On the Debate Concerning the Proper Characterisation of Quantum Dynamical Evolution

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June 17, 2012

Abstract

There has been a long-standing and sometimes passionate debate between physicists over whether a dynamical framework for quantum systems should incorporate not completely positive (NCP) maps in addition to completely positive (CP) maps. Despite the reasonableness of the arguments for complete positivity, we argue that NCP maps should be allowed, with a qualification: these should be understood, not as reflecting 'not completely positive' evolution, but as linear extensions, to a system's entire state space, of CP maps that are only partially defined. Beyond the domain of definition of a partial-CP map, we argue, much may be permitted.

1 Introduction

Conventional wisdom has it that any evolution of a quantum system can be represented by a family of completely positive (CP) maps on its state space. Moreover, there seem to be good arguments that evolutions outside this class must be regarded as unphysical. But orthodoxy is not without dissent; several authors have argued for considering evolutions represented by maps that are not completely positive (NCP).

The debate has implications that have the potential to go deep. The possibility of incorporating NCP maps into our quantum dynamical framework may illuminate much regarding the nature of and relation between quantum entanglement and other types of quantum correlations (Devi et al., 2011). If the use of NCP maps is illegitimate however, such investigations must be dismissed without further ado.

In the following, we will argue for the proposition that NCP maps should be allowed—but we will add a caveat: one should not regard NCP dynamical maps as descriptions of the 'not completely positive evolution' of quantum systems. An 'NCP map', properly understood, is a linear extension, to a system's entire state space, of a CP map that is only defined on a subset of this state space. In fact, as we will see, not much constrains the extension of a partially defined CP map. Depending on the characteristics of the state preparation, such extensions may be not completely positive, inconsistent,[1] or even nonlinear.

The paper will proceed as follows: in Section 2 we review the essential aspects of the theory of open quantum systems and in Section 3 we present the standard argument for complete positivity. In Section 4 we consider the issues involved in the debate over NCP maps and in

[1] Strictly speaking, when an inconsistent map is used this should not be seen as an extension but as a change of state space. This will be clarified below.

Section 5 we present our interpretation of the debate and what we believe to be its resolution.

2 Evolution of a Quantum System

Consider a quantum system S that is initially in a state ρ_S^0, represented by a density operator $\hat{\rho}_S^0$. If the system is isolated, its evolution will be given by a one-parameter family of unitary operators $\{U^t\}$, via

$$\hat{\rho}_S^t = U^t\, \hat{\rho}_S^0\, U^{\dagger t}. \tag{1}$$

Suppose, now, that the system interacts with another system R, which may include some piece of experimental apparatus. We take R to include everything with which S interacts. Suppose that S is prepared in a state that is uncorrelated with the state of R (though it may be entangled with some other system, with which it doesn't interact), so that the initial state of the composite system $S + R$ is

$$\hat{\rho}_{SR}^0 = \hat{\rho}_S^0 \otimes \hat{\rho}_R^0. \tag{2}$$

The composite system will evolve unitarily:

$$\hat{\rho}_{SR}^t = U^t\, \hat{\rho}_{SR}^0\, U^{\dagger t}, \tag{3}$$

where now $\{U^t\}$ is a family of operators operating on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R$ of the composite system. It is easy to show (see, e.g., Nielsen and Chuang 2000, §8.2.3) that, for each t, there will be a set $\{W_i(t)\}$ of operators, which depend on the evolution operators $\{U^t\}$

and the initial state of R, such that

$$\hat{\rho}^t_S = \textstyle\sum_i W_i(t)\, \hat{\rho}^0_S\, W_i^\dagger(t);$$
$$\textstyle\sum_i W_i^\dagger(t) W_i(t) = I. \tag{4}$$

This is all in the Schrödinger picture, in which we represent a change of state by a change in the density operator used. We can also use the Heisenberg picture, which represents a state change via a transformation of the algebra of operators used to represent observables:

$$\rho^t_S(A) = \rho^0_S(A^t), \tag{5}$$

where

$$A^t = \sum_i W_i(t) A^0 W_i^\dagger(t). \tag{6}$$

In addition to unitary evolution of an undisturbed system, we also associate state changes with measurements, via the collapse postulate. In the case of a von Neumann measurement, there is a complete set $\{P_i\}$ of projections onto the eigenspaces of the observable measured, and the state undergoes one of the state transitions T_i given by

$$\mathcal{T}_i \hat{\rho} = \frac{P_i\, \hat{\rho}\, Pi}{\mathrm{Tr}(P_i\, \hat{\rho})}, \tag{7}$$

The probability that the state transition will be $\mathcal{T}_i$ is Tr$(P_i\, \hat{\rho})$. When a measurement has been performed, and we don't yet know the result, the state that represents our state of knowledge of the system is

$$\mathcal{T} \hat{\rho} = \sum_i P_i\, \hat{\rho}\, P_i. \tag{8}$$

Note that this, also, has the form (4).

One can also consider *selective* operations, that is, operations that take as input a state and yield a transformed state, not with certainty, but with some probability less than one, and fail, otherwise. One such operation is the procedure of performing a measurement and keeping the result only if the outcome lies in a specified set (for example, we could do a spin measurement and select only '+' outcomes); the operation fails (does not count as preparing a state at all) if the measurement yields some other result. A selective operation is represented by a transformation of the state space that does not preserve norm. A selective operation $\mathcal{T}$, applied to state ρ, produces a final state $\mathcal{T}\rho$ with probability $\mathcal{T}\rho(I)$, and no result otherwise.

Unitary evolution, evolution of a system interacting with an environment with which it is initially correlated, and measurement-induced collapse can all be represented in the form (4). The class of state transformations that can be represented in this form is precisely the class of *completely positive* transformations of the system's state space, to be discussed in the next section.

3 Completely Positive Maps

We will want to consider, not just transformations of a single system's state space, but also mappings from one state space to another. The operation of forming a reduced state by tracing out the degrees of freedom of a subsystem is one such mapping; as we will see below, assignment maps used in the theory of open systems are another.

We associate with any quantum system a C^*-algebra whose self-adjoint elements represent the observables of the system. For any C^*-algebra $\mathcal{A}$, let $\mathcal{A}^*$ be its dual space, that is, the set of bounded linear functionals on $\mathcal{A}$. The state space of $\mathcal{A}$, $\mathcal{K}(\mathcal{A})$, is the subset of $\mathcal{A}^*$ consisting of positive linear functionals of unit norm.

For any linear mapping $\mathcal{T} : \mathcal{A} \to \mathcal{B}$, there is a dual map $\mathcal{T}^* : \mathcal{A}^* \to \mathcal{B}^*$, defined by

$$\mathcal{T}^* \mu(A) = \rho(\mathcal{T} A) \text{ for all } A \in \mathcal{A}. \tag{9}$$

If $\mathcal{T}$ is positive and unital, then $\mathcal{T}^*$ maps states on $\mathcal{A}$ to states on $\mathcal{B}$. Similarly, for any mapping of the state space of one algebra into the state space of another, there is a corresponding dual map on the algebras.

For any n, let W_n be an n-state system that doesn't interact with our system S, though it may be entangled with S. Given a transformation $\mathcal{T}$ of the state space of S, with associated transformation $\mathcal{T}$ of S's algebra, we can extend this transformation to one on the state space of the composite system $S + W_n$, by stipulating that the transformation act trivially on observables of W_n.

$$(\mathcal{T}^* \otimes I_n)\rho(A \otimes B) = \rho(\mathcal{T}(A) \otimes B). \tag{10}$$

A mapping $\mathcal{T}^*$ is n-*positive* if $\mathcal{T}^* \otimes I_n$ is positive, and *completely positive* if it is n-positive for all n. If S is a k-state system, a transformation of S's state space is completely positive if it is k-positive.

It can be shown (Nielsen and Chuang, 2000, §8.2.4) that, for any completely positive map $\mathcal{T}^* : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{B})$, there are operators $W_i : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{B}}$ such that

$$\begin{aligned} &\mathcal{T}^* \rho(A) = \rho(\textstyle\sum_i W_i^\dagger \, A \, W_i); \\ &\textstyle\sum_i W_i^\dagger W_i \leq I. \end{aligned} \tag{11}$$

This is equivalent to a transformation of density operators representing the states,

$$\hat{\rho} \rightarrow \hat{\rho}' = \sum_i W_i \, \hat{\rho} \, W_i^{\dagger}. \tag{12}$$

The standard argument that any physically realisable operation on the state of a system S must be completely positive goes as follows. We should be able to apply the operation $\mathcal{T}^*$ to S regardless of its initial state, and the effect on the state of S will be the same whether or not S is entangled with a "witness" system W_n. Since S does not interact with the witness, applying operation T^* to S is equivalent to applying $\mathcal{T}^* \otimes I_n$ to the composite system $S + W_n$. Thus, we require each mapping $\mathcal{T}^* \otimes I_n$ to be a positive mapping, and this is equivalent to the requirement that $\mathcal{T}^*$ be completely positive.

To see what goes wrong if the transformation applied to S is positive but not completely positive, consider the simplest case, in which S is a qubit. Suppose that we could apply a transformation $\rho_S^0 \rightarrow \rho_S^1$ that left the expectation values of σ_x and σ_y unchanged, while flipping the sign of the expectation value of σ_z.

$$\rho_S^1(\sigma_x) = \rho_S^0(\sigma_x); \quad \rho_S^1(\sigma_y) = \rho_S^0(\sigma_y); \quad \rho_S^1(\sigma_z) = -\rho_S^0(\sigma_z). \tag{13}$$

Suppose that S is initially entangled with another qubit, in, e.g., the singlet state, so that

$$\rho_{SW}^0(\sigma_x \otimes \sigma_x) = \rho_{SW}^0(\sigma_y \otimes \sigma_y) = \rho_{SW}^0(\sigma_z \otimes \sigma_z) = -1. \tag{14}$$

If we could apply the transformation (13) to S when it is initially in a singlet state with W, this would result in a state ρ^1_{SW} of $S + W$ satisfying,

$$\rho^1_{SW}(\sigma_x \otimes \sigma_x) = \rho^1_{SW}(\sigma_y \otimes \sigma_y) = -1; \quad \rho^1_{SW}(\sigma_z \otimes \sigma_z) = +1. \tag{15}$$

This is disastrous. Suppose we do a Bell-state measurement. One of the possible outcomes is the state $|\Psi^+\rangle$, and the projection onto this state is

$$|\Psi^+\rangle\langle\Psi^+| = \frac{1}{4}\left(I + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right). \tag{16}$$

A state satisfying (15) would assign an expectation value of $-1/2$ to this projection operator, rendering it impossible to interpret this expectation value as the probability of a Bell-state measurement resulting in $|\Psi^+\rangle$.

Note that the set-up envisaged in the argument is one in which it is presumed that we can prepare the system S in a state that is uncorrelated with the active part of its environment R. This set-up includes the typical laboratory set-up, in which system and apparatus are prepared independently in initial states; it also includes situations in which we prepare a system in an initial state and then put it into interaction with an environment, such as a heat bath, that has been prepared independently.

4 The Debate Concerning Not Completely Positive Dynamical Maps

The early pioneering work of Sudarshan et al. (1961), and Jordan and Sudarshan (1961), did not assume complete positivity, but instead characterised the most general dynamical framework for quantum systems in terms of linear maps of density matrices. After the important work of, for instance, Choi (1972) and Kraus (1983), however, it became increasingly generally accepted that complete positivity should be imposed as an additional requirement. Yet despite the reasonableness of the arguments for complete positivity, the imposition of this additional requirement was not universally accepted. Indeed, the issue of whether the more general or the more restricted framework should be employed remains controversial among physicists. At times, the debate has been quite passionate (e.g., Simmons, Jr. and Park, 1981; Raggio and Primas, 1982; Simmons, Jr. and Park, 1982).

The issues involved in the debate were substantially clarified by an exchange between Pechukas and Alicki which appeared in a series of papers between 1994 and 1995. Pechukas and Alicki analysed the dynamical map, Λ, for a system into three separate components: an 'assignment map', a unitary on the combined state space, and a trace over the environment:

$$\rho_S \to \Lambda\rho_S = \mathrm{tr}_R(U\Phi\rho_S U^\dagger), \tag{17}$$

with S, R representing the system of interest and the environment (the 'reservoir') respectively, and the assignment map, Φ, given by

$$\rho_S \to \Phi\rho_S = \rho_{SR}. \tag{18}$$

Since the unitary and the partial trace map are both CP, whether or not Λ itself is CP is solely determined by the properties of Φ, the assignment map. Φ represents an assignment of 'initial conditions' to the combined system: it assigns a *single* state, ρ_{SR}, to each state ρ_S. My use of inverted commas here reflects the fact that such a unique assignment cannot be made in general, since in general the state of the reservoir will be unknown. It will make sense to use such a map in some cases, however; for instance if there is a class Γ of possible initial states $S + R$ that is such that, within this class, ρ_S uniquely determines ρ_{SR}. Or it might be that, even though there are distinct possible initial states in Γ that yield the same reduced state ρ_S, the evolution of ρ_S is (at least approximately) insensitive to which of these initial states is the actual initial conditions.

When Φ is linear:

$$\Phi(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda\Phi(\rho_1) + (1-\lambda)\Phi(\rho_2), \tag{19}$$

consistent:

$$\mathrm{tr}_R(\Phi\rho_S) = \rho_S, \tag{20}$$

and of product form, one can show that Φ is of necessity CP as well. Pechukas (1994) inquired into what follows from the assumption that Φ is linear, consistent, and positive. Pechukas showed that if Φ is defined everywhere on the state space, and is linear, consistent, and positive, *it must be a product map*: $\rho_S \xrightarrow{\Phi} \rho_{SR} = \rho_S \otimes \rho_R$, with ρ_R a fixed density operator on the state space of the reservoir (i.e., all ρ_S's are assigned the same ρ_R). This is undesirable as there are situations in which we would like to describe the open dynamics of systems that do not begin in a product state with their environment. For instance, consider a multi-partite entangled state of some number of qubits representing the initial conditions of a quantum computer, with one of the qubits representing a 'register' and playing the role of S, and the rest playing the role of the reservoir R. If we are restricted to maps that are CP on the system's entire state space then it seems we cannot describe the evolution of such a system.

Pechukas went on to show that when one allows correlated initial conditions, Λ, interpreted as a dynamical map defined on the entire state space of S, may be NCP. In order to avoid the ensuing negative probabilities, one can define a 'compatibility domain' for this NCP map; i.e., one stipulates that Λ is defined only for the subset of states of S for which $\Lambda\rho_S \geq 0$ (or equivalently, $\Phi\rho_S \geq 0$). He writes:

> The operator Λ is defined, via reduction from unitary $S + R$ dynamics, only on a subset of all possible ρ_S's. Λ may be extended—trivially, by linearity—to the set of *all* ρ_S, but the motions $\rho_S \rightarrow \Lambda\rho_S$ so defined may not be physically realizable ... Forget complete positivity; Λ, extended to all ρ_S, may not even be positive (1994).

In his response to Pechukas, Alicki (1995) conceded that the only initial conditions appropriate to an assignment map satisfying all three "natural" requirements—of linearity, consistency, and complete positivity—are product initial conditions. However, he rejected Pechukas's suggestion that in order to describe the evolution of systems coupled to their environments one must forego the requirement that Λ be CP on S's entire state space. Alicki calls this the "fundamental positivity condition." Regarding Pechukas's suggestion that one may use an NCP map with a restricted compatibility domain, Alicki writes:

> ... Pechukas proposed to restrict ourselves to such initial density matrices for which $\Phi\rho_S \geq 0$. Unfortunately, it is impossible to specify such a domain of positivity for a general case, and moreover there exists no physical motivation in terms of operational prescription which would lead to [an NCP assignment of initial conditions] (Alicki, 1995).

It is not clear exactly what is meant by Alicki's assertion that it is impossible to *specify* the domain of positivity of such a map in general, for does not the condition $\Phi\rho_S \geq 0$ itself constitute a specification of this domain? Most plausibly, what Alicki intends is that *determining* the compatibility domain will be exceedingly difficult for the general case. We will return to this question in the next section, as well as to the question of the physical motivation for utilising NCP maps.

In any case, rather than abandoning the fundamental positivity condition, Alicki submits that in situations where the system and environment are initially correlated one should relax either consistency or linearity. Alicki attempts to motivate this by arguing that in certain situations the preparation process may induce an instantaneous perturbation of S. One may then define an inconsistent or nonlinear, but still completely positive, assignment map in which this perturbation is represented.

According to Pechukas (1995), however, there is an important sense in which one should not give up the consistency condition. Consider an inconsistent linear assignment map that takes the state space of S to a convex subset of the state space of $S + R$. Via the partial trace it maps back to the state space of S, but since the map is not necessarily consistent, the traced out state, ρ'_S, will not in general be the same as ρ_S; i.e.,

$$\rho_S \xrightarrow{\Phi} \Phi\rho_S \xrightarrow{\mathrm{tr}_R} \rho'_S \neq \rho_S. \tag{21}$$

Now each assignment of initial conditions, $\Phi\rho_S$, will generate a trajectory in the system's state space which we can regard as a sequence of CP transformations of the form:

$$\rho_S(t) = \mathrm{tr}_R(U_t \Phi \rho_S U_t^\dagger). \tag{22}$$

At $t = 0$, however, the trajectory begins from ρ'_S, not ρ_S. ρ_S, in fact, is a fixed point that lies *off* the trajectory. This may not be completely obvious, prima facie, for is it not the case, the sceptical reader might object, that we can describe the system as evolving from ρ_S to ρ_{SR} via the assignment map and then via the unitary transformation to its final state? While this much may be true, it is important to remember that Φ is supposed to represent an assignment of *initial conditions* to S. On this picture the evolution through time of $\Phi\rho_S$ is a proxy for the evolution of ρ_S. When Φ is consistent, $\text{tr}_R(U\Phi\rho_S U^\dagger) = \text{tr}_R(U\rho_{SR}U^\dagger)$ and there is no issue; however when Φ is inconsistent, $\text{tr}_R(U\Phi\rho_S U^\dagger) \neq \text{tr}_R(U\rho_{SR}U^\dagger)$, and we can no longer claim to be describing the evolution of ρ_S through time but only the evolution of the distinct state $\text{tr}(\Phi\rho_S) = \rho'_S$. And while the evolution described by the dynamical map $\rho'_S(0) \xrightarrow{\Lambda} \rho'_S(t)$ is completely positive, it has *not* been shown that the transformation $\rho_S(0) \xrightarrow{\Lambda} \rho_S(t)$ must always be so.

What of Alicki's suggestion to drop the linearity condition on the assignment map? It is unclear that this can be successfully physically motivated, for it is prima facie unclear just what it would mean to accept nonlinearity as a feature of reduced dynamics. Bluntly put, quantum mechanics is linear in its standard formulation: the Schrödinger evolution of the quantum-mechanical wave-function is linear evolution. Commenting on the debate, Rodríguez-Rosario et al. (2010) write: "giving up linearity is not desirable: it would disrupt quantum theory in a way that is not experimentally supported."

5 Linearity, Consistency, and Complete Positivity

We saw in the last section that there are good reasons to be sceptical with respect to the legitimacy of violating any of the three natural conditions on assignment maps. We will now argue that there are nevertheless, in many situations, good, physically motivated, reasons to violate these conditions.

Let us begin with the CP requirement. *Pace* Alicki, one finds a clear physical motivation for violating complete positivity if one notes, as Shaji and Sudarshan (2005) do, that if the system S is initially entangled with R, then not all initial states of S are allowed—for instance,

$\rho_S = \mathrm{tr}_R \rho_{SR}$ cannot be a pure state, since the marginal of an entangled state is always a mixed state. Such states will be mapped to negative matrices by a linear, consistent, NCP map. On the other hand the map will be positive for all of the valid states of S; this is the so-called compatibility domain of of the map: the subset of states of S that are compatible with Λ.

In light of this we believe it unfortunate that such maps have come to be referred to as NCP maps, for strictly speaking it is not the map Λ but its linear extension to the entire state space of S that is NCP. Λ is indeed CP *within its compatibility domain*. In fact this misuse of terminology is in our view at least partly responsible for the sometimes acrid tone of the debate. From the fact that the linear extension of a partially defined CP map is NCP, it does not follow that "reduced dynamics need not be completely positive."[2] Alicki and others are right to object to this latter proposition, for given the arguments for complete positivity it is right to demand of a dynamical map that it be CP on the domain within which it is defined. On the other hand it is *not* appropriate to insist with Alicki that a dynamical map must be CP on the entire state space of the system of interest—come what may—for negative probabilities will only result from states that cannot be the initial state of the system. Thus we believe that 'NCP maps'—or more appropriately: *Partial-CP* maps with NCP linear extensions—can and should be allowed within a quantum dynamical framework.

What of Alicki's charge that the compatibility domain is impossible to "specify" in general? In fact, the determination of the compatibility domain is a well-posed problem (cf. Jordan et al., 2004); however, as Alicki alludes to, there may be situations in which actually determining the compatibility domain will be computationally exceedingly difficult. But in other cases[3]—when computing the compatibility domain *is* feasible—we see no reason why

[2]This is the title of Pechukas's 1994 article.

[3]For examples, see Jordan et al. (2004); Shaji and Sudarshan (2005).

one should bar the researcher from using a Partial-CP map whose linear extension is NCP if it is useful for her to do so. Indeed, given the clear physical motivation for it, this seems like the most sensible thing to do in these situations.

There may, on the other hand, be other situations where proceeding in this way will be inappropriate. For instance, consider a correlated bipartite system $S + R$ with the following possible initial states:

$$x_+ \otimes \psi_+, \quad x_- \otimes \psi_-, \quad z_+ \otimes \phi_+, \quad z_- \otimes \phi_-. \tag{23}$$

The domain of definition of Φ consists of the four states $\{x_+, x_-, z_+, z_-\}$. Suppose we want to extend Φ so that it is defined on all mixtures of these states, and is linear. The totally mixed state of S can be written as an equally weighted mixture of x_+ and x_-, and also as an equally weighted mixture of z_+ and z_-.

$$\frac{1}{2}I = \frac{1}{2}x_+ + \frac{1}{2}x_- = \frac{1}{2}z_+ + \frac{1}{2}z_-. \tag{24}$$

If Φ is defined on this state, and is required to be a linear function, we must have

$$\begin{aligned} \Phi(\frac{1}{2}I) &= \frac{1}{2}\Phi(x_+) + \frac{1}{2}\Phi(x_-) \\ &= \frac{1}{2}x_+ \otimes \psi_+ + \frac{1}{2}x_- \otimes \psi_-, \end{aligned} \tag{25}$$

$$\begin{aligned} \Phi(\frac{1}{2}I) &= \frac{1}{2}\Phi(z_+) + \frac{1}{2}\Phi(z_-) \\ &= \frac{1}{2}z_+ \otimes \phi_+ + \frac{1}{2}z_- \otimes \phi_-, \end{aligned} \tag{26}$$

from which it follows that

$$\frac{1}{2}x_+ \otimes \psi_+ + \frac{1}{2}x_- \otimes \psi_- = \frac{1}{2}z_+ \otimes \phi_+ + \frac{1}{2}z_- \otimes \phi_-, \tag{27}$$

which in turn entails that

$$\psi_+ = \psi_- = \phi_+ = \phi_-, \tag{28}$$

so Φ cannot be extended to a linear map on the entire state space of S unless it is a product map.

It would be misleading to say that assignment maps such as these violate linearity, for much the same reason as it would be misleading to say that Partial-CP maps with NCP linear extensions violate complete positivity. It is not that these maps are defined on a convex domain, and are nonlinear on that domain; rather, there are mixtures of elements of the domain on which the function is undefined. But since we cannot be said to have violated linearity, then *pace* Rodríguez-Rosario et al., in such situations we see no reason to bar the researcher from utilising these 'nonlinear' maps, for properly understood, they are partial-linear maps with nonlinear extensions.

Pace Pechukas, there may even be situations in which it is appropriate to use an inconsistent assignment map. Unlike the previous cases, in this case the assignment map will be defined on the system's entire state space. This will have the disadvantage, of course, that our description of the subsequent evolution will not be a description of the true evolution of the system, but in many situations one can imagine that the description will be "close enough," i.e., that

$$\mathrm{tr}_R(U_t \rho_{SR} U_t^\dagger) \approx \mathrm{tr}_R(U_t \rho'_{SR} U_t^\dagger). \tag{29}$$

6 Conclusion

Bohr warned us long ago against extending our concepts, however fundamental, beyond their domain of applicability. The case we have just looked at is an illustration of this important point. The debate over the properties one should ascribe to the extension of a partially-defined description is a debate over the properties one should ascribe to a phantom.

Whether or not we must use a map whose extension is nonlinear, or a map whose linear extension is NCP, or an inconsistent map, is not a decision that can be made a priori or that can be shown to follow from fundamental physical principles. The decision will depend on the particular situation and on the particular state preparation we are dealing with.

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