The Paraconsistent Logic of Quantum Superpositions

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Abstract

Physical superpositions exist both in classical and in quantum physics. However, what is exactly meant by 'superposition' in each case is extremely different. In this paper we discuss some of the multiple interpretations which exist in the literature regarding superpositions in quantum mechanics. We argue that all these interpretations have something in common: they all attempt to avoid 'contradiction'. We argue in this paper, in favor of the importance of developing a new interpretation of superpositions which takes into account contradiction, as a key element of the formal structure of the theory, "right from the start". In order to show the feasibility of our interpretational project we present an outline of a paraconsistent approach to quantum superpositions which attempts to account for the contradictory properties present in general within quantum superpositions. This approach must not be understood as a closed formal and conceptual scheme but rather as a first step towards a different type of understanding regarding quantum superpositions.

1   Introduction

There is an important link in the history of physics between the interpretation of theories and their formal development. Relativity would have not

been possible without non-euclidean geometry nor classical physics without infinitesimal calculus. In quantum mechanics, the formal scheme was elaborated —mainly by Schrödingier, Heisenberg, Born, Jordan and Dirac— almost in parallel to the orthodox interpretation. However, still today, the interpretation of quantum mechanics remains controversial regarding most of its non-classical characteristics: indeterminism, holism, contextuality, non-locality, etc. There is, still today, no consensus regarding the meaning of such expressions of the theory. In this paper we shall be concerned with a specific aspect of quantum mechanics, namely, the principle of superposition which, as it is well known, gives rise to the so called quantum superpositions. Physical superpositions exist both in classical and in quantum physics. However, what is exactly meant by "superposition" in each case is extremely different. In classical physics one can have superpositions of waves or fields. A wave (field) α can be added to a different wave (field) β and the sum will give a 'new' wave (field) $\mu = \alpha + \beta$. There is in this case no weirdness for the sum of multiple states gives as a result a new single state. In quantum mechanics on the contrary, a linear combination of multiple states, $\alpha + \beta$ is not reducible to one single state, and there is no obvious interpretation of such superposition of states. As a matter of fact, today quantum superpositions play a central role within the most outstanding technical developments such as quantum teleportation, quantum cryptography and quantum computation [21, 26]. The question we attempt to address in this paper regards the meaning and physical representation of quantum superpositions. There are many interpretations of quantum mechanics each of which provides an answer to this question. In the following we shall review some of these proposals. We shall then argue in favor of the possibility to develop a new interpretation which considers contradictory properties as a main aspect of quantum superpositions and present an outline of a formal approach based on paraconsistent logic which attempts to consider contradiction "right from the start". We must remark that we do not understand this approach as a closed formal and conceptual scheme but rather as a first step towards a different type of understanding regarding quantum superpositions.

Paraconsistent logics are the logics of inconsistent but nontrivial theories. The origins of paraconsistent logics go back to the first systematic studies dealing with the possibility of rejecting the principle of noncontradiction. Paraconsistent logic was elaborated, independently, by Stanislaw Jaskowski in Poland, and by the first author of this paper in Brazil, around the middle of the last century (on paraconsistent logic, see, for example: [5]). A theory T founded on the logic L, which contains a symbol for negation, is called

2

inconsistent if it has among its theorems a sentence A and its negation $\neg A$; otherwise, it is said to be consistent. T is called trivial if any sentence of its language is also a theorem of T; otherwise, T is said to be non-trivial. In classical logics and in most usual logics, a theory is inconsistent if, and only if, it is trivial. L is paraconsistent when it can be the underlying logic of inconsistent but non trivial theories. Clearly classical logic and all usual logics are not paraconsistent. The importance of paraconsistent logic is not limited to the realm of pure logic but has been extended to many fields of application such as robot control, air traffic control [24], control systems for autonomous machines [25], defeasible deontic reasoning [23], information systems [1] and medicine.

In the following, we attempt to call the attention to the importance of extending the realm of paraconsistent logic to the formal account of quantum superpositions. Firstly, we shall discuss the very different meanings of the term 'superposition' in both classical and quantum physics. In section 3, we shall present some of the very different interpretations of the meaning of a quantum superposition which can be found in the literature. In section 4, we shall argue in favor of the importance of considering an interpretation of superposition in terms of paraconsistent logic. In section 5, we present a formal scheme in terms of paraconsistent logic which attempts to account for the inner contradictions present within a quantum superposition. Finally, in section 6, we argue in favor of considering contradiction "right from the start".

2 What is a Quantum Superposition?

In classical physics, every physical system may be described exclusively by means of its *actual properties*, taking 'actuality' as expressing the *preexistent* mode of being of the properties themselves, independently of observation — the 'pre' referring to its existence previous to measurement. Each system has a determined state characterized mathematically in terms of a point in phase space. The change of the system may be described by the change of its actual properties. Potential or possible properties are considered as the points to which the system might arrive in a future instant of time. The occurrence of possibilities in such cases merely reflects our ignorance about what is actual. Contrary to what seems to happen in quantum mechanics, statistical states do not correspond to features of the actual system, but quantify our lack of knowledge of those actual features ([9], p. 125).

Classical mechanics tells us via the equation of motion how the state of the system moves along the curve determined by initial conditions in the phase space. The representation of the state of the physical system is given by a point in phase space Γ and the physical magnitudes are represented by real functions over Γ. These functions commute between each other and can be interpreted as possessing definite values independently of physical observation, i.e. each magnitude can be interpreted as being actually preexistent to any possible measurement. In the orthodox formulation of quantum mechanics, the representation of the state of a system is given by a ray in Hilbert space $\mathcal{H}$. But, contrary to the classical scheme, physical magnitudes are represented by operators on $\mathcal{H}$ that, in general, do not commute. This mathematical fact has extremely problematic interpretational consequences for it is then difficult to affirm that these quantum magnitudes are *simultaneously preexistent*. In order to restrict the discourse to different sets of commuting magnitudes, different Complete Sets of Commuting Operators (CSCO) have to be chosen. The choice of a particular representation (given by a CSCO) determines the basis in which the observables diagonalize and in which the ray can be expressed. Thus, the ray can be written as different linear combinations of states:

$$\alpha_i |\varphi_i^{B1}> + \alpha_j |\varphi_j^{B1}> = |\varphi_q^{B2}> = \beta_m |\varphi_m^{B3}> + \beta_n |\varphi_n^{B3}> + \beta_o |\varphi_o^{B3}> \quad (1)$$

The linear combinations of states are also called quantum superpositions. As it was clearly expressed by Dirac [6]: "The nature of the relationships which the superposition principle requires to exist between the states of any system is of a kind that cannot be explained in terms of familiar physical concepts. One cannot in the classical sense picture a system being partly in each of two states and see the equivalence of this to the system being completely in some other state." The formal difference of using vectors in $\mathcal{H}$ instead of points in Γ seems to imply that in quantum mechanics —apart from the 'possibility' which is encountered in classical mechanics— there is another, different realm which must be necessarily considered and refers, at each instant of time, to *contradictory properties*. To see this, consider the following example: given a spin 1/2 system whose state is $| \uparrow_z>$, we let it interact with a magnetic field in the z direction. All outcomes that can become actual in the future are potential properties of the system, in an analogous manner as all possible reachable positions of a pendulum are in the classical case. But at each instant of time, for example at the initial

4

instant, if we consider the z direction and the projection operator $| \uparrow_z > < \uparrow_z |$ as representing a preexistent actual property, there are other incompatible properties arising from considering projection operators of spin projections in other directions. For example, in the x direction, the projection operators $| \uparrow_x > < \uparrow_x |$ and $| \downarrow_x > < \downarrow_x |$ do not commute with $| \uparrow_z > < \uparrow_z |$ and thus, cannot be considered to possess definite values simultaneously. Since Born interpretation of the wave function, these properties are usually considered as *possible*. However, this possibility is essentially different from the idea of possibility discussed in classical physics which relates to the idea of a *process*. If we consider that the formalism of quantum mechanics provides a description of the world, a representation of *what there is* —and does not merely make reference to measurement outcomes—, at each instant of time the properties, $| \uparrow_z > < \uparrow_z |$, $| \uparrow_x > < \uparrow_x |$ and $| \downarrow_x > < \downarrow_x |$ must be taken into account independently of their future actualization for they all provide non trivial information about the state of affairs. In particular, the properties $| \uparrow_x > < \uparrow_x |$ and $| \downarrow_x > < \downarrow_x |$, which constitute the superposition and must be considered simultaneously are in general *contradictory properties*.

In the quantum logic approach one of the properties, namely, the one in which we can write the state of affairs as a single term, is considered as 'actual' while the others are taken to be 'potential' properties. Potential properties can become actual. These properties, e.g. $| \uparrow_x > < \uparrow_x |$, $| \downarrow_x > < \downarrow_x |$, $| \uparrow_y > < \uparrow_y |$ and $| \downarrow_y > < \downarrow_y |$ in our example, are always part of superpositions with more than one term and are constituted by contradictory properties. However, from a mathematical perspective, independently of their mode of existence, both potential and actual properties are placed at the same level in the algebraic frame which describes the state of affairs according to quantum mechanics: the projections of the spin in all directions are atoms of the lattice and there is no formal priority of the actual over the potential properties. This rises the question if one can consider quantum superpositions as preexistent entities, independently of their future actualization.

In the laboratory, it is precisely this contradictory potential realm which is necessary to be considered by the experimentalist in the developments which are taking place today regarding the processing of quantum information as quantum computing and quantum communication [21, 26]. This seems to point in the direction that these properties have an existence which cannot be reduced to their *becoming actual* at a future instant of time. Superpositions correspond to possible outcomes which occur on an equal footing in the superposition of the final state, so that there is no sign that any one of them is more real than any other ([9], p. 120). Taking these problems

5

into account there are many interpretations which attempt to provide an answer to the question: what is a quantum superposition? We shall discuss in the next section some of these proposals.

3   The Multiple Interpretations of Quantum Superpositions

As we have seen above, the formal description of quantum mechanics seems to imply a deep departure from the classical notion of possible or probable. This was cleverly exemplified by Erwin Schrödinger in his famous cat experiment [31], in which a half dead and half alive cat seemed to laugh of the idea of possessing a determined state. However, one can find in the literature, there are many different interpretations of quantum mechanics in general and of the meaning of a quantum superposition in particular. In this section we shall review some of these very distinct interpretations. We do not attempt to provide a complete review of interpretations but rather to analyze instead their specific understanding of quantum superpositions.

Although we must take into account the fact that 'state vector' and 'cat' are two concepts in different levels of discourse ([8], p. 189). From a realist perspective, which considers physics as providing a description or an expression of the world, the question still arises, if this formal or mathematical representation given by superpositions, namely equation 1 —which allow us to calculate the probability of the possible measurement outcomes—, can be related conceptually to a notion which can allow us to think, independently of measurement outcomes, about the 'superposition of states in Hilbert space' in an analogous manner as we think of a 'point in phase space' (in the formal level) as describing an 'object in space-time' (in the conceptual level). What is describing a mathematical superposition? Can we create or find adequate concepts which can provide a representational realistic account of a quantum superposition independent of measurement outcomes? Of course, from a general empiricist perspective one is not committed to answering these set of questions. The idea that the quantum wave function as related to a superposition is just a theoretical device with no ontological content goes back to Bohr's interpretation of quantum mechanics. The impossibility to interpret the quantum wave function in an ontological fashion can be understood in relation to his characterization of Ψ in terms of an algorithmic device which computes measurement results.[1] This position

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[1]According to Bohr ([36], p. 338) the Schrödinger wave equation is just an abstract

radically addressed seems to end up in the instrumentalistic account shared implicitly by many and developed explicitly by Fuchs and Peres [17]. Bas van Fraassen, whom we consider a close follower of Bohr's ideas, has also taken an anti-metaphysical position with respect to the interpretation of the quantum wave function. His justification stands on his empiricist account of both physics and philosophy (see [34], section 9.1).

From an empiricist perspective the formalism does not provide a description of *what there is*. Superpositions are thus, a theoretical device through which one can consider the actual observation *hic et nunc*. Empiricism can be linked to probability in terms of the frequency interpretation which rests, contrary to the original conception of probability, not on the idea that probability describes in terms of ignorance an existent state of affairs, but rather in a set of empirical results found in a series of measurements. However, and independently of the problems encountered within such empiricist stances, if superpositions are considered just as a *theoretical device*, then the question of interpretation seems to loose its strength. For why should we pursue an interpretation if, like Fuchs and Peres remark, quantum mechanics does the job and already provides an algorithm for computing probabilities for the macroscopic events? There are other reasons which one could put forward to account for the importance of interpretation even from an empiricist perspective (see for example van Fraassen [33]), however these reasons must remain only secondary in the quest of science.

On the contrary, from a realist position, there is need to provide an answer to the link between the theory and its conceptual understanding of the world. To put it in a nutshell: what is quantum mechanics telling us about the world? As noticed by Bacciagaluppi ([2], p. 74), the hidden variable program attempts to "restore a classical way of thinking about *what there is*." In this sense, Bohm's proposal seems to restore the possibility of discussing in terms of a state of affairs described in terms of a set of definite valued properties. In Bohmian mechanics the state of a system is given by the wave function Ψ together with the configuration of particles X. The quantum wave function must be understood in analogy to a classical field that moves the particles in accordance with the following functional relation: $\frac{dx}{dt} = \nabla S$, where $S = \hbar \delta$ (δ being the phase of ψ). Thus, particles always have a well defined position together with the rest of their properties and the evolution depends on the quantum field. It then follows that, there are

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method of calculus and it does not designate in itself any phenomena. See also [3] for discussion.

no superpositions of states, the superposition is given only at the level of the field and remains as mysterious as the superposition of classical fields. The field does not only have a dynamical character but also determines the epistemic probability of the configuration of particles *via* the usual Born rule.

A different approach, which starts from a particular interpretation of quantum superpositions is the so called many worlds interpretation (MW), considered to be a direct conclusion from Everett's first proposal in terms of 'relative states' [12]. Everett's idea was to let quantum mechanics find its own interpretation, making justice to the symmetries inherent in the Hilbert space formalism in a simple and convincing way [7]. MW interpretations are no-collapse interpretations which respect the orthodox formulation of quantum mechanics. The main idea behind many worlds interpretations is that superpositions relate to collections of worlds, in each of which exactly one value of an observable, which corresponds to one of the terms in the superposition, is realized. Apart from being simple, the claim is that it possesses a natural fit to the formalism, respecting its symmetries. The solution proposed to the measurement problem is provided by assuming that each one of the terms in the superposition is *actual* in its own correspondent world. Thus, it is not only the single value which we see in 'our world' which gets actualized but rather, that a branching of worlds takes place in every measurement, giving rise to a multiplicity of worlds with their corresponding actual values. The possible splits of the worlds are determined by the laws of quantum mechanics but each world becomes again 'classical'. Quantum superpositions are interpreted as expressing the existence of multiple worlds, each of which exists in (its own) actuality. However, there are no superpositions in this, our actual world, for each world becomes again a "classical world". The many worlds interpretation seems to be able to recover these islands of classicality at the price of multiplying the 'actual realm'. In this case, the quantum superposition is expelled from each actual world and recovered only in terms of the relation between the multiple worlds.

The Geneva school to quantum logic and similar approaches such as that of Foulis and Randall [16] attempt to consider quantum physics as related to the realms of actuality and potentiality in analogous manner to classical physics. According to the Geneva school, both in classical and quantum physics measurements will provoke fundamental changes of the state of the system.[2] Continuing Heisenberg's considerations in the new physics, Con-

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[2]What is special for a classical system, is that 'observables' can be described by func-

stantin Piron has been one of the leading figures in developing the notion of potentiality within the logical structure of quantum mechanics [27, 28]. Following [32], a physical property, never mind whether a classical or quantum one, is specified as what corresponds to a set of definite experimental projects. A *definite experimental project* (DEP) is an experimental procedure (in fact, an equivalence class of experimental procedures) consisting in a list of actions and a rule that specifies in advance what has to be considered as a *positive* result, in correspondence with the *yes* answer to a dichotomic question. Each DEP tests a property. A given DEP is called *certain* (correspondingly, a dichotomic question is called *true*) if it is sure that the positive response would be obtained when the experiment is performed or, more precisely, in case that whenever the system is placed in a measurement situation then it produces certain definite phenomenon to happen. A physical property is called *actual* in case the DEPs which test it are certain and it is called *potential* otherwise. Whether a property is actual or potential depends on the state in which one considers the system to be. Though in this approach both actuality and potentiality are considered as modes of being, actual properties are considered as attributes that *exist*, in the EPR sense as elements of physical reality[3], while potential properties are not conceived as existing in the same way as real ones. They are thought as *possibilities* with respect to actualization, because potential properties may be actualized due to some change in the state of the system. In this case the superposition provides a measure —given by the real numbers which appear in the same term as the state— over the potential properties which could become actual in a given situation.

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tions on the state space. This is the main reason that, a measurement corresponding to such an observable, can be left out of the description of the theory 'in case one is not interested in the change of state provoked by the measurement', but 'only interested in the values of the observables'. It is in this respect that the situation is very different for a quantum system. Observables can also be described, as projection valued measures on the Hilbert space, but 'no definite values can be attributed to such a specific observable for a substantial part of the states of the system'. For a quantum system, contrary to a classical system, it is not true that 'either a property or its negation is actual'.

[3]Einstein designed, in the by now famous EPR 'paper' [11], a definition of when a physical quantity could be considered an *element of physical reality* within quantum mechanics. By using this definition Einstein, Podolsky and Rosen argued against the completeness of the quantum theory. For a general discussion see [15].

4 Quantum Superpositions and the 'Contradiction' of Properties?

Although the interpretations we have discussed in the previous section from both their formal and metaphysical commitments have many differences, there is still something they all share in common: they all attempt to avoid contradictions. Indeed 'contradiction' has been regarded with disbelief in Western thought due to certain metaphysical presuppositions which go back to Plato, Aristotle, Leibniz and Kant. Even after the development of paraconsistent logic in the mid XX century and the subsequent technical progress this theory has allowed, the aversion towards contradiction is still present today within science and philosophy. The famous statement of Popper that the acceptance of inconsistency "would mean the complete breakdown of science" remains an unfortunate prejudice within present philosophy of science (see [4], Chap. 5).

Leaving instrumentalist positions aside, one of us has argued elsewhere [29] that one can find in the vast literature regarding the interpretation of quantum mechanics, two main strategies which attempt to provide an answer to the riddle of 'what is quantum mechanics talking about'. The first strategy is to begin with a presupposed set of metaphysical principles and advance towards a new formalism. Examples of this strategy are Bohmian mechanics, which has been discussed above, and the collapse theory proposed by Ghirardi, Rimini and Weber (also called 'GRW theory') [18], which introduces non-linear terms in the Schrödinger equation. The second strategy is to accept the orthodox formalism of quantum mechanics and advance towards the creation and elucidation of the metaphysical principles which would allow us to answer the question: 'what is quantum mechanics talking about'? Examples of this second strategy are quantum logic and its different lines of development such as the just described Geneva School of Jauch and Piron, and the modal interpretation (see for example [10, 30, 35]). From this perspective, the importance is to focus in the formalism of the theory and try to learn about the symmetries, the logical features and structural relations. The idea is that, by learning about such aspects of the theory we can also develop the metaphysical conditions which should be taken into account in a coherent ontological interpretation of quantum mechanics.

But even independently of the choice of this strategy, it seems quite clear that technical developments which are taking place today regarding quantum mechanics have advanced quite independently of the commitments to

any classical metaphysical background. Quantum computation makes use of the multiple flow of information in the superposition even considering (in principle) contradictory paths. Also quantum cryptography uses the relation between contradictory terms in order to send messages avoiding classical spies. At a formal level, the path integral approach takes into account the multiple contradictory paths within two points [14]. Thus, since both the formalism and experiments seem to consider 'contradictory elements' within quantum mechanics, we argue that it can be of deep interest to advance towards a formalism which takes contradiction into account "right from the start".[4] Evidently, such a formalism could open paths not only to continue the technical developments just mentioned but also to understand the meaning of quantum superpositions from a new perspective. Our proposal is twofold, firstly, to call the attention of the importance of considering contradictory properties within the formalism and interpretation of quantum superpositions; and secondly, to show that paraconsistent logics can open a formal line of research. In the next section we make a first step in this same direction, providing an outline to an approach based on paraconsistent logic.

5   An Outline of a Paraconsistent Approach to Quantum Superpositions

We bring into, now, a paraconsistent logical system ZF_1, that is a strong set theory, even stronger than common ZF (Zermelo Frenkel set theory). On ZF_1 and related matters, see [5]. In what follows, we employ the terminology, notations and conventions of Kleene [19]

The basic symbols of the langugae ZF_1 are the following: 1) Propositional connectives: implication ($\rightarrow$), conjunction ($\wedge$), disjunction ($\vee$) and (weak) negation ($\neg$), equivalence ($\leftrightarrow$) is defined as usual. 2) Individual variables: a demmuerable set of variables, that are represented by smal Latin letters of the end of the aplphabet. 3) The quantifiers $\forall$ (for all) and $\exists$ (there exists). 4) The binary predicate symbols $\in$ (membership) and $=$ (identity). 5) Auxiliary symbols: parenthesis.

Syntactic notions, for example those of formula, closed formula or sentence, and free occurrence of a variable in a formula, are defined as customary. Russell's symbol for description (ι) is introduced by contextual defi-

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[4]In an analogous fashion as Décio Krause has developed a Q-set theory which accounts for indistinguishable particles with a formal calculus "right from the start" [20].

nition and with the help of the description, the classifier $\{x; F(x)\}$, where $F(x)$ is a formula and x a variable.

Definition 5.1 A° abbreviates $\neg(A \wedge \neg A)$.

Loosely speaking, A° means that A is a well-behaved formula, i.e., that it is not the case that one has A and $\neg A$ both true (or, what is the same thing, that the contradiction $A \wedge \neg A$ is false).

Definition 5.2 $\neg^* A$ abbreviates $\neg A \wedge A^\circ$.

$\neg^*$ functions like a strong negation, a kind of classical negation in our logic. On the other hand, $\neg$ is the weak (or paraconsistent) negation.

Postulates of ZF_1

a) Propositional postulates:

1. $A \rightarrow (B \rightarrow A)$

2. $(A \rightarrow B) \rightarrow (A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)$

3. $\dfrac{A \quad A \rightarrow B}{B}$

4. $(A \wedge B) \rightarrow A$

5. $(A \wedge B) \rightarrow B$

6. $A \rightarrow (B \rightarrow (A \wedge B))$

7. $A \rightarrow (A \vee B)$

8. $B \rightarrow (A \vee B)$

9. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$

10. $A \vee \neg A$

11. $\neg\neg A \rightarrow A$

12. $B^\circ \rightarrow ((A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A))$

13. $(A^\circ \wedge B^\circ) \rightarrow (A \rightarrow B)^\circ$

14. $(A^\circ \wedge B^\circ) \to (A \wedge B)^\circ$

15. $(A^\circ \wedge B^\circ) \to (A \vee B)^\circ$

  b) Quantificational Postulaes:

1. $\dfrac{C \to A(x)}{C \to \forall x A(x)}$

2. $\forall x A(x) \to A(t)$

3. $A(t) \to \exists x A(x)$

4. $\dfrac{A(x) \to C}{\exists x A(x) \to C}$

5. $\forall x (A(x))^\circ \to (\forall x A(x))^\circ$

6. $\forall x (A(x))^\circ \to (\exists x A(x))^\circ$

   The preceding postulates are subject to the usual restrictions. In classical logic, as well as in most usual logics, we are allowed to reletter bind variables and suppress vacuous variables, but it seems that this is not probable in connection with our system. Therefore, we introduce the postulate:

7. If B is a formula obtained from A by relettering bound variables or by the suppression of void quantifiers, then $A \leftrightarrow B$ is an axiom.

Remark. Postulates 1-15 constitute a propositional system of paraconsistent logic and adding the quantificational postulates we obtain a first-order quantificational paraconsistent logic.

  c) Set-Theoretic Postulates:
They are all those of classical ZF in whose formulations the symbol of negation is replaced by the symbol of strong negation $\neg*$. The postulates can be formulated supposing that ZF_1 is a pure set theory or a theory that contains *Urelemente* (objects that are not sets). Our results don't depend on the version employed of ZF_1. Moreover the existence of some sets that cause problems (do not exist) in ZF, like Russell's collection, could be postulated as existing in ZF_1; however, this possibility is here excluded. (ZF is studied, for instance, in [22]).

From now on, capital letters stand for formulas. We have (see [5]):

13

Definition 5.3 $\vdash A$ stands for A is a theorem of ZF_1; $\nvdash A$ is the negation of $\vdash A$.

Definition 5.4 A^* is the formula obtained from A by replacing any occurrence of $\neg$ by an occurrence of $\neg^*$.

Theorem 5.5 *In* ZF_1:
$\vdash A \vee \neg A, \vdash \neg\neg A \rightarrow A, \vdash (A \wedge \neg^* A) \rightarrow B, \nvdash (A \wedge \neg A) \rightarrow B, \nvdash A \rightarrow \neg\neg A, \nvdash \neg(A \wedge \neg A)$.

Theorem 5.6 *If A is provable in ZF, then A^* is probable in ZF_1 (ZF is included in ZF_1).*

Theorem 5.7 *ZF is inconsistent if and only if ZF_1 is non trivial.*

ZF is, in a certain sense, contained in ZF_1. So, in ZF_1 it is possible to systematize extant classical mathematics; in consequence, ZF_1 encompasses all mathematical analysis required for the treatment of standard quantum mechanics, and this treatment is similar to the one with classical logic (and set theory) as the basic logic.

In the study of a quantum system S in NF_1, we note that its states behave as in classical quantum mechanics, in the sense that

$$ZF_1 \nvdash \exists s(s \in \hat{S} \wedge \neg^*(s \in \hat{S})) \tag{2}$$

and some other similar formulas are (apparently) not probable, where $\hat{S}$ is the set of the states of S included in a given superposition.

When S is in the state of superposition of, say, the states s_1 and s_2 (classically inconsistent), we introduce in ZF_1 the extra predicate K and expand the system with the postulates

$$K(S, s_1) \text{ and } \neg K(S, s_1) \tag{3}$$

as well as:

$$K(S, s_2) \text{ and } \neg K(S, s_2) \tag{4}$$

Informally, for instance $K(S_1, s_1)$ means that "S has the superposition predicate associated to s_1" (or the "paraconsistent predicate associated to s_1"). In other words, superposition creates a contradictory situation, giving rise to contradictory relations. In ZF_1, we can not directly assume that the

14

linear combination of two classically incompatible states is an 'inconsistent' state; this is so because the mathematics of usual quantum mechanics is classical, and such kind of inconsistency would make our system trivial.

To cope with this situation, we appeal to a new postulate:

Postulate of inconsistency. Let S be a quantum system which is in the superposition of the (classically incompatible) states s_1 and s_2. Under the hipothesys, we have:

$$K(S, s_1) \wedge \neg K(S, s_1) \wedge K(S, s_2) \wedge \neg K(S, s_2) \qquad (5)$$

This means that superposition implies contradiction. Similarly when superposition involves more than two states.

Therefore, ZF_1 constitutes the underlaying logic of an inconsistent, but apparently non trivial, quantum mechanics that we denote by QM_1; usual quantum mechanics will be denoted by QM. Thus, QM is in a certain sense, contained in QM_1. But the details of the construction of a paraconsistent quantum mechanics will be left to a future series of technical works.

6   Discussion: Considering Contradiction "Right from the Start"

Our proposal focuses on the idea that it would be worthwhile to develop a new interpretation of quantum superpositions which considers contradiction "right from the start". We have provided an outline of a paraconsistent approach to quantum superpositions which shows the possibility to consider contradictions also from a formal perspective. However, it should be clear that we do not take paraconsistent logic to be the "true logic" which should replace classical logic; in the same way as we do not regard quantum mechanics as a theory that should replace classical mechanics [4, 30]. From our perspective we argue that physicists should recognize the possibility to use new forms of logic —such as paraconsistent logic— which might help us understanding features of different domains of reality; features which might not be necessarily accommodated by means of classical logic. We do not believe there is a "true logic", but rather that distinct logical systems can be of use to develop and understand complementary aspects of reality. Recalling the words of Albert Einstein: "It is only the theory which can tell

you what can be observed"[5] it could be argued that only within a theory it is possible to consider and account for phenomena. From this standpoint the development of the formalism can be regarded not only as a merely technical improvement, but also as a way to open new paths of understanding and even of development of new phenomena. Formal development is not understood here as going beyond the theory, as improving and showing something that "was not there before" in the formalism —as it is the case of the GRW theory or Bohmian mechanics. Rather, this development is understood as taking seriously the features which the theory seems to show us, exposing them in all their strength, "right from the start".

We also have to stress that non relativistic quantum mechanics, based on classical logic and on the common specific postulates, seems to be consistent in the strict logical meaning. So, a paraconsistent version of it has to postulate, in some way or other, the inconsistent character of determinable situations. It appears to be that there is no possibility that someone can "deduce", employing any one of a majority of extant logics, contradictory consequences of the specific axioms of non relativistic quantum mechanics.

Evidently, we have to develop and to explore the ideas here sketched. One of the important points to take into consideration is that there are numerous ways to obtain such kind of inconsistent quantum mechanics. In addition, we should verify if there are really important new results of QM_1 which are not valid in QM. The philosophical meaning of QM_1 also deserves detailed analysis.

Acknowledgments

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[5]These words, according to Heisneberg himself, led him to the development of the indetermination principle in his foundational paper of 1927.

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