

# Is an electron a charge cloud? A reexamination of Schrödinger's charge density hypothesis

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## Abstract

This article re-examines Schrödinger's charge density hypothesis, according to which the charge of an electron is distributed in the whole space, and the charge density in each position is proportional to the modulus squared of the wave function of the electron there. It is demonstrated that the charge distribution of a quantum system can be measured by protective measurements as expectation values of certain observables, and the results as predicted by quantum mechanics confirm Schrödinger's original hypothesis. Moreover, the physical origin of the charge distribution is also investigated. It is argued that the charge distribution of a quantum system is effective, that is, it is formed by the ergodic motion of a localized particle with the charge of the system.

## 1 Introduction

In quantum mechanics, an electron has an electric charge represented by  $-e$  in the potential term of the Schrödinger equation,  $-e\varphi\psi(x,t)$ , where  $\psi(x,t)$  is the wave function of the electron, and  $\varphi$  is an external electric scalar potential. An intriguing question is: how is the charge of the electron distributed in space? Historically, Schrödinger first gave an explicit answer to this question: the charge of the electron is distributed in the whole space, and the charge density in position  $x$  at instant  $t$  is  $-e|\psi(x,t)|^2$  (Schrödinger 1926). Schrödinger's purpose was not to simply answer this question, but to interpret the wave function of the electron in terms of the charge density of the electron. It is well known that this semiclassical interpretation of the wave function was soon rejected and replaced by Born's probability interpretation (Born 1926). The rejection seemed to have a well-founded basis. Unfortunately, however, the above question and Schrödinger's answer to it have also been largely ignored since then.

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In this paper, we will re-examine the above question and argue that Schrödinger's answer is actually right. In particular, we will show that protective measurements (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993), which are distinct from conventional projective measurements, can directly measure the charge distribution of an electron assumed by Schrödinger. Moreover, we will also investigate the physical origin of the charge distribution. The investigation may not only help unveil the existent form of the distribution but also provide a possible clue to the physical meaning of the wave function.

The plan of this paper is as follows. In Section 2, we first give a historical analysis of Schrödinger's heuristic hypothesis on the existence of the charge distribution of an electron. It is argued that although Schrödinger's semiclassical interpretation of the wave function in terms of charge density meets serious problems and is unsatisfactory, this does not imply that the charge distribution of an electron does not exist. Moreover, it is also pointed out that the charge distribution is an incomplete manifestation of the wave function for many-body systems, and even for one-body systems it also has some puzzling features, e.g. the charge distribution of a single electron has no electrostatic self-interaction. Section 3 then presents a more convincing argument for the existence of the charge distribution of a quantum system. It is demonstrated that the charge distribution of a quantum system can be directly measured by protective measurements as expectation values of certain observables. The results as predicted by quantum mechanics show that the charge of a charged quantum system is distributed throughout space, and the charge density in each position is proportional to the modulus squared of the wave function of the system there. This confirms Schrödinger's original hypothesis. In Section 4, we further investigate the physical origin of the charge distribution in order to explain its puzzling features and solve the incompleteness problem. It is argued that the charge distribution of a quantum system is effective, that is, it is formed by the ergodic motion of a localized particle with the charge of the system. This may explain the non-existence of electrostatic self-interaction for the charge distribution of a single quantum system. Moreover, the picture of ergodic motion of a single particle can be extended to many particles, which may further solve the incompleteness problem. The extension may also provide a possible clue to the physical meaning of the wave function. Conclusions are given in the last section.

## 2 A historical review

The charge of a classical system always localizes in a definite position in space at each moment. For a quantum system described by the wave function  $\psi(x, t)$ , how does its charge distribute in space then? We can measure

the total charge of the quantum system by electromagnetic interaction and find it in certain region of space. Thus it seems that the charge of a quantum system must also exist in space with a certain distribution. When Schrödinger introduced the wave function and founded his wave mechanics, he also suggested a plausible answer to the above question (Schrödinger 1926). Schrödinger assumed that the charge of an electron is distributed in the whole space, and the charge density in position  $x$  at instant  $t$  is  $-e|\psi(x, t)|^2$ , where  $-e$  is the charge of the electron, and  $\psi(x, t)$  is the wave function of the electron. In the following, we will give a more detailed historical analysis of Schrödinger's charge density hypothesis.

In his paper on the equivalence between wave mechanics and matrix mechanics (Schrödinger 1926a), Schrödinger suggested that it might be possible to give an extraordinarily anschaulich interpretation of the intensity and polarization of radiation by assuming the wave function, which was then called mechanical field scalar, is the source of the radiation. In particular, he assumed that the charge density of an electron as the source of radiation is given by the real part of  $-e\psi\frac{\partial\psi^*}{\partial t}$ , where  $\psi$  is the wave function of the electron. In his third paper on wave mechanics (Schrödinger 1926b), which deals with perturbation theory and its application to the Stark effect, Schrödinger noted in an addendum in proof that the correct charge density of an electron was given by  $-e|\psi|^2$ . Then in his fourth paper on wave mechanics and his 1927 Solvey report (Schrödinger 1926, 1928), Schrödinger further showed how this gives rise to a sensible notion of charge density for several electrons, each contribution being obtained by integrating over the other electrons. Concretely speaking, for a many-particle system, select one particle and keep the coordinates of the particle that describe its position fixed at a given position and integrate  $|\psi|^2$  over all the rest of the coordinates of the system and multiply the charge of the particle, and do a similar thing for each particle, in each case fixing the selected particle at the same given position. Then the sum of all these partial results gives the charge density at the given position. In other words, the charge density is  $\rho(x, t) = \sum_i \int \dots \int Q_i |\psi(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_N, t)|^2 dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_N$ .

At the 1927 Solvey conference, Born posed an objection relating to quadrupole moments for Schrödinger's heuristic hypothesis about charge density (Bacciagaluppi and Valentini 2009, p.426). Born considered two particles with charge  $e$  whose wave function is  $\psi(x_1, x_2)$ . According to Schrödinger, the charge density is

$$\rho(x) = e \int |\psi(x, x_2)|^2 dx_2 + e \int |\psi(x_1, x)|^2 dx_1 . \quad (1)$$

But the quadrupole moment

$$\int \int x_1 x_2 |\psi(x_1, x_2)|^2 dx_1 dx_2$$

cannot be expressed using the function  $\rho(x)$ . As a result, one cannot reduce the radiation of the quadrupole to the motion of a charge distribution  $\rho(x)$  in the usual three-dimensional space. Born then concluded that interpreting the quantity  $|\psi|^2$  as a charge density leads to difficulties in the case of quadrupole moments.

However, it can be seen by a more careful analysis that the above problem is not really about the charge density itself, but about Schrödinger's interpretation of the wave function in terms of the charge density. In fact, Schrödinger also clearly realized this problem. As early as in his equivalence paper (Schrödinger 1926a), Schrödinger already noticed the difficulty relating to the problem of several electrons, which lies in the fact that the wave function is a function in configuration space, not in real space. Although the charge density in three-dimensional space can be consistently defined for an N-particle system, it does not reflect all information encoded in the wave function of the system which lives in the  $3N$ -dimensional configuration space. As a result, although the existence of the charge density may provide an approximate, classical explanation for some phenomena of radiation, it cannot account for all predictions of quantum mechanics and experimental observations, e.g., as Born rightly pointed out, the motion of a charge distribution cannot explain the radiation of the quadrupole.

Besides this incompleteness problem for many-body systems, Schrödinger also realized that the charge distribution of a single quantum system such as an electron cannot be purely classical because his equation does not include the usual Coulomb interaction between the distributions (the interaction between two systems is described by the potential terms in the equation) (Schrödinger 1926). In particular, there is no electrostatic self-interaction of the charge distribution of a single charged quantum system (Schrödinger 1928). Moreover, according to the Schrödinger equation, the interacting systems should be treated as a whole, and its wave function is defined in the multi-dimensional configuration space, which cannot be decomposed into a direct product of the wave functions of all interacting systems. This makes the interaction between two quantum systems more complex than that between two classical charges.

Schrödinger's interpretation of the wave function in terms of charge density was later investigated and extended by a few authors (see, e.g. Madelung 1926, 1927; Janossy 1962; Jaynes 1973; Barut 1988). Due to the above problems, however, this semiclassical interpretation cannot be satisfactory in the final analysis. Moreover, although this fact does not imply the non-existence of the charge density, the very limited success of the interpretation does not provide a convincing argument for the existence of the charge density either. Presumably because of these reasons and because the charge density cannot be directly measured and its existence also lacks a consistent physical explanation, Schrödinger's charge density hypothesis has been largely ignored after his interpretation of the wave function was

replaced by Born's probability interpretation (Born 1926).

### 3 The existence of charge density

As we have noted above, the failure of Schrödinger's interpretation of the wave function in terms of charge density does not imply that the charge distribution of an electron as assumed by him does not exist. The charge density hypothesis is not refuted, but only ignored. There is still a great puzzle about the charge distribution of an electron. Does the charge distribution really exist as Schrödinger suggested? Can it be directly measured? In this section, we will demonstrate that the charge distribution of a quantum system can indeed be measured by protective measurements (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993), and the results as predicted by quantum mechanics will also confirm Schrödinger's charge density hypothesis.

According to quantum mechanics, a quantum system being in a position eigenstate has a definite position in space. Moreover, since the system has properties such as mass and charge, the mass and charge of the system also exist in the definite position. Then for a position eigenstate of a charged quantum system, the total charge of the system concentrates in a definite position. This charge distribution can also be measured by a projective position measurement, which is realized by electrostatic interaction between the system and a charged measuring device. The result of the measurement (which does not disturb the measured system) will indicate that the measured system, which is in a position eigenstate, has its total charge in its position.

Then how can we know the charge distribution of a quantum system being in a position superposition state? It is well known that a projective position measurement of the superposition state can only tell us that the system is in one of the positions in superposition (and the charge of the system concentrates in this position) *after* the measurement. Since the projective measurement destroys the measured state through the collapse of the wave function, its result cannot tell us the actual charge distribution of the measured system *before* the measurement. If projective measurements were the only possible measurements, then it would be quite understandable that the charge distribution of a quantum system, even if it exists, cannot be directly measured.

Fortunately, it has been known that projective measurements are only one kind of measurements, for which the coupling between the measuring device and the measured system is very strong and almost instantaneous (and thus the measurement results are the eigenvalues of the measured observable), and the coupling strength and the measuring time can both be adjusted for a standard measurement procedure, and thus there also exist

other kinds of measurements such as weak measurements (Aharonov, Albert and Vaidman 1988) and protective measurements (Aharonov and Vaidman 1993; Aharonov, Anandan and Vaidman 1993)<sup>1</sup>. Protective measurement uses a weak and long duration coupling interaction and an appropriate procedure to protect the measured system from being disturbed. A general scheme is to let the measured system be in a nondegenerate eigenstate of the whole Hamiltonian using a suitable protective interaction (in some situations the protection is provided by the measured system itself), and then make the measurement adiabatically so that the state of the system neither collapses nor becomes entangled with the measuring device appreciably. In this way, such protective measurements can measure the expectation values of observables on a single quantum system<sup>2</sup>, and in particular, the charge distribution of the system can be measured as expectation values of certain observables.

Before showing how protective measurements can measure the charge distribution of a quantum system, we will first give a brief analysis of measurements (which do not disturb the measured system), which may help to understand the existence and measurability of the charge distribution of a quantum system. No matter how to define measurement, a measurement must be realized by certain interaction between the measured system and the measuring device. Concretely speaking, the measuring device is influenced by the measured system through an interaction that depends on the measured property, and the change of the measuring system then reflects the measured property of the measured system. For example, a position

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<sup>1</sup>Note that weak measurements have been implemented in experiments (see, e.g. Lundeen et al 2011), and it can be reasonably expected that protective measurements can also be implemented in the near future with the rapid development of quantum technologies.

<sup>2</sup>An immediate implication of protective measurements is that the result of a protective measurement, namely the expectation value of the measured observable in the measured state, reflects the actual physical state of the measured system (see also Aharonov and Vaidman 1993; Anandan 1993; Dickson 1995), as the system is not disturbed after this result has been obtained. This is in accordance with the fundamental assumption that the result of a measurement that does not disturb the measured system reflects the actual property or state of the system. Moreover, since the wave function can be reconstructed from the expectation values of a sufficient number of observables, the wave function of a quantum system is a representation of the physical state of the system. Note that for a realistic protective measurement whose measuring interval  $T$  is finite, there is always a tiny probability proportional to  $1/T^2$  to obtain a different result, and after obtaining the result the measured state also collapses to the state corresponding to the result. However, the key point is that when the measurement obtains the expectation value of the measured observable, the state of the measured system is not disturbed. Moreover, the above probability can be made arbitrarily small in principle when  $T$  approaches infinity, as well as negligibly small in practice by making  $T$  sufficiently large. It might be also worth noting that there appeared numerous objections to the validity of protective measurements (see, e.g. Unruh 1994; Rovelli 1994; Ghose and Home 1995; Uffink 1999), and these objections have been answered (Aharonov, Anandan and Vaidman 1996; Dass and Qureshi 1999; Vaidman 2009; Gao 2012). For a detailed introduction of protective measurement and its possible implications see Gao (2013a).

measurement must depend on the existence of certain position-dependent interaction between the system and the device such as electrostatic interaction between two electric charges. The existence of an electrostatic interaction between a charged measuring device and a measured system then tells us that the measured system has electric charge. Moreover, since the strength of the interaction relates to the distance between the two interacting systems, the measurement result may also reflect the charge distribution of the measured system in space. For example, as noted above, for a projective position measurement of a position eigenstate of an electron, which does not disturb the measured electron, if the measurement interaction is electrostatic interaction between the electron and a charged measuring device, then the result of the measurement will indicate that the electron, which is in a position eigenstate, has charge  $-e$  in its position. Similarly, as we will demonstrate below, when a protective measurement (which does not disturb the measured system) is realized by electromagnetic interaction between the measured system and the measuring device, the measurement can also measure the charge distribution of the system, which may be in a general position superposition state.

Consider a protective measurement of the charge of a quantum system with charge  $Q$  in a small spatial region  $V$  having volume  $v$ . This is equivalent to measuring the following observable:

$$A = \begin{cases} Q, & \text{if } x \in V, \\ 0, & \text{if } x \notin V. \end{cases} \quad (2)$$

A protective measurement of  $A$  in a general superposition state  $\psi(x, t)$  yields

$$\langle A \rangle = Q \int_V |\psi(x, t)|^2 dx, \quad (3)$$

which gives the charge of the system in the region  $V$ . When  $v \rightarrow 0$  and after performing measurements in sufficiently many regions  $V$ , we can find the charge density everywhere in space, which is  $\rho_Q(x, t) = Q|\psi(x, t)|^2$ <sup>3</sup>.

This result can be illustrated by a specific example. Consider a quantum system with charge  $Q$  whose wave function is

$$\psi(x, t) = a\psi_1(x, t) + b\psi_2(x, t), \quad (4)$$

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<sup>3</sup>Similarly, we can protectively measure another observable  $B = \frac{\hbar}{2mi}(A\nabla + \nabla A)$ . The measurements will give the electric flux density  $j_Q(x, t) = \frac{\hbar Q}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*)$  everywhere in space. Moreover, we can also protectively measure the charge density (and electric flux density) of a many-body system, and the density turns out to be the same as that given by Schrödinger. For example, a protective measurement of  $A_1 + A_2$  in a two-particle state  $\psi(x_1, x_2, t)$  yields  $\langle A_1 + A_2 \rangle = Q_1 \int_V \int_{-\infty}^{+\infty} |\psi(x_1, x_2, t)|^2 dx_2 dx_1 + Q_2 \int_V \int_{-\infty}^{+\infty} |\psi(x_1, x_2, t)|^2 dx_1 dx_2$ . When  $v \rightarrow 0$  we can find the charge density is  $\rho_Q(x, t) = Q_1 \int_{-\infty}^{+\infty} |\psi(x, x_2, t)|^2 dx_2 + Q_2 \int_{-\infty}^{+\infty} |\psi(x_1, x, t)|^2 dx_1$ .

where  $\psi_1(x, t)$  and  $\psi_2(x, t)$  are two normalized wave functions respectively localized in their ground states in two small boxes 1 and 2, and  $|a|^2 + |b|^2 = 1$ . A measuring electron, whose initial state is a Gaussian wave packet narrow in both position and momentum, is shot along a straight line near box 1 and perpendicular to the line of separation between the boxes. The electron is detected on a screen after passing by box 1. Suppose the separation between the boxes is large enough so that a charge  $Q$  in box 2 has no observable influence on the electron. Then if the system is in box 2, namely  $|a|^2 = 0$ , the trajectory of the electron wave packet will be a straight line as indicated by position “0” in Fig.1, indicating that there is no charge in box 1. If the system is in box 1, namely  $|a|^2 = 1$ , the trajectory of the electron wave packet will be deviated by the electric field of the system by a maximum amount as indicated by position “1” in Fig.1, indicating that there is a charge  $Q$  in box 1. These two measurements are conventional measurements of the eigenstates of the system’s charge in box 1, and their results can reveal the actual charge distribution in box 1. However, when  $0 < |a|^2 < 1$ , i.e. when the measured system is in a superposition of two eigenstates of its charge in box 1, it is well known that such conventional measurements cannot detect the actual charge distribution in box 1.

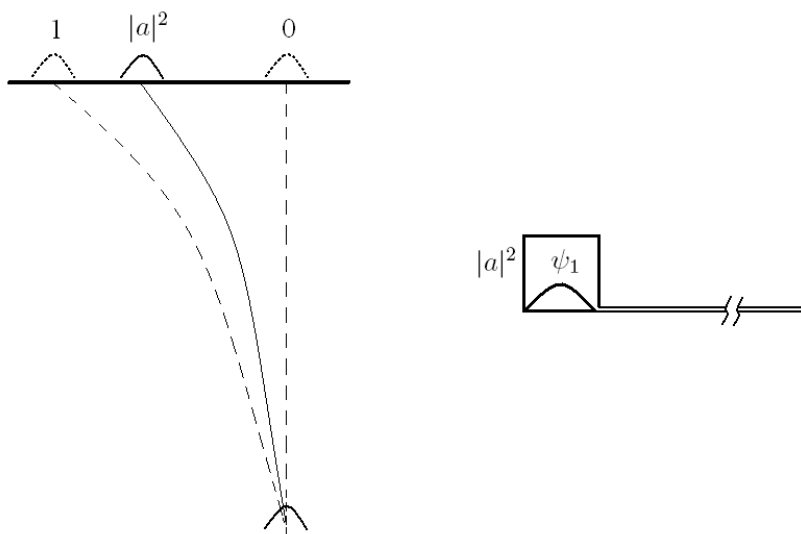


Fig.1 Scheme of a protective measurement of the charge distribution of a quantum system

Now let’s make a protective measurement of the charge of the system in box 1 for the general superposition state  $\psi(x, t)$ . Since the state  $\psi(x, t)$  is degenerate with its orthogonal state  $\psi'(x, t) = b^*\psi_1(x, t) - a^*\psi_2(x, t)$ , we need an artificial protection procedure to remove the degeneracy, e.g.



joining the two boxes with a long tube whose diameter is small compared to the size of the box<sup>4</sup>. By this protection  $\psi(x, t)$  will be a nondegenerate energy eigenstate. The adiabaticity condition and the weakly interacting condition, which are required for a protective measurement, can be further satisfied when assuming that (1) the measuring time of the electron is long compared to  $\hbar/\Delta E$ , where  $\Delta E$  is the smallest of the energy differences between  $\psi(x, t)$  and the other energy eigenstates, and (2) at all times the potential energy of interaction between the electron and the system is small compared to  $\Delta E$ . Then the measurement by means of the electron trajectory is a protective measurement, and the trajectory of the electron wave packet is only influenced by the expectation value of the charge of the system in box 1. As a result, the electron wave packet will reach the position “ $|a|^2$ ” between “0” and “1” on the screen as denoted in Fig.1, indicating that there is a charge  $|a|^2 Q$  in box 1.

Here it may be necessary to further clarify the meaning of charge distribution as a property of a quantum system. As noted before, any physical measurement is necessarily based on some interaction between the measured system and the measuring system. One basic form of interaction is the electrostatic interaction between two electric charges as in the above example, and the existence of this interaction during a measurement, which is indicated by the deviation of the trajectory of the charged measuring system such as an electron, means that the measured system also has the charge responsible for the interaction. Then at least in the sense that any part of a physical entity has electrostatic interaction with another charged system, we can say that the physical entity has charge distribution in space<sup>5</sup>. In the above example, the definite deviation of the trajectory of the electron will reflect that there exists a definite amount of charge in box 1, and the extent of the deviation will further indicate how much charge there is there.

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<sup>4</sup>It is worth stressing that the added protection procedure depends on the measured state, and different states need different protection procedures in general. This means that a protective measurement with an artificial protection procedure requires that the wave function of the measured system is known beforehand.

<sup>5</sup>This is consistent with the anti-Humean position about laws of nature in contemporary philosophy. According to this view, laws are grounded in the ontology, and the theoretical terms (expressed in the language of mathematics) connect to the entities existing in the physical world. It is essential for a property to induce a certain behaviour of the objects that instantiate the property in question, while the law expresses that behaviour. For example, the parameter we call “charge” in the Schrödinger equation refers to a property of quantum systems. This property is not a pure quality, but a disposition whose manifestation is the electromagnetic interaction between the systems as expressed qualitatively and quantitatively by the Schrödinger equation. In this way, laws are suitable to figure in explanations answering why-questions, and they reveal the real connections that there are in nature. By contrast, according to Humeanism, the laws are mere means of economical description, and they do not have any explanatory function. They sum up what has happened in the world; but they do not answer the question why what has happened did in fact happen, given certain initial conditions. Note that there are a number of substantial philosophical objections against Humeanism (see e.g. Mumford 2004).

Lastly, we will clarify some possible misunderstandings regarding the above argument. One might doubt the existence of charge density because it can only be measured under the special conditions of a protective measurement. This doubt is unfounded. The reason is that the conditions of a protective measurement are exactly those needed to measure the properties of a quantum system including its charge density without disturbing the system. A general measurement such as a conventional impulsive measurement, which does not satisfy the conditions required for a protective measurement, will destroy the measured state by the collapse of the wave function, and the measurement result cannot reflect the actual properties of the measured system such as the charge density of the system. In other words, the wavefunction collapse during a general measurement prevents the charge density from manifesting itself explicitly.

However, one may further point out that when a quantum system interacts with another quantum system under general conditions (e.g. the interaction is not adiabatic), the charge density of each system does not manifest itself explicitly either. For example, the interaction between two charged quantum systems are not described by charge density terms but by the potential terms in the Schrödinger equation. Since during the interaction no wavefunction collapse happens, the above reason does not apply. This is indeed true. The key to understand this fact is to realize that, as we have noted in the last section, the charge density (and electric flux density) are not a complete manifestation of the physical state of a many-body system, e.g. they do not contain the entanglement between its sub-systems. Therefore, although the charge density also has its efficiency under general circumstances<sup>6</sup>, the efficiency is always combined with the efficiencies of other elements of the complete physical state, and thus the manifestation of the charge density is not so obvious. Moreover, even for a one-body system the charge density (and electric flux density) may also have a deeper physical origin and only be an appearance of the underlying physical state (see below). Therefore, it is understandable that only under special circumstances such as an adiabatic interaction can the charge density manifest itself more directly and explicitly.

It should be stressed again that the charge distribution of a quantum system is not necessarily classical, and its existence does not imply that two quantum systems interact directly by way of their charge distributions as

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<sup>6</sup>It seems that the existence of the charge density and its efficiency can also be seen from the potential terms in the Schrödinger equation. For instance, the electrostatic interaction term  $Q\varphi\psi(x, t)$  in the Schrödinger equation for a charged quantum system suggests that the electrostatic interaction exists in all regions where  $\psi(x, t)$  is nonzero, where  $Q$  is the charge of the system,  $\psi(x, t)$  is the wave function of the system, and  $\varphi$  is an external electric scalar potential. Thus it seems that the charge of the system should also distribute throughout these regions. If the charge did not distribute and have efficiency in regions where the wave function is nonzero, then there would not exist electrostatic interaction there.

in classical mechanics either. In other words, the existence of the charge distribution can be consistent with quantum mechanics, in which the interaction between two quantum systems is described by the potential terms in the Schrödinger equation. As we will see in the next section, however, the consistency will restrict and even determine the existing form of the charge distribution of a quantum system.

## 4 The origin of charge density

We have demonstrated that the charge of a quantum system is distributed throughout space, and the charge density in each position is equal to the modulus squared of the wave function of the system there multiplied by the charge of the system. This confirms Schrödinger's original charge density hypothesis. In this section, we will further investigate the physical origin of the charge distribution. As we have pointed out earlier, there are at least two good motivations for our further investigation. First, although the charge distribution can be consistently defined for a many-body system, the distribution contains no information about the entanglement between the sub-systems of the many-body system. This indicates that the charge distribution is an incomplete manifestation of the underlying physical state and has a deeper physical origin. Second, even for one-body systems the charge distribution also has some puzzling features, e.g. the charge distribution of a single electron has no electrostatic self-interaction. These puzzling aspects are in want of a reasonable physical explanation, which may be provided by the origin of the charge distribution. In addition, the charge distribution has two possible forms, and we need to determine which possible form is the actual one. Again, this is closely related to the physical origin of the distribution.

As we have argued in the last section, protective measurements show that the expectation values of observables are the properties of a quantum system. These properties are defined either at a precise instant or during an infinitesimal time interval. Correspondingly, the charge distribution of a quantum system, which can be protectively measured as the expectation values of certain observables, has two possible existent forms: it is either real or effective. The distribution is real means that it exists throughout space at the same time. The distribution is effective means that at every instant there is only a localized, point-like particle with the total charge of the system, and its motion during an infinitesimal time interval forms the effective distribution. Concretely speaking, at a particular instant the charge density of the particle in each position is either zero (if the particle is not there) or singular (if the particle is there), while the time average of the density during an infinitesimal time interval gives the effective charge density. Moreover, the motion of the particle is ergodic in the sense that the

integral of the formed charge density in any region is required to be equal to the expectation value of the total charge in the region.

In the following, we will determine the existent form of the charge distribution of a quantum system. If the charge distribution is real, then any two parts of the distribution (e.g. the two wavepackets in box 1 and box 2 in the example given in the last section), like two electrons, will have electrostatic interaction described by the potential terms in the Schrödinger equation<sup>7</sup>. The existence of such electrostatic self-interaction for individual quantum systems contradicts the superposition principle of quantum mechanics (at least for microscopic systems such as electrons). Moreover, the existence of the electrostatic self-interaction for the charge distribution of an electron is incompatible with experimental observations either. For example, for the electron in the hydrogen atom, since the potential of the electrostatic self-interaction is of the same order as the Coulomb potential produced by the nucleus, the energy levels of hydrogen atoms would be remarkably different from those predicted by quantum mechanics and confirmed by experiments if there existed such electrostatic self-interaction. By contrast, if the charge distribution is effective, there will exist no electrostatic self-interaction of the effective distribution, as there is only a localized particle at every instant. This is consistent with the superposition principle of quantum mechanics and the Schrödinger equation.

Here is a further clarification of the above analysis. It can be seen that, as noted before, the non-existence of electrostatic self-interaction for the charge distribution of a single quantum system poses a puzzle. According to quantum mechanics, two charge distributions such as two electrons, which exist in space at the same time, have electrostatic interaction described by the potential term in the Schrödinger equation, but in the example given in the last section, the two charges in box 1 and box 2 have no such electrostatic interaction. This puzzle is not so much dependent on the existence of the charge distribution as a property of a quantum system. It is essentially that according to quantum mechanics, the wavepacket  $\psi_1$  in box 1 has interaction with any test electron (e.g. deviating the trajectory of the electron wavepacket), so does the wavepacket  $\psi_2$  in box 2, but these two wavepackets, unlike two electrons, have no electrostatic interaction.

Facing this puzzle one may have two choices. The first one is simply admitting that the non-existence of the self-interaction of the charge distribution is a distinct feature of the laws of quantum mechanics, but insisting that the laws are what they are and no further explanation is needed. However, this choice seems to beg the question and is unsatisfactory in the final analysis. A more reasonable choice is to try to explain this puzzling feature, e.g. by analyzing its relationship with the existent form of the charge

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<sup>7</sup>Moreover, these two parts will be also entangled and their wave function be defined in a six-dimensional configuration space.

distribution. The charge distribution has two possible forms after all. On the one hand, the non-existence of the self-interaction of the distribution may help determine which possible form is the actual one. For example, one possible form is inconsistent with this distinct feature, while the other possible form is consistent with it. On the other hand, the actual existent form of the charge distribution may also help explain the non-existence of the self-interaction of the distribution.

This is just what the previous analysis has done. The analysis establishes a connection between the non-existence of the self-interaction of the charge distribution and the actual existent form of the distribution. The reason why two wavepackets of an electron, each of which has part of the electron's charge, have no electrostatic interaction is that these two wavepackets do not exist at the same time, and their charges are formed by the motion of a localized particle with the total charge of the electron. Since there is only a localized particle at every instant, there exists no electrostatic self-interaction of the charge distribution formed by the motion of the particle. By contrast, if the two wavepackets with charges, like two electrons, existed at the same time, then they would also have the same form of electrostatic interaction as that between two electrons<sup>8</sup>.

Let's give a further analysis of the ergodic motion of a particle. In order that the ergodic motion of a particle forms the right charge distribution, for which the charge density in each position is proportional to the modulus squared of its wave function there, the probability density for the particle to appear in each position must be proportional to the modulus squared of its wave function there too (and for normalized wave functions they should be equal)<sup>9</sup>. This is understandable. As noted above, at a particular instant the charge density of the particle in each position is either zero (if the particle is not there) or singular (if the particle is there), and during an infinitesimal time interval the time average of the density generates the effective charge density. Therefore, that the (effective) charge density is larger in a position requires that the spending time of the particle near the position is longer or the frequency of the particle appearing there is higher. Loosely speaking, the ergodic motion of a particle will form a particle "cloud" extending throughout space (during an infinitesimal time interval), and the density of the cloud is  $|\psi(x, t)|^2$ , representing the probability density for the particle to appear in position  $x$  at instant  $t$ , where  $\psi(x, t)$  is the wave function of the particle. For a charged particle such as an electron, the cloud will be an electric cloud, and the density  $|\psi(x, t)|^2$ , when multiplied by the charge of

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<sup>8</sup>Note that this argument does not assume that real charges which exist at the same time are classical charges and they have classical interaction. By contrast, the Schrödinger-Newton equation, which was proposed by Diósi (1984) and Penrose (1998), treats the mass distribution of a quantum system as classical.

<sup>9</sup>Besides, for normalized wave functions, the probability current density must also equal to the formed charge flux density divided by the charge of the particle.

the particle, will be the charge density of the cloud.

This picture of ergodic motion of a single particle can be extended to many particles. The extension may help explain the multi-dimensionality of the wave function and solve the incompleteness problem for charge density. At a given instant, a quantum system of  $N$  particles can be represented by a point in a  $3N$ -dimensional configuration space. During an infinitesimal time interval near the instant, the representative point performs the ergodic motion in the configuration space and forms a cloud there. Then, similar to the single particle case, the representative point is required to spend in each volume element in the configuration space a time that is proportional to the modulus squared of the wave function of the system there. In other words, the density of the cloud in the configuration space is  $\rho(x_1, x_2, \dots, x_N, t) = |\psi(x_1, x_2, \dots, x_N, t)|^2$ , where  $\psi(x_1, x_2, \dots, x_N, t)$  is the wave function of the system. The density represents the probability density of particle 1 appearing in position  $x_1$  and particle 2 appearing in position  $x_2$ , ..., and particle  $N$  appearing in position  $x_N$ . When these  $N$  particles are independent, the density can be reduced to the direct product of the density for each particle, namely  $\rho(x_1, x_2, \dots, x_N, t) = \prod_{i=1}^N |\psi(x_i, t)|^2$ .

## 5 Conclusions

Historically, it is Schrödinger who first assumed the existence of the charge distribution of an electron and gave the mathematical formulation of the distribution. Schrödinger clearly realized that the charge distribution cannot be of classical nature because his equation does not include the usual classical interaction between the distributions. A more puzzling behavior is that the charge distribution of a quantum system has no electrostatic self-interaction. Moreover, although the charge distribution can be consistently defined for a many-body system, it is an incomplete manifestation of the wave function of the system. Presumably since people thought that the charge distribution could not be measured and also lacked a consistent physical picture, Schrödinger's charge density hypothesis has been largely ignored, though not refuted.

In this paper, we have re-examined Schrödinger's original hypothesis and argued that it is actually in accordance with quantum mechanics. It is demonstrated that the charge distribution of a quantum system can be directly measured by protective measurements as expectation values of certain observables, and the results as predicted by quantum mechanics show that the charge of a charged quantum system is distributed throughout space, and the charge density in each position is proportional to the modulus squared of the wave function of the system there. This confirms Schrödinger's charge density hypothesis. Moreover, we have argued that the charge distribution may also have a consistent physical picture. It is demonstrated that the

superposition principle of quantum mechanics requires that the charge distribution of a quantum system is effective, that is, it is formed by the ergodic motion of a localized particle with the total charge of the system. This picture may explain the puzzling behavior of the charge distribution such as the non-existence of electrostatic self-interaction for the distribution. Besides, the ergodic motion of a particle can be extended to many particles, which may further solve the incompleteness problem. The extension may also provide a possible clue to the physical meaning of the wave function. It has been suggested that the wave function represents the state of the ergodic motion of particles, which is arguably random and discontinuous in nature (Gao 2013b).

In his 1927 Solvay report, Schrödinger said:

The classical system of material points does not really exist, instead there exists something that continuously fills the entire space and of which one would obtain a ‘snapshot’ if one dragged the classical system, with the camera shutter open, through *all* its configurations, the representative point in  $q$ -space spending in each volume element  $d\tau$  a time that is proportional to the *instantaneous* value of  $\psi\psi^*$  (Schrödinger 1928).

It will be interesting to see if Schrödinger already envisaged the picture of ergodic motion of particles suggested here. We will leave this intriguing question for future research.

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