now we use simplex method to solve the problem. The process of operation can be seen from the table of simplex method.

	i			ii						iii	iv	v	vi
	Basic variables			Nonbasic variables					b	Basic dispels	x_l	b''_k / a''_{kl}	x_k
	ξ_1	ξ_2	ξ_3	A_s	A_C	x_1	x_2	x_3					
	1	0	0	1	3.967e-2	-1	0	0	5.4667e 3	ξ_1=5.4667e3		5.4667e 3	
	0	1	0	1	-4e-3	0	-1	0	0	ξ_2 =0		0	
	0	0	1	[1]	-5e-2	0	0	1	0	ξ_2 =0		0	ξ_3
	50e-5	50e-5	50e-5	2.42e-5	3e-7	0	0	0	z				
el	0 0	50e-5 0	50e-5 50e-5	-47.58e-5 -97.58e-5	-1.954e-5 -1.754e-5	50e-5 50e-5	0 50e-5	0 0	z-2.7334 z-2.7334				
	0	0	0	(-147.58e-5)	0.7465e-5	50e-5	50e-5	50e-5	z-2.7334	Z=2.7334	A_s		

1	ξ_1	ξ_2	A_s	ξ_3	A_C	x_1	x_2	x_3			x_l	b_k'' / a_{kl}''	x_k
	1	0	0	-1	8.967e-2	-1	0	-1	5.4667e3	=5.4667e3		60964.31	
	0	1	0	-1	[4.6e-2]	0	-1	-1	0	=0		0	
	0	0	1	1	-5e-2	0	0	1	0	=0		--	ξ_2
	0	0	0	147.58e-5	(-6.6e-5)	50e-5	50e-5	97.58e-5	z-2.7334	Z=2.7334	A_C		
	ξ_1	A_C	A_s	ξ_3	ξ_2	x_1	x_2	x_3					
	1	0	0	0.94935	-1.94935	-1	[1.94935]	0.94935	5.4667e3	=5.4667e3		2804.35	ξ_1
	0	1	0	-21.74	21.74	0	-21.74	-21.74	0	=0		--	
	0	0	1	-8.696e-2	1.08696	0	-1.08696	8.696e-2	0	=0		--	
	0	0	0	4.102e-5	143.473e-5	50e-5	(-93.478)	-45.898	z-2.7334	Z=2.7334	x_2		

- 1

	x_2	A_C	A_s	ξ_3	ξ_2	x_1	ξ_1	x_3	b		x_l	b_k'' / a_{kl}''	x_k
	1	0	0	0.487	-1	-0.513	0.513	[0.487]	2.804e3	=2.804e3		5.778 e 2	
	0	1	0	-11.152	0	-11.152	11.152	-11.152	6.964e3	=6.964e4		--	
	0	0	1	0.4424	0	-0.5576	0.5576	0.4424	3.048e3	=3.048e3		6.89e 3	
	0	0	0	49.63e-5	50e-5	2.064e-5	47.95e-5	(0.3736)	z-0.1120				
	x_3	A_C	A_s	ξ_3	ξ_2	x_1	ξ_1	x_2					
	1 0 0 0	0 1 0 0	0 0 1 0	1 0 0 50e-5	-2.092 -22.9 0.908 49.23e-5	-1.053 -22.9 -0.0916 1.65e-5	1.053 -22.9 0.0916 48.3e-5	2.092 22.9 -0.908 0.77e-5	5.758e3 22.9 5.01e3 z-0.1335	Z=0.1335			

The optimal point

$$A_{cmm} = 1.2518e5mm^2, A_s^* = 5.01e3mm^2, z^* = 0.1335yuan$$

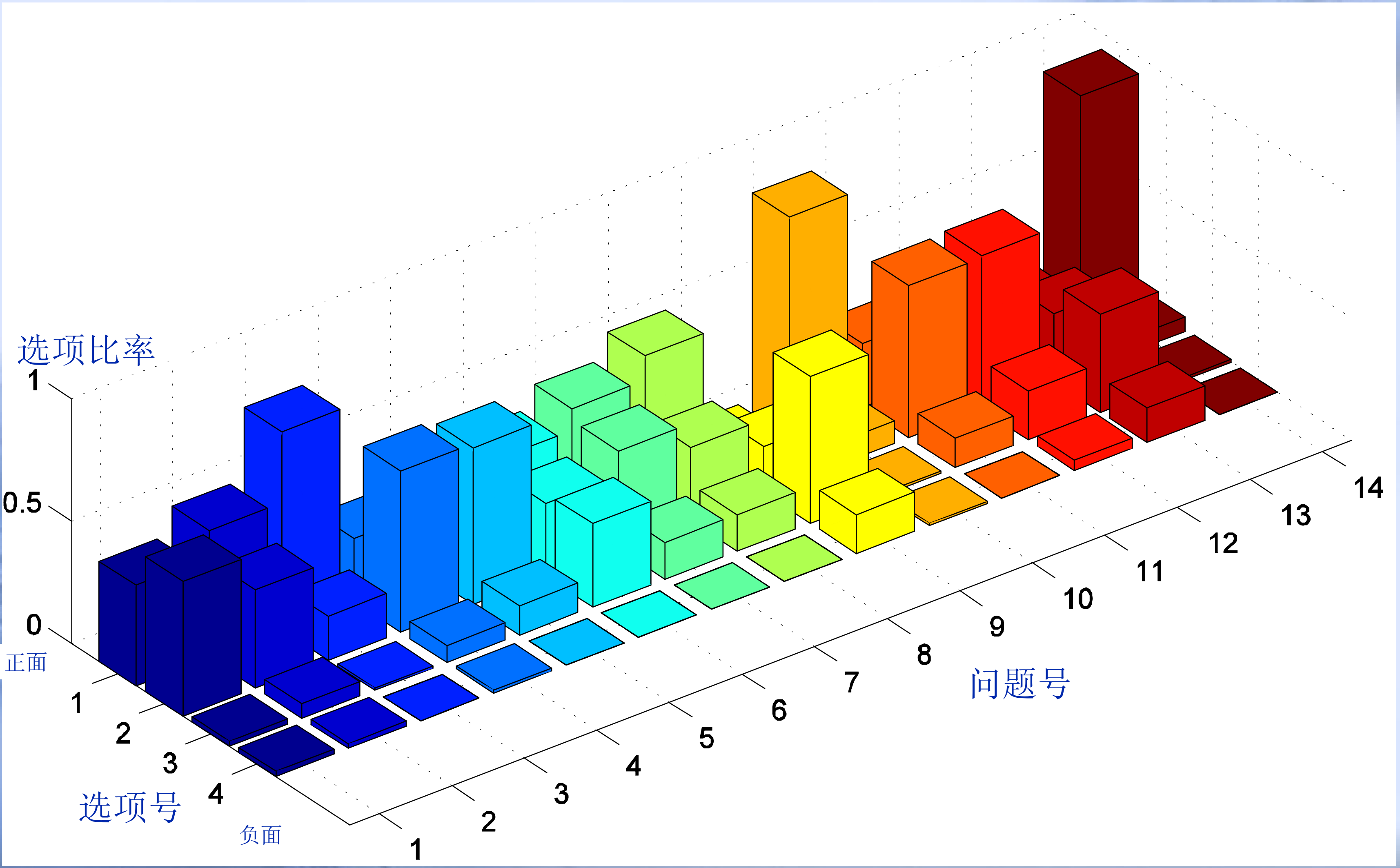
问卷调查结果

各问题的百分数（%）

选项	问题号													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A	41	52	81	26	24	36	45	55	5	89	26	23	18	93
B	55	40	18	66	64	30	40	30	19	9	62	63	28	6
C	2	6	1	7	12	34	15	15	60	1	12	20	40	1
D	2	2	0	1	0	0	0	0	16	1	0	4	14	0

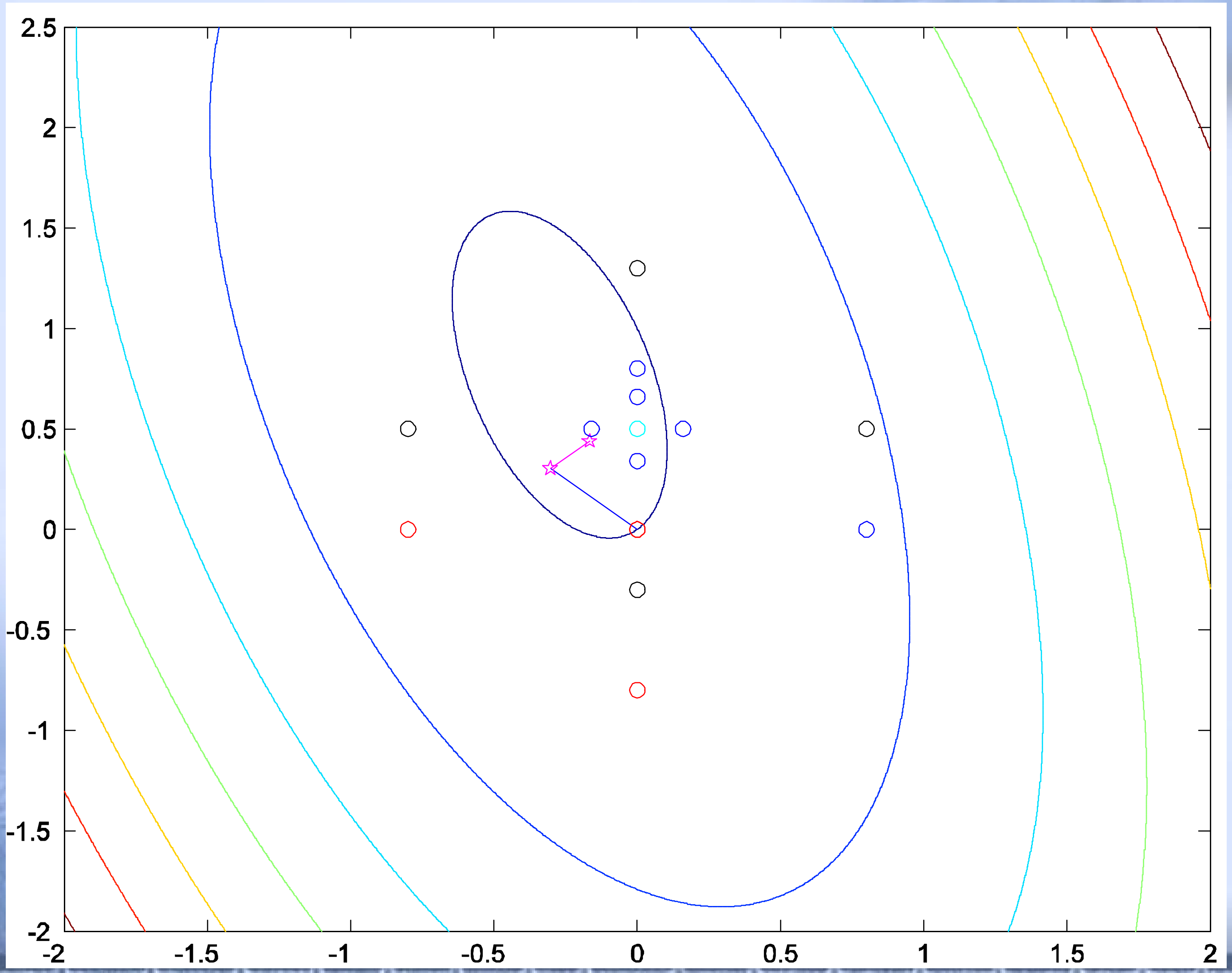
备注：总调查人数 100 人。

问卷调查结果



课外讲作业

- 第3章



- 下面是团队课外竞赛活动

课外竞赛活动团队积分动态

- 第一组: Leader: Wanrun Li 团队长 陈晓
  - 李云飞 陈 强 耿继芳 李光辉 刘晓东 白春宁
  - LC 积 285
- 第二组: Leader: Xiaoming Chen 团队长 陈晓明
  - 麻文娜 马晓斌 王欣欣 彭绍刘 吴 扬 黄彪健
  - LC 积 180
- 第三组: Leader: Pingong Fan 团队长 范萍萍
  - 吴忠铁 毕东涛 柳 肯 吴永生 张怀振
  - LC 积 240

课外竞赛活动团队积分动态

Task	第一团队	第二团队	第三团队
Initial	300	300	300
BD	5	5	5
Cum Tot	305	305	305